Nihonkai Math. J.
Vol.25(2014), 151–172

THE DUNKL-WILLIAMS CONSTANT OF SYMMETRIC OCTAGONAL NORMS ON $\mathbb{R}^2$ II

HIROYASU MIZUGUCHI

Abstract. Recently, the author and two other researchers constructed a calculation method for the Dunkl-Williams constant $DW(X)$ of a normed linear space X. Using the method, we determined the constant of $\mathbb{R}^2$ with symmetric octagonal norms. In this paper, we calculate the Dunkl-Williams constant of its dual space. As the result, the space $\mathbb{R}^2$ with symmetric octagonal norm becomes an example for which the Dunkl-Williams constant of the own space and the dual space have same value.

1. Introduction and preliminaries

This paper is a continuation of [18]. A norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be absolute if $\|(a,b)\| = \|(|a|,|b|)\|$ for all $(a,b) \in \mathbb{R}^2$, and normalized if $\|(1,0)\| = \|(0,1)\| = 1$. Let AN_2 be the family of all absolute normalized norms on $\mathbb{R}^2$, and let Ψ_2 be the set of all continuous convex functions on $[0,1]$ satisfying $\max\{1-t,t\} \le \psi(t) \le 1$ for $t \in [0,1]$. According to [3], AN_2 and Ψ_2 are in a one-to-one correspondence with $\psi(t) = \|(1-t,t)\|$ for $t \in [0,1]$ and

$$\|(a,b)\|_\psi = \begin{cases} (|a|+|b|)\psi\left(\dfrac{|b|}{|a|+|b|}\right) & \text{if } (a,b) \neq (0,0), \\ 0 & \text{if } (a,b) = (0,0) \end{cases}$$

(see also [20]).

2010 *Mathematics Subject Classification.* 46B20.
Key words and phrases. Dunkl-Williams constant, absolute normalized norm, norming functional.

For each $\beta \in (1/2, 1)$, let $\psi_\beta(t) = \max\{1-t, t, \beta\}$. Then, $\psi_\beta \in \Psi_2$. The norm $\|\cdot\|_{\psi_\beta}$ associated with ψ_β is given by

$$\begin{aligned}\|(a,b)\|_{\psi_\beta} &= \max\{|a|, |b|, \beta(|a|+|b|)\} \\ &= \begin{cases} |a| & \left(|b| \le \dfrac{1-\beta}{\beta}|a|\right), \\ \beta(|a|+|b|) & \left(\dfrac{1-\beta}{\beta}|a| \le |b| \le \dfrac{\beta}{1-\beta}|a|\right), \\ |b| & \left(\dfrac{\beta}{1-\beta}|a| \le |b|\right). \end{cases}\end{aligned}$$

Remark that the unit sphere of $(\mathbb{R}^2, \|\cdot\|_{\psi_\beta})$ is an octagon, and that the norm $\|\cdot\|_{\psi_\beta}$ is symmetric, that is, $\|(a,b)\|_{\psi_\beta} = \|(b,a)\|_{\psi_\beta}$ for all $(a,b) \in \mathbb{R}^2$.

Throughout this paper, the term "normed linear space" always means a real normed linear space which has two or more dimension. Let X be a normed linear space, and let B_X and S_X denote the unit ball and the unit sphere of X, respectively. In [12], the Dunkl-Williams constant $DW(X)$ of a normed linear space X was introduced:

$$DW(X) = \sup\left\{\frac{\|x\|+\|y\|}{\|x-y\|}\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| : x, y \in X \setminus \{0\},\ x \ne y\right\}.$$

We collect some basic properties of the Dunkl-Williams constant:

(i) $2 \le DW(X) \le 4$ for any normed linear space X ([5]).
(ii) X is an inner product space if and only if $DW(X) = 2$ ([5, 14]).
(iii) X is uniformly non-square if and only if $DW(X) < 4$ ([1, 12]).

However, it is very hard to calculate the Dunkl-Williams constant. It is not known for almost all normed linear spaces.

In [18], we determined the Dunkl-Williams constant of $\mathbb{R}^2$ with $\|\cdot\|_{\psi_\beta}$ for all $\beta \in (1/2, 1)$:

$$DW((\mathbb{R}^2, \|\cdot\|_{\psi_\beta})) = \begin{cases} \dfrac{2}{\beta^2}\left\{(1-\beta)^2 + \beta^2\right\} & (1/2 < \beta \le 1/\sqrt{2}), \\ 4\left\{(1-\beta)^2 + \beta^2\right\} & (1/\sqrt{2} \le \beta < 1). \end{cases}$$

Our aim in this paper is to calculate the Dunkl-Williams constant of its dual space. Finally, we obtain that the Dunkl-Williams constant of $\mathbb{R}^2$ with $\|\cdot\|_{\psi_\beta}$ always coincide with that of its dual space.

2. The dual norm of $\|\cdot\|_{\psi_\beta}$

For $\psi \in \Psi_2$, a function ψ^* on $[0,1]$ is defined by

$$\psi^*(s) = \sup\left\{\frac{(1-t)(1-s)+ts}{\psi(t)} : t \in [0,1]\right\}$$

for $s \in [0,1]$. It was proved that $\psi^* \in \Psi_2$ and that $\|\cdot\|_{\psi^*} \in AN_2$ is the dual norm of $\|\cdot\|_\psi$, that is, $(\mathbb{R}^2, \|\cdot\|_\psi)^*$ is identified with $(\mathbb{R}^2, \|\cdot\|_{\psi^*})$ (cf. [15, 16, 17]). A norming functional f of $x = (x_1, x_2) \in (\mathbb{R}^2, \|\cdot\|_\psi)$ is identified with an element $(\alpha_1, \alpha_2) \in (\mathbb{R}^2, \|\cdot\|_{\psi^*})$ such that

$$\|(\alpha_1,\alpha_2)\|_\psi^* = 1 \quad \text{and} \quad \langle (x_1,x_2),(\alpha_1,\alpha_2)\rangle = \|(x_1,x_2)\|_\psi. \tag{1}$$

We denote by $D((\mathbb{R}^2, \|\cdot\|_\psi), x)$ the set of all elements $(\alpha_1,\alpha_2) \in (\mathbb{R}^2, \|\cdot\|_\psi^*)$ satisfying the condition (1).

For $\beta \in (1/2, 1)$, we determine the convex function $\psi_\beta^* \in \Psi_2$ and the dual norm $\|\cdot\|_{\psi_\beta^*}$ of $\|\cdot\|_{\psi_\beta}$.

Proposition 2.1. *Let $\beta \in (1/2,1)$. Then*

$$\psi_\beta^*(s) = \begin{cases} 1 - \dfrac{2\beta-1}{\beta}s & (0 \le s \le 1/2), \\ \dfrac{1-\beta}{\beta} + \dfrac{2\beta-1}{\beta}s & (1/2 \le s \le 1). \end{cases}$$

Proof. Fix $s \in [0,1]$. We define the function f_s from $[0,1]$ into $\mathbb{R}$ by

$$f_s(t) = \frac{(1-t)(1-s)+ts}{\psi_\beta(t)}.$$

We note that $\psi_\beta^*(s) = \max\{f_s(t) : 0 \le t \le 1\}$ and calculate the maximum of f_s on $[0,1]$. By the definition of ψ_β, we have

$$f_s(t) = \begin{cases} 1 - s + \dfrac{st}{1-t} & (0 \le t \le 1-\beta), \\ \dfrac{1-s-(1-2s)t}{\beta} & (1-\beta \le t \le \beta), \\ s + \dfrac{(1-s)(1-t)}{t} & (\beta \le t \le 1). \end{cases}$$

If $0 \le s \le 1/2$, then the function $f_s(t)$ is increasing on $[0, 1-\beta]$ and is decreasing on $[1-\beta, 1]$. Hence we have

$$\psi_\beta^*(s) = f_s(1-\beta) = 1 - \frac{2\beta-1}{\beta}s.$$

Suppose that $1/2 \le s \le 1$. Then the function $f_s(t)$ is increasing on $[0, \beta]$ and is decreasing on $[\beta, 1]$. Hence we have

$$\psi_\beta^*(s) = f_s(\beta) = \frac{1-\beta}{\beta} + \frac{2\beta - 1}{\beta} s.$$

Thus we obtain this proposition. □

From this result, we easily obtain the following

Proposition 2.2. *Let* $\beta \in (1/2, 1)$. *Then*

$$\|(a,b)\|_{\psi_\beta^*} = \begin{cases} |a| + \dfrac{1-\beta}{\beta}|b| & (|a| \ge |b|), \\ \dfrac{1-\beta}{\beta}|a| + |b| & (|a| \le |b|). \end{cases}$$

The Dunkl-Williams constant of $(\mathbb{R}, \|\cdot\|_{\psi_\beta})^*$ coincides with that of $(\mathbb{R}^2, \|\cdot\|_{\psi_\beta^*})$ and so we calculate $DW((\mathbb{R}^2, \|\cdot\|_{\psi_\beta^*}))$ in the following sections.

