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NORMAL LOG CANONICAL DEL PEZZO SURFACES OF RANK ONE WITH UNIQUE SINGULAR POINTS

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Abstract. Normal del Pezzo surfaces of rank one with only rational log canonical singularities are studied. We classify such surfaces with unique singular points. Moreover, by using the classification result and the results in [11], we study the fundamental groups of their smooth parts.

1. Introduction

Throughout this paper, we work over the complex number field $\mathbb{C}$. A normal del Pezzo surface means a normal projective surface whose anticanonical divisor is an ample $\mathbb{Q}$-Cartier divisor. A normal del Pezzo surface is said to have rank one if its Picard number equals one.

Normal del Pezzo surfaces have been studied by many mathematicians and we have many significant results on such surfaces. For details, see [14, §1] and papers given in References of [14]. Here we recall some results on the fundamental groups of the smooth parts of normal del Pezzo surfaces. Let X be a normal del Pezzo surface and $X_{\mathrm{reg}} := X \setminus \mathrm{Sing}X$ its smooth part. In [8] and [9], Gurjar and Zhang proved that, if X has only log terminal singularities, then the fundamental group of X_{reg} is finite. Fujiki, Kobayashi and Lu [5] gave an another and very short proof of the main result of [8] and [9]. Zhang [22] proved that, if X has only log canonical singularities, then X_{reg} has finite fundamental group or is affine ruled, i.e., it contains $\mathbb{A}^1 \times C$ for an affine curve C as a Zariski open subset. In fact, he proved more general results.

Recently, the author and Takahashi studied normal del Pezzo surfaces of rank one with only rational log canonical singularities. In [13], they proved some elementary results on normal del Pezzo surfaces of rank one with only rational singularities, which are generalizations of some results in [21, §2], and classified the minimal compactifications of the affine plane $\mathbb{A}^2$ with only log canonical singularities. In

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[14], they proved that every normal del Pezzo surface of rank one with only rational log canonical singularities has at most five singular points. Later on, the author [12] determined such surfaces with five singular points. Here we note that Belousov [3] (see [4] for another proof) proved that every normal del Pezzo surface of rank one with only log terminal singularities can have at most four singular points.

In this paper, we classify the normal del Pezzo surfaces of rank one with only rational log canonical singularities and with unique singular points. Moreover, by using the classification result, we study the fundamental groups of some open rational surfaces. The main result of this paper is the following.

Theorem 1.1. *Let X be a normal del Pezzo surface of rank one with only rational log canonical singularities and $\pi : (V, D) \to X$ the minimal resolution of X, where D is the reduced exceptional divisor. Assume that $\#\mathrm{Sing}X = 1$ and the unique singular point on X is not log terminal. Then the following assertions hold true.*

(1) *The weighted dual graph of D is given as (n) for $n = 1, 2, 3$ in Fig. 1.1, where we omit the weight corresponding to a (-2)-curve.*
(2) *There exists a $\mathbb{P}^1$-fibration $\Phi : V \to \mathbb{P}^1$ in such a way that the configuration of D as well as all singular fibers is given as (n) for $n = 1, 2, 3$ in Fig. 1.2, where a dotted line stands for a (-1)-curve; a solid line stands for a component of D; the self-intersection number of a (-2)-curve is omitted; a line with $*$ on it is not a fiber component of the vertical $\mathbb{P}^1$-fibration on V.*
(3) *The fundamental group of X_{reg} is $\mathbb{Z}/2\mathbb{Z}$ (resp. $\mathbb{Z}/2\mathbb{Z}$, (1)) if the dual graph of D is given as (1) (resp. (2), (3)) in Fig. 1.1.*
(4) *All the cases are realizable.*

In Theorem 1.1, we assume further that the unique singular point on X is not log terminal because the case where the unique singular point on X is log terminal was determined by the author [11].

By using Theorem 1.1, the results in [11] and the minimal model theory for normal projective surfaces with only log canonical singularities due to Fujino [6] and Tanaka [20], we obtain the following result.