3. The calculation method

In [19], we obtain a calculation method of the Dunkl-Williams constant. When we make use of the calculation method, the notion of Birkhoff orthogonality plays an important role. We recall that $x \in X$ is said to be Birkhoff orthogonal to $y \in X$, denoted by $x \perp_B y$, if $\|x\| \le \|x + \lambda y\|$ for all $\lambda \in \mathbb{R}$. This notion has been studied in [2, 6, 7, 9, 10, 11] and so on.

To construct a calculation method, we introduced some notations related to Birkhoff orthogonality (cf. [18, 19]): For each $x \in S_X$, we define the subset $V(x)$ of X by $V(x) = \{y \in X : x \perp_B y\}$. For each $x \in S_X$ and each $y \in V(x)$, we put

$$\Gamma(x,y) = \left\{ \frac{\lambda + \mu}{2} : \lambda \le 0 \le \mu,\ \|x + \lambda y\| = \|x + \mu y\| \right\}$$

and $m(x,y) = \sup\{\|x + \gamma y\| : \gamma \in \Gamma(x,y)\}$. We define the positive number $M(x)$ by

$$M(x) = \sup\{m(x,y) : y \in V(x)\}.$$

Using these notions, we obtained a calculation method for the Dunkl-Williams constant.

Proposition 3.1 ([19]). *Let X be a normed linear space. Then,*

$$DW(X) = 2 \sup\{M(x) : x \in S_X\}.$$

For two-dimensional spaces, Proposition 3.1 has the following improvement.

Proposition 3.2 ([19]). *Let X be a two-dimensional normed linear space. Then,*

$$DW(X) = 2\sup\{M(x) : x \in \mathrm{ext}(B_X)\},$$

where $\mathrm{ext}(B_X)$ *denotes the set of all extreme points of* B_X.

From Proposition 3.2 and [18, Proposition 2.5], we obtain the following result concerning $(\mathbb{R}^2, \|\cdot\|_{\psi_\beta^*})$.

Proposition 3.3. *Let $\beta \in (1/2, 1)$. Then*

$$DW((\mathbb{R}^2, \|\cdot\|_{\psi_\beta^*})) = 2\max\{M((1,0)), M((\beta,\beta))\}.$$

Proof. It is easy to see that $\mathrm{ext}(B_{(\mathbb{R}^2,\|\cdot\|_{\psi_\beta^*})})$ is the set of all vertices of the octagon $S_{(\mathbb{R}^2,\|\cdot\|_{\psi_\beta^*})}$, that is,

$$\mathrm{ext}(B_{(\mathbb{R}^2,\|\cdot\|_{\psi_\beta^*})}) = \{(\pm 1, 0), (0, \pm 1)\} \cup \{(\varepsilon_1\beta, \varepsilon_2\beta) : |\varepsilon_1| = |\varepsilon_2| = 1\}$$

Since $\|\cdot\|_{\psi_\beta^*}$ is a symmetric absolute normalized norm on $\mathbb{R}^2$, the map $(x_1, x_2) \mapsto (-x_2, x_1)$ is an isometric isomorphism from $(\mathbb{R}^2, \|\cdot\|_{\psi_\beta^*})$ onto itself. Hence, by [18, Proposition 2.5], we have

$$M((0,1)) = M((-1,0)) = M((0,-1)) = M((1,0))$$

and

$$M((\varepsilon_1\beta, \varepsilon_2\beta)) = M((\beta,\beta)).$$

Thus, we obtain

$$\begin{aligned} DW((\mathbb{R}^2, \|\cdot\|_{\psi_\beta^*})) &= 2\sup\{M(x) : x \in \mathrm{ext}(B_{(\mathbb{R}^2,\|\cdot\|_{\psi_\beta^*})})\} \\ &= 2\max\{M((1,0)), M((\beta,\beta))\} \end{aligned}$$

by Proposition 3.2. □

For simplicity, we write $\|\cdot\|_\beta^*$ for $\|\cdot\|_{\psi_\beta^*}$ and let $X_\beta^* = (\mathbb{R}^2, \|\cdot\|_\beta^*)$. In addition, we put $e_1 = (1,0)$ and $x_\beta = (\beta,\beta)$. Then, by the preceding lemma, we have $DW(X_\beta^*) = 2\max\{M(e_1), M(x_\beta)\}$. To determine $DW(X_\beta^*)$, we calculate $M(e_1)$ and estimate $M(x_\beta)$.

4. The calculation of $M(e_1)$

In this section, we calculate $M(e_1)$ under the assumption $1/2 < \beta \le 1/\sqrt{2}$. We first determine the set $V(e_1)$.

The following is an important characterization of Birkhoff orthogonality.

Lemma 4.1 (James, 1947 [11]). *Let X be a normed linear space, and let x and y be two elements of X. Then, $x \perp_B y$ if and only if there exists a norming functional f of x such that $f(y) = 0$.*

From this lemma, one can easily have that

$$V(e_1) = \{(y_1, y_2) : \langle (y_1, y_2), (\alpha_1, \alpha_2)\rangle = 0 \text{ for some } (\alpha_1, \alpha_2) \in D(X_\beta^*, e_1)\}.$$

Henceforth, let $k_\beta = \frac{1-\beta}{\beta}$. Then $\sqrt{2} - 1 \le k_\beta < 1$ since $1/2 < \beta \le 1/\sqrt{2}$, and $\beta = (1 + k_\beta)^{-1}$.

Lemma 4.2. $V(e_1) = \{\alpha(c(1 + s), 1) : s \in [-1, -(1 - k_\beta)], |c| = 1, \alpha \in \mathbb{R}\}$.

Proof. It is easy to see that $(\psi_\beta^*)'_R(0) = -(1 - k_\beta)$, where $(\psi_\beta^*)'_R(0)$ is the right derivative of ψ_β^* at $t = 0$. According to [3, 16], we have

$$D(X_\beta^*, e_1) = \{(1, c(1 + s)) : s \in [-1, -(1 - k_\beta)], |c| = 1\}.$$

Thus we have

$$\begin{aligned} V(e_1) &= \{(y_1, y_2) : \langle (y_1, y_2), (\alpha_1, \alpha_2)\rangle = 0 \text{ for some } (\alpha_1, \alpha_2) \in D(X_\beta^*, e_1)\} \\ &= \{\alpha(-c(1 + s), 1) : s \in [-1, -(1 - k_\beta)], |c| = 1, \alpha \in \mathbb{R}\} \\ &= \{\alpha(c(1 + s), 1) : s \in [-1, -(1 - k_\beta)], |c| = 1, \alpha \in \mathbb{R}\}, \end{aligned}$$

as desired. □

To reduce the amount of calculation, we make use of some results used in [18] (cf. [19]). We note that

$$2 + k_\beta = \frac{1 + \beta}{\beta} \ge \frac{\beta}{1 - \beta} = k_\beta^{-1}$$

since $1/2 < \beta \le 1/\sqrt{2}$.

Lemma 4.3. $M(e_1) = \sup\{m(e_1, (1, -t)) : t \in (k_\beta^{-1}, \infty) \setminus \{2 + k_\beta\}\}$.

Proof. By the preceding lemma, $\{\alpha(c(1+s), 1) : s \in (-1, -(1-k_\beta)), |c| = 1, \alpha \in \mathbb{R}\}$ is a dense subset of $V(e_1)$. On the other hand,

$$\begin{aligned} &\{\alpha(c(1 + s), 1) : s \in (-1, -(1 - k_\beta)), |c| = 1, \alpha \in \mathbb{R}\} \\ &= \left\{\alpha\left(1, \frac{c}{1 + s}\right) : s \in (-1, -(1 - k_\beta)), |c| = 1, \alpha \in \mathbb{R}\right\}. \end{aligned}$$

Since the function $s \mapsto 1/(1+s)$ is continuous and decreasing, it maps $(-1, -(1-k_\beta))$ onto (k_β^{-1}, ∞). Thus one can have that

$$\begin{aligned} &\left\{\alpha\left(1, \frac{c}{1 + s}\right) : s \in (-1, -(1 - k_\beta)), |c| = 1, \alpha \in \mathbb{R}\right\} \\ &= \{\alpha(1, ct) : t \in (k_\beta^{-1}, \infty), |c| = 1, \alpha \in \mathbb{R}\}. \end{aligned}$$

From this, it follows that $\{\alpha(1, ct) : t \in (k_\beta^{-1}, \infty) \setminus \{2 + k_\beta\}, |c| = 1, \alpha \in \mathbb{R}\}$ is also a dense subset of $V(e_1)$. Since the map $(x_1, x_2) \mapsto (x_1, -x_2)$ is an isometric isomorphism from X_β^* onto itself, we have

$$M(e_1) = \sup\{m(e_1, \alpha(1, -t)) : t \in (k_\beta^{-1}, \infty) \setminus \{2 + k_\beta\}, \alpha \in \mathbb{R}\}$$

by [18, Proposition 2.5 and Lemma 2.7]. Finally, applying [18, Lemma 2.4], we obtain $M(e_1) = \sup\{m(e_1, (1, -t)) : t \in (k_\beta^{-1}, \infty) \setminus \{2 + k_\beta\}\}$. □

For each $t \in \mathbb{R}$, put $y_t = (1, -t)$. We give the formula of $\|e_1 + \lambda y_t\|_\beta^*$ for all $t \in (k_\beta^{-1}, \infty)$ and all $\lambda \in \mathbb{R}$.

Lemma 4.4. *Let $t \in (k_\beta^{-1}, \infty)$, and let*

$$a_t = \frac{1}{t-1} \quad \text{and} \quad b_t = -\frac{1}{t+1}.$$

Then

$$\|e_1 + \lambda y_t\|_\beta^* = \begin{cases} -k_\beta - (t + k_\beta)\lambda & (\lambda \leq -1), \\ k_\beta - (t - k_\beta)\lambda & (-1 \leq \lambda \leq b_t), \\ 1 - (k_\beta t - 1)\lambda & (b_t \leq \lambda \leq 0), \\ 1 + (k_\beta t + 1)\lambda & (0 \leq \lambda \leq a_t), \\ k_\beta + (t + k_\beta)\lambda & (a_t \leq \lambda). \end{cases}$$

Proof. First we note that $e_1 + \lambda y_t = (1 + \lambda, -t\lambda)$ and that

$$-1 < -\frac{1}{t+1} = b_t < 0 < \frac{1}{t-1} = a_t.$$

By the definition of $\|\cdot\|_\beta^*$, we have

$$\|e_1 + \lambda y_t\|_\beta^* = \begin{cases} |1 + \lambda| + k_\beta|-t\lambda| & (|1 + \lambda| \geq |-t\lambda|), \\ k_\beta|1 + \lambda| + |-t\lambda| & (|1 + \lambda| \leq |-t\lambda|). \end{cases}$$

On the other hand, one has

$$(1 + \lambda)^2 - (-t\lambda)^2 = -(t + 1)(t - 1)(\lambda - a_t)(\lambda - b_t).$$

Thus, we obtain this lemma. □

By the preceding lemma, we immediately have the following

Lemma 4.5. *Let $t \in (k_\beta^{-1}, \infty)$. Then the function $\lambda \mapsto \|e_1 + \lambda y_t\|_\beta^*$ is strictly decreasing on $(-\infty, 0]$ and strictly increasing on $[0, \infty)$.*

We consider the relationship among $\|e_1 + a_t y_t\|_\beta^*$, $\|e_1 + b_t y_t\|_\beta^*$ and $\|e_1 - y_t\|_\beta^*$.

Lemma 4.6. *Let $t \in (k_\beta^{-1}, \infty) \setminus \{2 + k_\beta\}$. Then the following hold:*

(i) *If $t \in (k_\beta^{-1}, 2 + k_\beta)$, then $\|e_1 + b_t y_t\|_\beta^* < \|e_1 - y_t\|_\beta^* < \|e_1 + a_t y_t\|_\beta^*$.*
(ii) *If $t \in (2 + k_\beta, \infty)$, then $\|e_1 + b_t y_t\|_\beta^* < \|e_1 + a_t y_t\|_\beta^* < \|e_1 - y_t\|_\beta^*$.*

Proof. By Lemma 4.4, we have

$$\|e_1 - y_t\|_\beta^* = t \quad \text{and} \quad \|e_1 + a_t y_t\|_\beta^* = 1 + \frac{k_\beta t + 1}{t-1}$$

which implies that

$$\|e_1 - y_t\|_\beta^* - \|e_1 + a_t y_t\|_\beta^* = t - 1 - \frac{k_\beta t + 1}{t-1} = \frac{t}{t-1}\{t - (2 + k_\beta)\}.$$

Thus, $\|e_1 + a_t y_t\|_\beta^* > \|e_1 - y_t\|_\beta^*$ if $t < 2 + k_\beta$, and $\|e_1 + a_t y_t\|_\beta^* < \|e_1 - y_t\|_\beta^*$ if $t > 2 + k_\beta$.

Suppose that $t \in (k_\beta^{-1}, 2 + k_\beta)$. Then, as was mentioned above, $\|e_1 - y_t\|_\beta^* < \|e_1 + a_t y_t\|_\beta^*$. Moreover, since $-1 < b_t < 0$, by Lemma 4.5, we have $\|e_1 + b_t y_t\|_\beta^* < \|e_1 - y_t\|_\beta^*$.

Next we assume that $t \in (2 + k_\beta, \infty)$. Then we have $\|e_1 + a_t y_t\|_\beta^* < \|e_1 - y_t\|_\beta^*$. Further, by Lemma 4.4, we obtain

$$\|e_1 + b_t y_t\|_\beta^* = 1 + \frac{k_\beta t - 1}{t+1} < 1 + \frac{k_\beta t + 1}{t-1} = \|e_1 + a_t y_t\|_\beta^*.$$

This shows (ii). □

Let $t \in (k_\beta^{-1}, \infty) \setminus \{2 + k_\beta\}$. Then, the intermediate value theorem guarantees that the function $\lambda \mapsto \|e_1 + \lambda y_t\|_\beta^*$ maps $(-\infty, 0]$ onto $[1, \infty)$ and $[0, \infty)$ onto $[1, \infty)$. Thus, for any $\mu \in [0, \infty)$, there exists $\lambda \in (-\infty, 0]$ such that $\|e_1 + \lambda y_t\|_\beta^* = \|e_1 + \mu y_t\|_\beta^*$. Furthermore, by Lemma 4.5, this gives a one-to-one correspondence between $[0, \infty)$ and $(-\infty, 0]$. Let p_t, q_t, r_t be real numbers such that $p_t < 0 < q_t, r_t$, $\|e_1 + a_t y_t\|_\beta^* = \|e_1 + p_t y_t\|_\beta^*$, $\|e_1 + b_t y_t\|_\beta^* = \|e_1 + q_t y_t\|_\beta^*$, and $\|e_1 - y_t\|_\beta^* = \|e_1 + r_t y_t\|_\beta^*$. Then we have the following

Lemma 4.7. *Let $t \in (k_\beta^{-1}, \infty) \setminus \{2 + k_\beta\}$. Then the following hold:*

(i) *If $t \in (k_\beta^{-1}, 2 + k_\beta)$, then $p_t < -1 < b_t < 0 < q_t < r_t < a_t$ and*

$$p_t = -a_t - \frac{2k_\beta}{k_\beta + t}, \quad q_t = -\frac{k_\beta t - 1}{k_\beta t + 1} b_t, \quad \text{and} \quad r_t = \frac{t-1}{k_\beta t + 1}.$$

(ii) *If $t \in (2 + k_\beta, \infty)$, then $-1 < p_t < b_t < 0 < q_t < a_t < r_t$ and*

$$p_t = -\frac{t + k_\beta}{t - k_\beta} a_t, \quad q_t = -\frac{k_\beta t - 1}{k_\beta t + 1} b_t, \quad \text{and} \quad r_t = \frac{t - k_\beta}{t + k_\beta}.$$

Proof. Suppose that $t \in (k_\beta^{-1}, 2 + k_\beta)$. Then we clearly have $-1 < b_t < 0 < a_t$. Using Lemma 4.6 (i), we have the following diagram:

$$\begin{array}{ccccccc} + : & \|e_1 + q_t y_t\|_\beta & < & \|e_1 + r_t y_t\|_\beta & < & \|e_1 + a_t y_t\|_\beta \\ & \| & & \| & & \| \\ - : & \|e_1 + b_t y_t\|_\beta & < & \|e_1 - y_t\|_\beta & < & \|e_1 + p_t y_t\|_\beta. \end{array}$$

Thus, by Lemma 4.5, it follows that $p_t < -1 < b_t < 0 < q_t < r_t < a_t$. Then we have

$$\begin{aligned}
-k_\beta - (t+k_\beta)p_t &= \|e_1 + p_t y_t\|_\beta^* = \|e_1 + a_t y_t\|_\beta^* = k_\beta + (t+k_\beta)a_t,\\
1 + (k_\beta t + 1)q_t &= \|e_1 + q_t y_t\|_\beta^* = \|e_1 + b_t y_t\|_\beta^* = 1 - (k_\beta t - 1)b_t \text{ and}\\
1 + (k_\beta t + 1)r_t &= \|e_1 + r_t y_t\|_\beta^* = \|e_1 - y_t\|_\beta^* = t.
\end{aligned}$$

Thus one can obtain (i). One can show (ii) similarly, so we omit the proof. □

Next, we consider the set $\Gamma(e_1, y_t)$. As was mentioned, for each $\mu \in [0, \infty)$ there exists a unique $\lambda_\mu \in (-\infty, 0]$ such that $\|e_1 + \lambda_\mu y_t\|_\beta^* = \|e_1 + \mu y_t\|_\beta^*$. Then it follows that

$$\Gamma(e_1, y_t) = \left\{ \frac{\lambda_\mu + \mu}{2} : \mu \in [0, \infty) \right\}.$$

Lemma 4.8. *Let $t \in (k_\beta^{-1}, 2 + k_\beta)$. Then,*

$$\Gamma(e_1, y_t) = \left[\frac{-1 + r_t}{2}, 0 \right].$$

Proof. By Lemma 4.7 (i), we have $p_t < -1 < b_t < 0 < q_t < r_t < a_t$.