Theorem 1.2. *Let X be a normal complete rational surface with unique singular point. Suppose that the singular point on X is log canonical and $\overline{\kappa}(X_{\mathrm{reg}}) = -\infty$. Then the fundamental group of X_{reg} is finite and is a residue group of $\mathbb{Z} \oplus \mathbb{Z}$. In particular, the fundamental group of X_{reg} is abelian.*

Let X be a normal projective rational surface with unique singular point. Gurjar and Zhang [10] proved that if $\overline{\kappa}(X_{\mathrm{reg}}) \leq 1$ and the singular point on X is log terminal

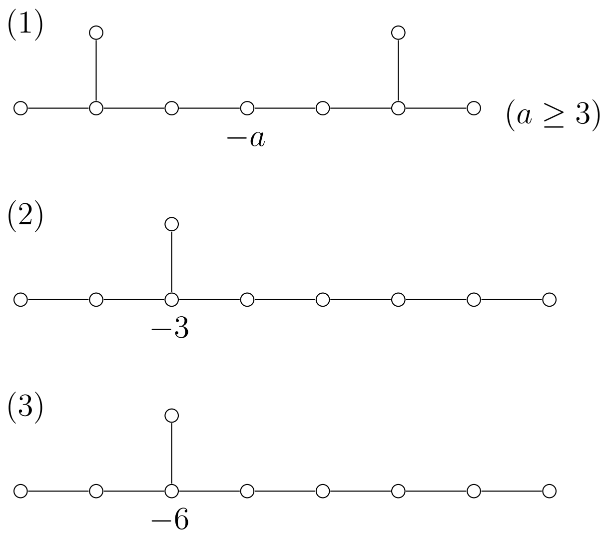

Fig. 1.1

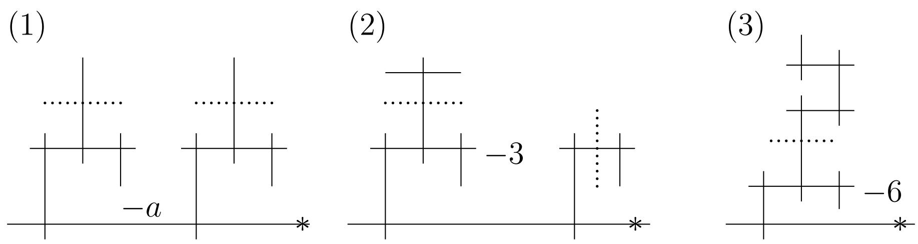

Fig. 1.2

then X_{reg} has finite fundamental group. Theorem 1.2 includes their result in the case $\overline{\kappa}(X_{\mathrm{reg}}) = -\infty$.

2. Preliminaries

2.1. A $(-n)$-curve is a smooth projective rational curve with self-intersection number $-n$. A reduced effective divisor D is called an SNC-divisor if it has only simple normal crossings. In this paper, we employ the following notations:

$\pi_1(T)$: the fundamental group of T.

K_X: the canonical divisor on X.

$\rho(X)$: the Picard number of X.

X_{reg} : the smooth locus of X.

$\#D$: the number of all irreducible components in Supp$D$.

$\overline{\kappa}(S)$: the logarithmic Kodaira dimension of S (see, e.g., [16] and [17] for the definition).

2.2. In order to prove the results in this paper, we frequently use some results on normal del Pezzo surfaces of rank one with only rational singularities given in [14]. The results in [14, §2] except for [14, Lemma 2.9] are originally given in [13, §3].

Let X be a normal del Pezzo surface of rank one with only rational singularities and $\pi : (V, D) \to X$ the minimal resolution of X, where D is the reduced exceptional divisor. Then the canonical divisor K_X on X is $\mathbb{Q}$-Cartier and there exists a unique effective $\mathbb{Q}$-divisor $D^\#$ such that $\mathrm{Supp} D^\# \subset \mathrm{Supp} D$ and $\pi^*(K_X) \equiv D^\# + K_V$. Let $\mathrm{MV}(V, D)$ be the set of all irreducible curves C such that $-C(D^\# + K_V)$ attains the smallest positive value (cf. [14, p. 55]). Since X has only rational singularities, D is an SNC-divisor (cf. [2]) and X is a rational surface by [14, Lemma 2.1 (1)].

Definition 2.1. The pair (V, D) is said to be of the first kind if there exists a curve $C \in \mathrm{MV}(V, D)$ such that $|C + D + K_V| \neq \emptyset$. It is said to be of the second kind if it is not of the first kind, namely, $|C + D + K_V| = \emptyset$ for every curve $C \in \mathrm{MV}(V, D)$.

2.3. In order to prove Theorem 1.2, we will use the minimal model theory for log surfaces due to Fujino [6] and Tanaka [20] in §4. For the definition of $\overline{NE}(X)$ of a normal variety X, an extremal ray, a Mori fiber space, etc., see [6] and [20]. We can use the minimal model theory for log surfaces in [6] and [20] since only normal projective surfaces with at most rational log canonical singularities are studied in this paper.

We have very useful intersection theory for normal surfaces due to Mumford [18] and Sakai [19]. The theory can be applied also for the normal surfaces with non $\mathbb{Q}$-factorial singularities. However, in this paper, we consider the intersection numbers only for $\mathbb{Q}$-Cartier divisors.

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1.

Let X, V and D be the same as in Theorem 1.1. Let C be a curve of $\mathrm{MV}(V, D)$ (see §2 for the definition of $\mathrm{MV}(V, D)$). Since the unique singular point on X is not log terminal but rational, the weighted dual graph of D is one of the dual graphs (6)–(8) in [15, p. 58]. We infer from [14, Lemmas 2.5 and 2.6] that C is a (-1)-curve and $|C + D + K_V| = \emptyset$. Since the dual graph of $C + D$ is a tree by [16, Lemma I.2.1.3 (p. 7)], we have $CD = 1$.

We consider the following cases separately.

Case 1. The dual graph of D is one of (7) and (8) in [15, p. 58]. Let $D = \sum_{i=1}^{r} D_i$ be the decomposition of D into irreducible components and set $a_i = -(D_i)^2$ for $i = 1, \ldots, r$. In this case, $r \geq 5$ and the dual graph of D looks like that in Fig. 3.1.

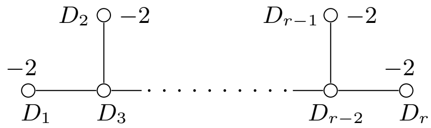

Fig. 3.1

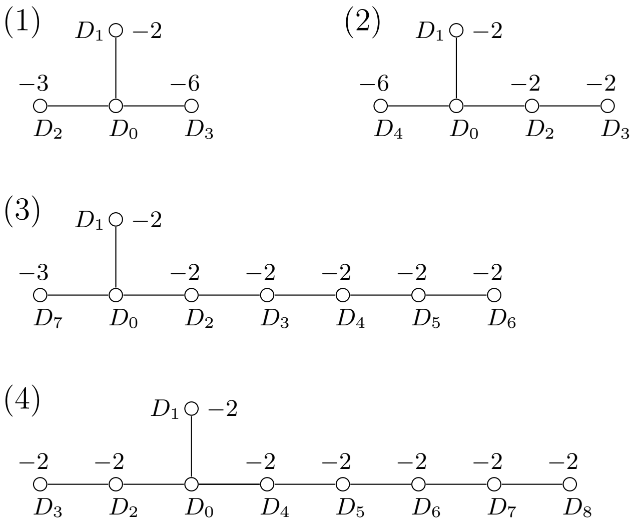

Fig. 3.2

In this case, we have $D^{\#} = \frac{1}{2}(D_1 + D_2 + D_{r-1} + D_r) + \sum_{i=3}^{r-2} D_i$ (for the definition of $D^{\#}$, see §2). Since $CD = 1$ and $CD^{\#} < -CK_V = 1$, we may assume that $CD = CD_1 = 1$. Since the intersection matrix of $C + D$ (resp. D) is not negative definite (resp. negative definite), we know that $a_3 = 2$ and $r \geq 6$. Moreover, since $C + D$ is a big divisor, we know that $a_4 = 2$ and $r \geq 7$. Then the divisor $F := 2(C + D_1 + D_3) + D_2 + D_4$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F|} : V \to \mathbb{P}^1$, D_5 becomes a section of Φ and $D - D_5$ is contained in fibers of Φ.