Suppose that $0 \leq \mu \leq q_t$. Then $b_t \leq \lambda_\mu \leq 0$, and so we have

$$1 - (k_\beta t - 1)\lambda_\mu = \|e_1 + \lambda_\mu y_t\|_\beta^* = \|e_1 + \mu y_t\|_\beta^* = 1 + (k_\beta t + 1)\mu.$$

Thus we have

$$\lambda_\mu = -\frac{k_\beta t + 1}{k_\beta t - 1}\mu$$

and

$$\frac{\lambda_\mu + \mu}{2} = -\frac{\mu}{k_\beta t - 1}.$$

Since $t \in (k_\beta^{-1}, 2 + k_\beta)$, the function $\mu \mapsto \frac{\lambda_\mu + \mu}{2}$ is decreasing on $[0, q_t]$. Thus we have

$$\left\{ \frac{\lambda_\mu + \mu}{2} : \mu \in [0, q_t] \right\} = \left[\frac{b_t + q_t}{2}, 0 \right].$$

Next, we suppose that $q_t \leq \mu \leq r_t$. Then $-1 \leq \lambda_\mu \leq b_t$, and so we have

$$k_\beta - (t - k_\beta)\lambda_\mu = \|e_1 + \lambda_\mu y_t\|_\beta^* = \|e_1 + \mu y_t\|_\beta^* = 1 + (k_\beta t + 1)\mu.$$

From this we have

$$\lambda_\mu = -\frac{1 - k_\beta}{t - k_\beta} - \frac{k_\beta t + 1}{t - k_\beta}\mu$$

and

$$\frac{\lambda_\mu + \mu}{2} = -\frac{1 - k_\beta}{2(t - k_\beta)} + \frac{(1 - k_\beta)t - (1 + k_\beta)}{2(t - k_\beta)}\mu.$$

Since $t \in (k_\beta^{-1}, 2+k_\beta)$, we have $(1-k_\beta)t - (1+k_\beta) < 0$ and hence the function $\mu \mapsto \frac{\lambda_\mu+\mu}{2}$ is decreasing on $[q_t, r_t]$, and hence

$$\left\{\frac{\lambda_\mu+\mu}{2} : \mu \in [q_t, r_t]\right\} = \left[\frac{-1+r_t}{2}, \frac{b_t+q_t}{2}\right].$$

In the case of $r_t \le \mu \le a_t$, we have $p_t \le \lambda_t \le -1$. Then we have

$$-k_\beta - (t+k_\beta)\lambda_\mu = \|e_1 + \lambda_\mu y_t\|_\beta^* = \|e_1 + \mu y_t\|_\beta^* = 1 + (k_\beta t + 1)\mu.$$

It follows that

$$\lambda_\mu = -\frac{1+k_\beta}{t+k_\beta} - \frac{k_\beta t+1}{t+k_\beta}\mu$$

and

$$\frac{\lambda_\mu+\mu}{2} = -\frac{1+k_\beta}{2(t+k_\beta)} + \frac{(1-k_\beta)(t-1)}{2(t+k_\beta)}\mu.$$

Since $1 < k_\beta^{-1} < t$, the function $\mu \mapsto \frac{\lambda_\mu+\mu}{2}$ is increasing on $[r_t, a_t]$. Thus we have

$$\left\{\frac{\lambda_\mu+\mu}{2} : \mu \in [r_t, a_t]\right\} = \left[\frac{-1+r_t}{2}, \frac{a_t+p_t}{2}\right].$$

Finally, we assume $a_t \le \mu$. Then $\lambda_\mu \le p_t$ and hence

$$-k_\beta - (t+k_\beta)\lambda_\mu = \|e_1 + \lambda_\mu y_t\|_\beta^* = \|e_1 + \mu y_t\|_\beta^* = k_\beta + (t+k_\beta)\mu.$$

Thus we have

$$\frac{\lambda_\mu+\mu}{2} = -\frac{k_\beta}{t+k_\beta} = \frac{a_t+p_t}{2}.$$

Since the function $\mu \mapsto \frac{\lambda_\mu+\mu}{2}$ is continuous, one has that

$$\begin{aligned}\Gamma(e_1, y_t) &= \left\{\frac{\lambda_\mu+\mu}{2} : \mu \in [0,\infty)\right\}\\ &= \left[\frac{b_t+q_t}{2}, 0\right] \cup \left[\frac{-1+r_t}{2}, \frac{b_t+q_t}{2}\right] \cup \left[\frac{-1+r_t}{2}, \frac{a_t+p_t}{2}\right]\\ &= \left[\frac{-1+r_t}{2}, 0\right].\end{aligned}$$

□

We remark that

$$2 + k_\beta = \frac{1+\beta}{\beta} \le \frac{1}{2\beta-1} = \frac{1+k_\beta}{1-k_\beta}$$

since $1/2 < \beta \le 1/\sqrt{2}$.

Lemma 4.9. *Let $t \in (2+k_\beta, \infty)$. Then*

$$\Gamma(e_1, y_t) = \left[\frac{-1+r_t}{2}, 0\right].$$

Proof. By Lemma 4.7 (ii), we have $-1 < p_t < b_t < 0 < q_t < a_t < r_t$. Suppose that $0 \leq \mu \leq q_t$. Then $b_t \leq \lambda_\mu \leq 0$ and so

$$1 - (k_\beta t - 1)\lambda_\mu = \|e_1 + \lambda_\mu y_t\|_\beta^* = \|e_1 + \mu y_t\|_\beta^* = 1 + (k_\beta t + 1)\mu.$$

As in the proof of the preceding lemma, we have

$$\frac{\lambda_\mu + \mu}{2} = -\frac{\mu}{k_\beta t - 1},$$

which implies that the function $\mu \mapsto \frac{\lambda_\mu + \mu}{2}$ is decreasing on $[0, q_t]$. Thus we obtain

$$\left\{\frac{\lambda_\mu + \mu}{2} : \mu \in [0, q_t]\right\} = \left[\frac{b_t + q_t}{2}, 0\right].$$

In the case of $q_t \leq \mu \leq a_t$, we have $p_t \leq \lambda_\mu \leq b_t$ and hence

$$k_\beta - (t - k_\beta)\lambda_\mu = \|e_1 + \lambda_\mu y_t\|_\beta^* = \|e_1 + \mu y_t\|_\beta^* = 1 + (k_\beta t + 1)\mu.$$

Thus, we have

$$\frac{\lambda_\mu + \mu}{2} = -\frac{1 - k_\beta}{2(t - k_\beta)} + \frac{(1 - k_\beta)t - (1 + k_\beta)}{2(t - k_\beta)}\mu.$$

This implies that the function $\mu \mapsto \frac{\lambda_\mu + \mu}{2}$ is decreasing on $[q_t, a_t]$ if $t \leq (1 + k_\beta)/(1 - k_\beta)$, and is increasing if $t \geq (1 + k_\beta)/(1 - k_\beta)$. Hence we have

$$\left\{\frac{\lambda_\mu + \mu}{2} : \mu \in [q_t, a_t]\right\} = \begin{cases} \left[\frac{a_t + p_t}{2}, \frac{b_t + q_t}{2}\right] & \left(2 + k_\beta < t \leq \frac{1 + k_\beta}{1 - k_\beta}\right), \\ \left[\frac{b_t + q_t}{2}, \frac{a_t + p_t}{2}\right] & \left(\frac{1 + k_\beta}{1 - k_\beta} \leq t < \infty\right). \end{cases}$$

Assume that $a_t \leq \mu \leq r_t$. Then we have $-1 \leq \lambda_\mu \leq p_t$ and so

$$k_\beta - (t - k_\beta)\lambda_\mu = \|e_1 + \lambda_\mu y_t\|_\beta^* = \|e_1 + \mu y_t\|_\beta^* = k_\beta + (t + k_\beta)\mu.$$