Suppose that $r = 7$. Then the fiber F_1 of Φ containing D_6 is different from that F_2 containing D_7. Since $C \in \mathrm{MV}(V, D)$ and $\mathrm{Supp}F$ consists only of C and four (-2)-curves, it follows from [14, Lemma 2.8] that F_i $(i = 1, 2)$ consists only of one (-2)-curve and two (-1)-curves. This contradicts $\rho(V) = 1 + \#D$. Hence $r \geq 8$ and the divisor $D_6 + \cdots + D_r$ is contained in a fiber, say G, of Φ.

We infer from $\rho(V) = 1 + \#D$ that $\mathrm{Supp}G$ consists of $D_6, \ldots, D_r$ and one (-1)-curve E and that F and G exhaust the singular fibers of Φ. Then $ED = E(D_6 + \cdots + D_r) = 1$. So $ED = ED_j = 1$ for $j = r - 1$ or r because $E(D^{\#} + K_V) < 0$. Thus we know that the dual graph of D (resp. the configuration of $C + D$ and all singular fibers of Φ) is given as (1) in Fig. 1.1 (resp. Fig. 1.2). Here we note that $a \geq 3$, where a is the number in Fig. 1.1 (1) and Fig. 1.2 (1), because the intersection matrix of D is negative definite.

Case 2. The dual graph of D is one of (6) in [15, p. 58]. We treat only the case where $(\Delta_1, \Delta_2, \Delta_3) = (2, 3, 6)$ with the notations in [15, p. 58]. The other cases can be treated easier. In fact, we see that the other cases do not take place by using the following arguments.

From now on, we assume that $(\Delta_1, \Delta_2, \Delta_3) = (2, 3, 6)$. Then the dual graph of D is given as one of (1)–(4) in Fig. 3.2.

We consider the following cases separately.

Case 2-1. The dual graph of D is given as (1) in Fig. 3.2, where $D = D_0 + D_1 + D_2 + D_3$ is the decomposition of D into irreducible components. Then $\rho(V) = 5$ and $D^{\#} = D_0 + \frac{1}{2}D_1 + \frac{2}{3}D_2 + \frac{5}{6}D_3$. Since $CD = 1$ and $CD^{\#} < -CK_V = 1$, we know that $CD = CD_i = 1$ for some i, $1 \leq i \leq 3$. Since $a_0 = -(D_0)^2 \geq 2$, the intersection matrix of $C + D$ is then negative definite. This is a contradiction.

Case 2-2. The dual graph of D is given as (2) in Fig. 3.2, where $D = D_0 + D_1 + \cdots + D_4$ is the decomposition of D into irreducible components. Then $\rho(V) = 6$ and $D^{\#} = D_0 + \frac{1}{2}D_1 + \frac{2}{3}D_2 + \frac{1}{3}D_3 + \frac{5}{6}D_4$. Since $CD = 1$, $CD^{\#} < -CK_V = 1$, $a_0 = -(D_0)^2 \geq 2$ and the intersection matrix of $C + D$ is not negative definite, we know that $CD = CD_2 = 1$ and $a_0 = 2$. Then the divisor $F_0 := 2(C + D_2) + D_0 + D_3$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_0|} : V \to \mathbb{P}^1$, D_1 and D_4 become sections of Φ and $D - (D_1 + D_4)$ is contained in fibers of Φ. Since $\rho(V) = 6 > 2 + (\#F_0 - 1) = 5$, there exists another singular fiber F_1 of Φ. We infer from [14, Lemma 2.2 (3)] that $F_1 = E_1 + E_1'$, where E_1 and E_1' are (-1)-curves and $E_1E_1' = 1$. Since D_1 is a section of Φ, we may assume that $E_1D_1 = 0$. Then $E_1D = E_1D_4 \leq 1$ and so the intersection matrix of $E_1 + D$ is negative definite. This is a contradiction.

Case 2-3. The dual graph of D is given as (3) in Fig. 3.2, where $D = D_0 + \cdots + D_7$ is the decomposition of D into irreducible components. Then $\rho(V) = 9$ and $D^{\#} = D_0 + \frac{1}{2}D_1 + \frac{5}{6}D_2 + \frac{2}{3}D_3 + \frac{1}{2}D_4 + \frac{1}{3}D_5 + \frac{1}{6}D_6 + \frac{2}{3}D_7$. Since $CD = 1$, $CD^{\#} < -CK_V = 1$, $a_0 = -(D_0)^2 \geq 2$ and the intersection matrix of $C + D$ is not negative definite, we know that $CD = CD_i = 1$ for some i, $1 \leq i \leq 5$. We consider the following subcases separately.

Subcase 1. $i = 1$. Then $a_0 = 2$ because the intersection matrix of $C + D$ is not negative definite. So the divisor $F_0 := 3(C + D_1 + D_0) + 2D_2 + D_3 + D_7$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_0|} : V \to \mathbb{P}^1$, D_4 becomes a section of Φ and $D - D_4$ is contained in fibers of Φ. Let F_1 be the fiber of Φ containing $D_5 + D_6$. We infer from [14, Lemma 2.2 (3)] that $\mathrm{Supp}F_1$ consists of the (-2)-curves D_5 and D_6 and some (-1)-curves. So $\#F_1 = 4$. Then we have

$$\rho(V) = 9 \geq 2 + (\#F_0 - 1) + (\#F_1 - 1) = 10,$$

a contradiction.

Subcase 2. $3 \leq i \leq 5$. Then the divisor $F_0 := 2(C + D_i) + D_{i-1} + D_{i+1}$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_0|} : V \to \mathbb{P}^1$, D_{i-2} and D_{i+2} become sections of Φ and $D - (D_{i-2} + D_{i+2})$ is contained in fibers of Φ. Here we note that, if $i = 5$, then D_{i+2} does not exist. Since $C \in \mathrm{MV}(V, D)$ and $\mathrm{Supp}F_0$ consists only of C and three (-2)-curves, it follows from [14, Lemma 2.8] that every singular fiber of Φ consists only of (-1)-curves and (-2)-curves. This is a contradiction because D_7 is a fiber component of Φ.

Subcace 3. $i = 2$. Since the intersection matrix of $C + D$ is not negative definite and $a_0 \geq 2$, we see that $2 \leq a_0 \leq 5$.

Suppose that $a_0 = 5$. Then the divisor $F_0 := 5(C+D_2)+4D_3+3D_4+2D_5+D_0+D_6$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_0|} : V \to \mathbb{P}^1$, D_1 and D_7 become sections of Φ and $D - (D_1 + D_7)$ is contained in fibers of Φ. Since $\rho(V) = 9 > 2 + (\#F_0 - 1) = 8$, there exists another singular fiber F_1 of Φ. We infer from [14, Lemma 2.2 (3)] that $F_1 = E_1 + E_1'$, where E_1 and E_1' are (-1)-curves and $E_1E_1' = 1$. Since D_1 is a section of Φ, we may assume that $E_1'D_1 > 0$. Then $E_1D = E_1D_7 \leq 1$ and so the intersection matrix of $E_1 + D$ is negative definite, which is a contradiction.

Suppose that $a_0 = 4$. Then the divisor $F_0 := 4(C + D_2) + 3D_3 + 2D_4 + D_0 + D_5$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_0|} : V \to \mathbb{P}^1$, D_1, D_6 and D_7 become sections of Φ and $D - (D_1 + D_6 + D_7)$ is contained in fibers of Φ. Since $\rho(V) = 9 > 2 + (\#F_1 - 1) = 7$, there exists another singular fiber F_1 of Φ. We infer from [14, Lemma 2.2 (3)] that $F_1 = E_1 + E_1'$, where E_1 and E_1' are (-1)-curves and $E_1E_1' = 1$. Since D_1 is a section of Φ, we may assume that $E_1D_1 = 1$. Since $E_1(D - (D_1 + D_6 + D_7)) = 0$ and the intersection matrix of $E_1 + D$ is not negative definite, E_1 must meet at least one of D_6 and D_7. Then the divisor $E_1' + D$ has negative definite intersection matrix because $E_1'D = E_1'(D_6 + D_7) \leq 1$. This is a contradiction.