Thus, we obtain

$$\lambda_\mu = -\frac{t + k_\beta}{t - k_\beta}\mu$$

and

$$\frac{\lambda_\mu + \mu}{2} = -\frac{k_\beta}{t - k_\beta}\mu.$$

It follows that the function $\mu \mapsto \frac{\lambda_\mu + \mu}{2}$ is decreasing on $[a_t, r_t]$, and hence

$$\left\{\frac{\lambda_\mu + \mu}{2} : \mu \in [a_t, r_t]\right\} = \left[\frac{-1 + r_t}{2}, \frac{a_t + p_t}{2}\right].$$

In the case of $r_t \leq \mu$, we have $\lambda_\mu \leq -1$. Thus we have

$$-k_\beta - (t + k_\beta)\lambda_\mu = \|e_1 + \lambda_\mu y_t\|_\beta^* = \|e_1 + \mu y_t\|_\beta^* = k_\beta + (t + k_\beta)\mu$$

and
$$\frac{\lambda_\mu+\mu}{2}=-\frac{k_\beta}{t+k_\beta}=\frac{-1+r_t}{2}.$$

Finally, if $2+k_\beta<t\le(1+k_\beta)/(1-k_\beta)$, then
$$\begin{aligned}\Gamma(e_1,y_t)&=\left\{\frac{\lambda_\mu+\mu}{2}:\mu\in[0,\infty)\right\}\\&=\left[\frac{b_t+q_t}{2},0\right]\cup\left[\frac{a_t+p_t}{2},\frac{b_t+q_t}{2}\right]\cup\left[\frac{-1+r_t}{2},\frac{a_t+p_t}{2}\right]\\&=\left[\frac{-1+r_t}{2},0\right].\end{aligned}$$

On the other hand, if $(1+k_\beta)/(1-k_\beta)\le t<\infty$, then
$$\begin{aligned}\Gamma(e_1,y_t)&=\left\{\frac{\lambda_\mu+\mu}{2}:\mu\in[0,\infty)\right\}\\&=\left[\frac{b_t+q_t}{2},0\right]\cup\left[\frac{b_t+q_t}{2},\frac{a_t+p_t}{2}\right]\cup\left[\frac{-1+r_t}{2},\frac{a_t+p_t}{2}\right]\\&=\left[\min\left\{\frac{b_t+q_t}{2},\frac{-1+r_t}{2}\right\},\max\left\{0,\frac{a_t+p_t}{2}\right\}\right].\end{aligned}$$

However, we have
$$\frac{a_t+p_t}{2}=-\frac{k_\beta}{t-k_\beta}a_t<0$$

and
$$\begin{aligned}\frac{b_t+q_t}{2}-\frac{-1+r_t}{2}&=\frac{t\{k_\beta^2t+k_\beta(1+k_\beta)-1\}}{(t+k_\beta)(k_\beta t-1)(t+1)}\\&>\frac{t(k_\beta^2+2k_\beta-1)}{(t+k_\beta)(k_\beta t-1)(t+1)}\ge 0\end{aligned}$$

since $k_\beta^{-1}\le 2+k_\beta<t$ and $\sqrt{2}-1\le k_\beta<1$. Thus we obtain this lemma. □

Now, we calculate $M(e_1)$. We note that the formulas of $\frac{-1+r_t}{2}$ in Lemmas 4.8 and 4.9 are not the same.

Proposition 4.10. $M(e_1)=1+k_\beta^2$.

Proof. By Lemma 4.3, $M(e_1)=\sup\{m(e_1,y_t):t\in(k_\beta^{-1},\infty)\setminus\{2+k_\beta\}\}$. In the case of $t\in(k_\beta^{-1},2+k_\beta)$, we have $b_t<\frac{-1+r_t}{2}<0$. Indeed, one has
$$0>\frac{-1+r_t}{2}=\frac{-2+(1-k_\beta)t}{2(k_\beta t+1)}$$

and

$$\frac{-1+r_t}{2} - b_t = \frac{(1-k_\beta)t-2}{2(k_\beta t+1)} + \frac{1}{1+t} = \frac{(1-k_\beta)(t-1)t}{2(k_\beta t+1)(1+t)} > 0.$$

It follows from $1 < k_\beta^{-1} < t$ that

$$\left\| e_1 + \frac{-1+\tau_t}{2} y_t \right\|_\beta^* = 1 + \frac{(k_\beta t-1)\{-(1-k_\beta)t+2\}}{2(k_\beta t+1)}$$

$$< 1 + \frac{(k_\beta t-1)(1+k_\beta)}{2(k_\beta t+1)}.$$

Since the function $t \mapsto (k_\beta t-1)/(1+k_\beta t)$ is strictly increasing,

$$\frac{(k_\beta t-1)(1+k_\beta)}{2(k_\beta t+1)} < \frac{\{k_\beta(2+k_\beta)-1\}(1+k_\beta)}{2\{1+k_\beta(2+k_\beta)\}} = \frac{k_\beta^2+2k_\beta-1}{2(1+k_\beta)}.$$

On the other hand, we have

$$k_\beta^2 - \frac{k_\beta^2+2k_\beta-1}{2(1+k_\beta)} = \frac{2k_\beta^3+(1-k_\beta)^2}{2(1+k_\beta)} > 0.$$

Thus we obtain

$$\left\| e_1 + \frac{-1+\tau_t}{2} y_t \right\|_\beta^* < 1 + k_\beta^2,$$

and hence

$$m(e_1, y_t) = \max\left\{ \left\| e_1 + \frac{-1+r_t}{2} y_t \right\|_\beta^*, \|e_1\|_\beta^* \right\} < 1 + k_\beta^2$$

by [18, Lemma 2.6].

Let $t \in (2+k_\beta, \infty)$. Since $k_\beta < 1$, we have

$$b_t = -\frac{1}{t+1} < -\frac{k_\beta}{t+k_\beta} = \frac{-1+r_t}{2} < 0,$$

and so

$$\left\| e_1 + \frac{-1+r_t}{2} y_t \right\|_\beta^* = 1 + \frac{k_\beta(k_\beta t-1)}{t+k_\beta}.$$

From the fact that the function $t \mapsto (k_\beta t-1)/(t+k_\beta)$ is strictly increasing, it follows that

$$\frac{k_\beta(k_\beta t-1)}{t+k_\beta} < k_\beta^2.$$

Hence, by [18, Lemma 2.6], we have

$$m(e_1, y_t) = \max\left\{ \left\| e_1 + \frac{-1+r_t}{2} y_t \right\|_\beta^*, \|e_1\|_\beta^* \right\} < 1 + k_\beta^2.$$

Thus, by Lemma 4.3, we obtain $M(e_1) \leq 1 + k_\beta^2$.

Finally, since

$$M(e_1) \geq 1 + \frac{k_\beta(k_\beta t - 1)}{t + k_\beta}$$

for each $t \in (2 + k_\beta, \infty)$, we have $M(e_1) \geq 1 + k_\beta^2$. This implies that $M(e_1) = 1 + k_\beta^2$. □

5. The estimation of $M(x_\beta)$

As in the above section, we suppose that $1/2 < \beta \leq 1/\sqrt{2}$. We prove $M(x_\beta) \leq 1 + k_\beta^2$. To do this, we start with determining the set $V(x_\beta)$.

Lemma 5.1. $V(x_\beta) = \{\alpha y_t : t \in [k_\beta, k_\beta^{-1}], \alpha \in \mathbb{R}\}$.

Proof. First we note that

$$x_\beta = (\beta, \beta) = \frac{1}{\psi_\beta^*(1/2)} \left(\frac{1}{2}, \frac{1}{2} \right).$$

One can have $(\psi_\beta^*)'_L(1/2) = -(1-k_\beta)$ and $(\psi_\beta^*)'_R(1/2) = 1-k_\beta$, where $(\psi_\beta^*)'_L(1/2)$ and $(\psi_\beta^*)'_R(1/2)$ are respectively the left and right derivative of ψ_β^* at $t = 1/2$. According to [3, 16], we have

$$D(X_\beta^*, x_\beta) = \left\{ \frac{1}{2}(1 + k_\beta - s, 1 + k_\beta + s) : s \in [-(1 - k_\beta), 1 - k_\beta] \right\}.$$

Thus,

$$V(x_\beta) = \{\alpha(1 + k_\beta + s, -(1 + k_\beta - s)) : s \in [-(1 - k_\beta), 1 - k_\beta], \alpha \in \mathbb{R}\}$$

$$= \left\{ \alpha \left(1, -\frac{1 + k_\beta - s}{1 + k_\beta + s} \right) : s \in [-(1 - k_\beta), 1 - k_\beta], \alpha \in \mathbb{R} \right\}.$$

Since the function $s \mapsto (1 + k_\beta - s)/(1 + k_\beta + s)$ is continuous and decreasing, it maps $[-(1 - k_\beta), 1 - k_\beta]$ onto $[k_\beta, k_\beta^{-1}]$. Therefore one can obtain $V(x_\beta) = \{\alpha y_t : t \in [k_\beta, k_\beta^{-1}], \alpha \in \mathbb{R}\}$. □

As in Lemma 4.3, we reduce the amount of calculation.

Lemma 5.2. $M(x_\beta) = \sup\{m(x_\beta, y_t) : t \in (1, k_\beta^{-1})\}$

Proof. By Lemma 5.1, it is clear that $\{\alpha y_t : t \in (k_\beta, k_\beta^{-1}) \setminus \{1\}, \alpha \in \mathbb{R}\}$ is the dense subset of $V(x_\beta)$. Since an isometric isomorphism $(x_1, x_2) \mapsto (x_2, x_1)$ maps αy_t to $\alpha(-t, 1) = -\alpha t y_{1/t}$, we have

$$M(x_\beta) = \sup\{m(x_\beta, \alpha y_t) : t \in (1, k_\beta^{-1}), \alpha \in \mathbb{R}\}$$

by [18, Proposition 2.5 and Lemma 2.7]. Thus we obtain

$$M(x_\beta) = \sup\{m(x_\beta, y_t) : t \in (1, k_\beta^{-1})\}$$

by [18, Lemma 2.4]. □

Next we give the formula of $\|x_\beta + \lambda y_t\|_\beta^*$ for all $t \in (1, k_\beta^{-1})$ and all $\lambda \in \mathbb{R}$.

Lemma 5.3. *Let $t \in (1, k_\beta^{-1})$, and let*

$$c_t = \frac{1}{(1+k_\beta)t} \quad \text{and} \quad d_t = \frac{2}{(1+k_\beta)(t-1)}.$$

Then

$$\|x_\beta + \lambda y_t\|_\beta^* = \begin{cases} \dfrac{1-k_\beta}{1+k_\beta} - (k_\beta + t)\lambda & (\lambda \le -(1+k_\beta)^{-1}), \\ 1 - (t - k_\beta)\lambda & (-(1+k_\beta)^{-1} \le \lambda \le 0), \\ 1 + (1 - k_\beta t)\lambda & (0 \le \lambda \le c_t), \\ \dfrac{1-k_\beta}{1+k_\beta} + (1 + k_\beta t)\lambda & (c_t \le \lambda \le d_t), \\ -\dfrac{1-k_\beta}{1+k_\beta} + (k_\beta + t)\lambda & (d_t \le \lambda). \end{cases}$$

Proof. First we note that

$$x_\beta + \lambda y_t = \big((1+k_\beta)^{-1} + \lambda, (1+k_\beta)^{-1} - t\lambda\big)$$

and that

$$-(1+k_\beta)^{-1} < 0 < c_t = \frac{1}{(1+k_\beta)t} < \frac{2}{(1+k_\beta)(t-1)} = d_t.$$

It follows from the definition of $\|\cdot\|_\beta^*$ that

$$\|x_\beta + \lambda y_t\|_\beta^* = \begin{cases} |(1+k_\beta)^{-1} + \lambda| + k_\beta |(1+k_\beta)^{-1} - t\lambda| \\ \qquad\qquad (|(1+k_\beta)^{-1} + \lambda| \ge |(1+k_\beta)^{-1} - t\lambda|), \\ k_\beta |(1+k_\beta)^{-1} + \lambda| + |(1+k_\beta)^{-1} - t\lambda| \\ \qquad\qquad (|(1+k_\beta)^{-1} + \lambda| \le |(1+k_\beta)^{-1} - t\lambda|). \end{cases}$$

On the other hand, we have

$$\big\{(1+k_\beta)^{-1} + \lambda\big\}^2 - \big\{(1+k_\beta)^{-1} - t\lambda\big\}^2 = (t+1)(t-1)(d_t - \lambda)\lambda.$$

From this, one can obtain this lemma. □

The following lemma is an easy consequence of Lemma 5.3.

Lemma 5.4. *Let $t \in (1, k_\beta^{-1})$. Then the function $\lambda \mapsto \|x_\beta + \lambda y_t\|_\beta^*$ is strictly decreasing on $(-\infty, 0]$ and strictly increasing on $[0, \infty)$.*

We clarify the relationship among $\|x_\beta + c_t y_t\|_\beta^*$,$\|x_\beta + d_t y_t\|_\beta^*$ and $\|x_\beta - (1 + k_\beta)^{-1} y_t\|_\beta^*$.

Lemma 5.5. *Let $t \in (1, k_\beta^{-1})$. Then*

$$\|x_\beta + c_t y_t\|_\beta^* < \|x_\beta - (1+k_\beta)^{-1} y_t\|_\beta^* < \|x_\beta + d_t y_t\|_\beta^*.$$

Proof. By Lemma 5.3, we have

$$\|x_\beta + c_t y_t\|_\beta^* = \frac{1+t}{(1+k_\beta)t} \quad \text{and} \quad \|x_\beta - (1+k_\beta)^{-1} y_t\|_\beta^* = \frac{1+t}{1+k_\beta}.$$

Since $t > 1$, we have $\|x_\beta + c_t y_t\|_\beta^* < \|x_\beta - (1+k_\beta)^{-1} y_t\|_\beta^*$. Moreover,

$$\|x_\beta + d_t y_t\|_\beta^* = \frac{1}{1+k_\beta}\left\{1 - k_\beta + \frac{2(1+k_\beta t)}{t-1}\right\}$$

and so

$$\begin{aligned}\|x_\beta + d_t y_t\|_\beta^* - \|x_\beta - (1+k_\beta)^{-1} y_t\|_\beta^* &= \frac{1}{1+k_\beta}\left\{-(k_\beta + t) + \frac{2(1+k_\beta t)}{t-1}\right\} \\ &= \frac{(2+k_\beta - t)(t+1)}{(1+k_\beta)(t-1)}.\end{aligned}$$

On the other hand, since $t < k_\beta^{-1}$, we have

$$2 + k_\beta - t > 2 + k_\beta - k_\beta^{-1} \geq 0.$$

Thus we obtain $\|x_\beta - (1+k_\beta)^{-1} y_t\|_\beta^* < \|x_\beta + d_t y_t\|_\beta^*$. □

Let $t \in (1, k_\beta^{-1})$. Then, the function $\lambda \mapsto \|x_\beta + \lambda y_t\|_\beta^*$ maps $(-\infty, 0]$ onto $[1, \infty)$ and $[0, \infty)$ onto $[1, \infty)$. Thus by Lemma 5.4, for any $\mu \in [0, \infty)$, there exists a unique $\lambda \in (-\infty, 0]$ such that $\|x_\beta + \lambda y_t\|_\beta^* = \|x_\beta + \mu y_t\|_\beta^*$. Now, let ρ_t, σ_t, τ_t be real numbers such that $\rho_t, \tau_t < 0 < \sigma_t$, $\|x_\beta + c_t y_t\|_\beta^* = \|x_\beta + \rho_t y_t\|_\beta^*$, $\|x_\beta - (1+k_\beta)^{-1} y_t\|_\beta^* = \|x_\beta + \sigma_t y_t\|_\beta^*$, and $\|x_\beta + d_t y_t\|_\beta^* = \|x_\beta + \tau_t y_t\|_\beta^*$. Then, we have the following lemma. The proof is similar to that of Lemma 4.7 (i) and so we omit it.

Lemma 5.6. *Let $t \in (1, k_\beta^{-1})$. Then $\tau_t < -(1+k_\beta)^{-1} < \rho_t < 0 < c_t < \sigma_t < d_t$ and*

$$\rho_t = -\frac{1 - k_\beta t}{t - k_\beta} c_t, \quad \sigma_t = \frac{k_\beta + t}{(1+k_\beta t)(1+k_\beta)}, \quad \text{and} \quad \tau_t = \frac{2(1-k_\beta)}{(1+k_\beta)(k_\beta + t)} - d_t.$$

We consider the set $\Gamma(x_\beta, y_t)$. As was mentioned, for each $\mu \in [0, \infty)$ there exists a unique $\lambda_\mu \in (-\infty, 0]$ such that $\|x_\beta + \lambda_\mu y_t\|_\beta^* = \|x_\beta + \mu y_t\|_\beta^*$. Then it follows that

$$\Gamma(x_\beta, y_t) = \left\{\frac{\lambda_\mu + \mu}{2} : \mu \in [0, \infty)\right\}.$$

We remark that

$$1 < \frac{k_\beta(1+k_\beta)}{3k_\beta - 1} = \frac{1-\beta}{\beta(3-4\beta)} \leq \frac{\beta}{1-\beta} = k_\beta^{-1}$$

since $1/2 < \beta \leq 1/\sqrt{2}$.

Lemma 5.7. *Let* $t \in (1, k_\beta^{-1})$. *Then*

$$\Gamma(x_\beta, y_t) = \begin{cases} \left[0, \dfrac{d_t + \tau_t}{2}\right] & \left(1 < t \le \dfrac{k_\beta(1+k_\beta)}{3k_\beta - 1}\right), \\ \left[0, \dfrac{c_t + \rho_t}{2}\right] & \left(\dfrac{k_\beta(1+k_\beta)}{3k_\beta - 1} \le t < k_\beta^{-1}\right). \end{cases}$$

Proof. By Lemma 5.6, we have $\tau_t < -(1+k_\beta)^{-1} < \rho_t < 0 < c_t < \sigma_t < d_t$. Suppose that $0 \le \mu \le c_t$. Then Lemma 5.4 guarantees that $\rho_t \le \lambda_\mu \le 0$, and so

$$1 - (t - k_\beta)\lambda_\mu = \|x_\beta + \lambda_\mu y_t\|_\beta^* = \|x_\beta + \mu y_t\|_\beta^* = 1 + (1 - k_\beta t)\mu.$$

Hence we have

$$\lambda_\mu = -\frac{1 - k_\beta t}{t - k_\beta}\mu,$$

which implies that

$$\frac{\lambda_\mu + \mu}{2} = \frac{(1+k_\beta)(t-1)}{2(t-k_\beta)}\mu.$$

Since $t \in (1, k_\beta^{-1})$, the function $\mu \mapsto \frac{\lambda_\mu + \mu}{2}$ is increasing on $[0, c_t]$, and hence

$$\left\{\frac{\lambda_\mu + \mu}{2} : \mu \in [0, c_t]\right\} = \left[0, \frac{c_t + \rho_t}{2}\right].$$

Next, we suppose that $c_t \le \mu \le \sigma_t$. Then we have $-(1+k_\beta)^{-1} \le \lambda_\mu \le \rho_t$, and so

$$1 - (t - k_\beta)\lambda_\mu = \|x_\beta + \lambda_\mu y_t\|_\beta^* = \|x_\beta + \mu y_t\|_\beta^* = \frac{1 - k_\beta}{1 + k_\beta} + (1 + k_\beta t)\mu.$$

From this, we have

$$\lambda_\mu = \frac{2k_\beta}{(1+k_\beta)(t-k_\beta)} - \frac{1 + k_\beta t}{t - k_\beta}\mu$$

and

$$\frac{\lambda_\mu + \mu}{2} = \frac{k_\beta}{(1+k_\beta)(t-k_\beta)} + \frac{(1-k_\beta)t - (1+k_\beta)}{2(t-k_\beta)}\mu.$$

Since $t \in (1, k_\beta^{-1})$, $(1-k_\beta)t - (1+k_\beta) < 0$ and hence the function $\mu \mapsto \frac{\lambda_\mu + \mu}{2}$ is decreasing on $[c_t, \sigma_t]$. Thus we have

$$\left\{\frac{\lambda_\mu + \mu}{2} : \mu \in [c_t, \sigma_t]\right\} = \left[\frac{-(1+k_\beta)^{-1} + \sigma_t}{2}, \frac{c_t + \rho_t}{2}\right].$$

In the case of $\sigma_t \le \mu \le d_t$, we have $\tau_t \le \lambda_\mu \le -(1+k_\beta)^{-1}$. Then we have

$$\frac{1 - k_\beta}{1 + k_\beta} - (k_\beta + t)\lambda_\mu = \|x_\beta + \lambda_\mu y_t\|_\beta^* = \|x_\beta + \mu y_t\|_\beta^* = \frac{1 - k_\beta}{1 + k_\beta} + (1 + k_\beta t)\mu.$$

It follows that

$$\lambda_\mu = -\frac{1 + k_\beta t}{k_\beta + t}\mu$$

and

$$\frac{\lambda_\mu + \mu}{2} = \frac{(1-k_\beta)(t-1)}{2(k_\beta + t)}\mu.$$

This shows that the function $\mu \mapsto \frac{\lambda_\mu+\mu}{2}$ is increasing on $[\sigma_t, d_t]$, and hence

$$\left\{\frac{\lambda_\mu + \mu}{2} : \mu \in [\sigma_t, d_t]\right\} = \left[\frac{-(1+k_\beta)^{-1} + \sigma_t}{2}, \frac{d_t + \tau_t}{2}\right].$$

Finally, we assume that $d_t \le \mu$. Then it follows from $\lambda_\mu \le \tau_t$ that

$$\frac{1-k_\beta}{1+k_\beta} - (k_\beta + t)\lambda_\mu = \|x_\beta + \lambda_\mu y_t\|_\beta^* = \|x_\beta + \mu y_t\|_\beta^* = -\frac{1-k_\beta}{1+k_\beta} + (k_\beta + t)\mu.$$

Thus we have

$$\frac{\lambda_\mu + \mu}{2} = \frac{1-k_\beta}{(1+k_\beta)(k_\beta+t)} = \frac{d_t + \tau_t}{2}.$$

Since the function $\mu \mapsto \frac{\lambda_\mu+\mu}{2}$ is continuous, we obtain

$$\begin{aligned}
\Gamma(x_\beta, y_t) &= \left\{\frac{\lambda_\mu + \mu}{2} : \mu \in [0, \infty)\right\} \\
&= \left[0, \frac{c_t + \rho_t}{2}\right] \cup \left[\frac{-(1+k_\beta)^{-1} + \sigma_t}{2}, \frac{c_t + \rho_t}{2}\right] \\
&\quad \cup \left[\frac{-(1+k_\beta)^{-1} + \sigma_t}{2}, \frac{d_t + \tau_t}{2}\right] \\
&= \left[\min\left\{0, \frac{-(1+k_\beta)^{-1} + \sigma_t}{2}\right\}, \max\left\{\frac{c_t + \rho_t}{2}, \frac{d_t + \tau_t}{2}\right\}\right].
\end{aligned}$$

However, one has

$$\frac{-(1+k_\beta)^{-1} + \sigma_t}{2} = \frac{(1-k_\beta)(t-1)}{2(k_\beta + t)}\sigma_t > 0$$

and

$$\frac{d_t + \tau_t}{2} - \frac{c_t + \rho_t}{2} = \frac{(1+t)(3k_\beta - 1)}{2t(1+k_\beta)(t+k_\beta)(t-k_\beta)}\left\{\frac{k_\beta(1+k_\beta)}{3k_\beta - 1} - t\right\}.$$

Thus, we have this lemma. □

Now we estimate $M(x_\beta)$.

Proposition 5.8. $M(x_\beta) \le 1 + k_\beta^2$.

Proof. By Lemma 5.2, we have $M(x_\beta) = \sup\{m(x_\beta, y_t) : t \in (1, k_\beta^{-1})\}$.

First we suppose that $t \in (1, k_\beta(1+k_\beta)/(3k_\beta - 1)]$. Since

$$\frac{d_t + \tau_t}{2} = \frac{1-k_\beta}{(1+k_\beta)(k_\beta + t)} < \frac{1}{(1+k_\beta)t} = c_t,$$

we have $0 < \frac{d_t+\tau_t}{2} < c_t$. Hence we obtain

$$\left\|x_\beta + \frac{d_t+\tau_t}{2}y_t\right\|_\beta^* = 1 + \frac{(1-k_\beta)(1-k_\beta t)}{(1+k_\beta)(k_\beta+t)}.$$

From the fact that the function $t \mapsto (1-k_\beta t)/(k_\beta+t)$ is strictly decreasing, it follows that

$$\frac{(1-k_\beta)(1-k_\beta t)}{(1+k_\beta)(k_\beta+t)} < \frac{(1-k_\beta)^2}{(1+k_\beta)^2},$$

which implies

$$\left\|x_\beta + \frac{d_t+\tau_t}{2}y_t\right\|_\beta^* < 1 + \frac{(1-k_\beta)^2}{(1+k_\beta)^2} < 1 + k_\beta^2$$

since $(1-k_\beta)/(1+k_\beta) < k_\beta$. Thus for each $t \in (1, k_\beta(1+k_\beta)/(3k_\beta-1)]$, we have

$$m(x_\beta, y_t) = \max\left\{\|x_\beta\|_\beta^*, \left\|x_\beta + \frac{d_t+\tau_t}{2}y_t\right\|_\beta^*\right\} < 1 + k_\beta^2$$

by [18, Lemma 2.6].

Let $t \in [k_\beta(1+k_\beta)/(3k_\beta-1), k_\beta^{-1})$. Then we obtain

$$0 < \frac{c_t+\rho_t}{2} = \frac{t-1}{2t(t-k_\beta)} < \frac{1}{2t} < \frac{1}{(1+k_\beta)t} = c_t.$$

By Lemma 5.4, we obtain

$$\begin{aligned}
\left\|x_\beta + \frac{c_t+\rho_t}{2}y_t\right\|_\beta^* &< \left\|x_\beta + \frac{1}{2t}y_t\right\|_\beta^* \\
&= 1 + \frac{1-k_\beta t}{2t} \\
&< 1 + \frac{1-k_\beta^2(1+k_\beta)(3k_\beta-1)^{-1}}{2k_\beta(1+k_\beta)(3k_\beta-1)^{-1}} \\
&= 1 + \frac{(1-k_\beta)(k_\beta^2+2k_\beta-1)}{2k_\beta(1+k_\beta)}.
\end{aligned}$$

Since $\sqrt{2}-1 \leq k_\beta < 1$, we have

$$k_\beta^2 - \frac{(1-k_\beta)(k_\beta^2+2k_\beta-1)}{2k_\beta(1+k_\beta)} = \frac{2k_\beta^2(k_\beta^2+2k_\beta-1)+(1-k_\beta)^3}{2k_\beta(1+k_\beta)} > 0,$$

and so

$$\left\|x_\beta + \frac{c_t+\rho_t}{2}y_t\right\|_\beta^* < 1 + k_\beta^2.$$

Hence, by [18, Lemma 2.6], we have

$$m(x_\beta, y_t) = \max\left\{ \|x_\beta\|_\beta^*, \left\| x_\beta + \frac{c_t+\rho_t}{2} y_t \right\|_\beta^* \right\} < 1 + k_\beta^2.$$

Thus we obtain $M(x_\beta) \le 1 + k_\beta^2$. □

6. The Dunkl-Williams constant of $(\mathbb{R}^2, \|\cdot\|_{\psi_\beta})^*$

As was mentioned in Section 2, the equality $DW((\mathbb{R}^2, \|\cdot\|_{\psi_\beta})^*) = DW((\mathbb{R}^2, \|\cdot\|_{\psi_\beta^*}))$ holds for all $\beta \in (1/2, 1)$. From this fact, we obtain the main result.

Theorem 6.1. *Let $\beta \in (1/2, 1)$. Then the following hold:*

(i) *If $\beta \in (1/2, 1/\sqrt{2}]$, then*

$$DW((\mathbb{R}^2, \|\cdot\|_{\psi_\beta})^*) = DW((\mathbb{R}^2, \|\cdot\|_{\psi_\beta^*})) = \frac{2}{\beta^2}\left\{(1-\beta)^2 + \beta^2\right\}.$$

(ii) *If $\beta \in [1/\sqrt{2}, 1)$, then*

$$DW((\mathbb{R}^2, \|\cdot\|_{\psi_\beta})^*) = DW((\mathbb{R}^2, \|\cdot\|_{\psi_\beta^*})) = 4\left\{(1-\beta)^2 + \beta^2\right\}.$$

Proof. As in the above sections, we write X_β^* for $(\mathbb{R}^2, \|\cdot\|_{\psi_\beta^*})$.

(i) Suppose $\beta \in (1/2, 1/\sqrt{2}]$. Then by Proposition 3.3, we have

$$DW(X_\beta^*) = 2\max\{M(x_\beta), M(e_1)\}.$$

Thus, by Propositions 4.10 and 5.8, we obtain

$$DW((\mathbb{R}^2, \|\cdot\|_{\psi_\beta})^*) = DW(X_\beta^*) = 2(1 + k_\beta^2) = \frac{2}{\beta^2}\left\{(1-\beta)^2 + \beta^2\right\},$$

as desired.

(ii) For each $\beta \in (1/2, 1)$, it is easy to check that X_β^* is isometrically isomorphic to $X_{1/2\beta}^*$ under the identification

$$X_\beta^* \ni (x_1, x_2) \longleftrightarrow \frac{1}{2\beta}(x_1 + x_2, x_1 - x_2) \in X_{1/2\beta}^*$$

since $\max\{|x_1 + x_2|, |x_1 - x_2|\} = |x_1| + |x_2|$ for all $x_1, x_2 \in \mathbb{R}$. If $\beta \in [1/\sqrt{2}, 1)$, then $1/2\beta \in (1/2, 1/\sqrt{2}]$ and hence by (i)

$$\begin{aligned} DW((\mathbb{R}^2, \|\cdot\|_{\psi_\beta})^*) = DW(X_\beta^*) &= DW(X_{1/2\beta}^*) \\ &= \frac{2}{(1/2\beta)^2}\left\{(1 - (1/2\beta))^2 + (1/2\beta)^2\right\} \\ &= 4\left\{(1-\beta)^2 + \beta^2\right\}. \end{aligned}$$

Therefore we obtain this theorem. □

Remark 6.2. *From* Theorem 6.1 *and* [18, Theorem 3.1], $DW((\mathbb{R}^2, \|\cdot\|_{\psi_\beta}))$ *coincide with* $DW((\mathbb{R}^2, \|\cdot\|_{\psi_\beta})^*)$ *for all* $\beta \in (1/2, 1)$.

Let X^* *denote the dual space of a Banach space* X*. It is known that* $C_{NJ}(X) = C_{NJ}(X^*)$*, where* $C_{NJ}(X)$ *is the von Neumann-Jordan constant of* X [4, 13]. *On the other hand, the equality* $J(X) = J(X^*)$ *does not necessarily hold for the James constant* $J(X)$ [8, 21]. *It will be interesting to wonder if the equality* $DW(X) = DW(X^*)$ *holds for any Banach space* X.

References

[1] M. Baronti and P. L. Papini, *Up and down along rays*, Riv. Mat. Univ. Parma (6) **2*** (1999), 171–189.

[2] G. Birkhoff, *Orthogonality in linear metric spaces*, Duke Math. J. **1** (1935), 169–172.

[3] F. F. Bonsall and J. Duncan, *Numerical Ranges II*, London Math. Soc. Lecture Note Ser. 10, Cambridge University Press, Cambridge, 1973.

[4] J. A. Clarkson, *The von Neumann-Jordan constant for the Lebesgue spaces*, Ann. of Math. (2) **38** (1937), 114–115.

[5] C. F. Dunkl and K. S. Williams, *A simple norm inequality*, Amer. Math. Monthly **71** (1964), 53–54.

[6] M. M. Day, *Polygons circumscribed about closed convex curves*, Trans. Amer. Math. Soc. **62** (1947), 315–319.

[7] M. M. Day, *Some characterizations of inner-product spaces*, Trans. Amer. Math. Soc. **62** (1947), 320–337.

[8] J. Gao and K.-S. Lau, *On the geometry of spheres in normed linear spaces*, J. Austral. Math. Soc. Ser. A **48** (1990), 101–112.

[9] R. C. James, *Orthogonality in normed linear spaces*, Duke Math. J. **12** (1945), 291–302.

[10] R. C. James, *Inner product in normed linear spaces*, Bull. Amer. Math. Soc. **53** (1947), 559–566.

[11] R. C. James, *Orthogonality and linear functionals in normed linear spaces*, Trans. Amer. Math. Soc. **61** (1947), 265–292.

[12] A. Jiménez-Melado, E. Llorens-Fuster and E. M. Mazcuñán-Navarro, *The Dunkl-Williams constant, convexity, smoothness and normal structure*, J. Math. Anal. Appl. **342** (2008), 298–310.

[13] M. Kato and Y. Takahashi, *On the von Neumann-Jordan constant for Banach spaces*, Proc. Amer. Math. Soc. **125** (1997), 1055–1062.

[14] W. A. Kirk and M. F. Smiley, *Another characterization of inner product spaces*, Amer. Math. Monthly **71** (1964), 890–891.

[15] K.-I. Mitani and K.-S. Saito, *Dual of two dimensional Lorentz sequence spaces*, Nonlinear Anal. **71** (2009), 5238–5247.

[16] K.-I. Mitani, K.-S. Saito and T. Suzuki, *Smoothness of absolute norms on* $\mathbb{C}^n$, J. Convex Anal. **10** (2003), 89–107.

[17] K.-I. Mitani, S. Oshiro and K.-S. Saito, *Smoothness of* ψ*-ditect sums of Banach spaces*, Math. Inequal. Appl. **8** (2005), 147–157.

[18] H. Mizuguchi, K.-S. Saito and R. Tanaka, *The Dunkl-Williams constant of symmetric octagonal norms on* $\mathbb{R}^2$, Nihonkai Math. J. **23** (2012), 93–113.

[19] H. Mizuguchi, K.-S. Saito and R. Tanaka, *On the calculation of the Dunkl-Williams constant of normed linear spaces*, Cent. Eur. J. Math. **11** (2013), 1212–1227.

[20] K.-S. Saito, M. Kato and Y. Takahashi, *Von Neumann-Jordan constant of absolute normalized norms on* $\mathbb{C}^2$, J. Math. Anal. Appl. **244** (2000), 515–532.

[21] K.-S. Saito, N. Komuro and K.-I. Mitani, *How to calculate the James constants of Banach spaces*, Proc. of 4th International Symposium on Banach and Function Spaces **IV** (2013), 211–224.

(Hiroyasu Mizuguchi) Department of Mathematical Sciences, Graduate School of Science and Technology, Niigata University, Niigata 950–2181, Japan
E-mail address: `mizuguchi@m.sc.niigata-u.ac.jp`

Received October 2, 2014