Suppose that $a_0 = 3$. Then the divisor $F_0 := 3(C + D_2) + 2D_3 + D_0 + D_4$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_0|} : V \to \mathbb{P}^1$, D_1, D_5 and D_7 become sections of Φ and $D - (D_1 + D_5 + D_7)$ is contained in fibers of Φ. Let F_1 be the fiber of Φ containing D_6. By [14, Lemma 2.2 (3)], $\mathrm{Supp}F_1$ consists of D_6 and some (-1)-curves. So we know that $F_1 = E_1 + D_6 + E_1'$, where E_1 and E_1' are (-1)-curves, $E_1D_6 = E_1'D_6 = 1$ and $E_1E_1' = 0$. Since $E_1D^\#$, $E_1'D^\# < 1$ and D_1 and D_7 are sections of Φ, we may assume that $E_1D_1 = E_1'D_7 = 1$. Then $-E_1'(D^\# + K_V) = 1 - (\frac{2}{3} + \frac{1}{6}) = \frac{1}{6}$. Since $C \in \mathrm{MV}(V, D)$ and $-C(D^\# + K_V) = \frac{1}{6} = -E_1'(D^\# + K_V)$, we know that $E_1' \in \mathrm{MV}(V, D)$. This is a contradiction because $|E' + D + K_V| \neq \emptyset$ by [16, Lemma I.2.1.3 (p. 7)] and (V, D) is not of the first kind by [14, Lemma 2.5].

Suppose that $a_0 = 2$. Then the divisor $F_0 := 2(C + D_2) + D_0 + D_3$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_0|} : V \to \mathbb{P}^1$, D_1, D_4 and D_7 become sections of Φ, and $D - (D_1 + D_4 + D_7)$ is contained in fibers of Φ. Let F_1 be the fiber of Φ containing

D_5+D_6. Since $C \in \mathrm{MV}(V,D)$ and $\mathrm{Supp}F_0$ consists only of C and three (-2)-curves, it follows from [14, Lemma 2.8] that $F_1 = E_1 + D_5 + D_6 + E_1'$, where E_1 and E_1' are (-1)-curves, $E_1D_5 = D_6E_1' = 1$, $E_1E_1' = 0$ and $E_1, E_1' \in \mathrm{MV}(V,D)$. Since $E_1D^{\#}, E_1'D^{\#} < 1$ and D_1 and D_7 are sections of Φ, we know that $E_1D_1 = E_1'D_7 = 1$. Then $|E_1+D+K_V|, |E_1'+D+K_V| \neq \emptyset$. This is a contradiction because $E_1, E_1' \in \mathrm{MV}(V,D)$ and (V,D) is not of the first kind by [14, Lemma 2.5].

Therefore, we know that this subcase does not take place.

Case 2-4. The dual graph of D is given as (4) in Fig. 3.2, where $D = D_0 + D_1 + \cdots + D_8$ is the decomposition of D into irreducible components. Then $\rho(V) = 10$ and $D^{\#} = D_0 + \frac{1}{2}D_1 + \frac{2}{3}D_2 + \frac{1}{3}D_3 + \frac{5}{6}D_4 + \frac{2}{3}D_5 + \frac{1}{2}D_6 + \frac{1}{3}D_7 + \frac{1}{6}D_8$. Note that $a_0 = -(D_0)^2 \geq 3$ because the intersection matrix of D is negative definite. Since $CD = 1$, $CD^{\#} < 1$, $a_0 \geq 3$ and the intersection matrix of $C+D$ is not negative definite, we know that $CD = CD_i = 1$ for some $i \in \{2,4,5,6,7\}$. We consider the following subcases separately.

Subcase 1. $i = 2$. Then $a_0 = 3$ since the intersection matrix of $C+D$ is not negative definite. So the divisor $F_0 := 4(C+D_2) + 2(D_0+D_3) + D_1 + D_4$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_0|} : V \to \mathbb{P}^1$, D_5 becomes a section of Φ and $D - D_5$ is contained in fibers of Φ. Let F_1 be the fiber of Φ containing $D_6+D_7+D_8$. We infer from [14, Lemma 2.2 (3)] that every irreducible component of $\mathrm{Supp}F_1$ other than D_6, D_7 and D_8 is a (-1)-curve. By considering [14, Lemma 2.7 (1)], we know that $F_1 = 2(E_1+D_7) + D_6 + D_8$, where E_1 is a (-1)-curve with $E_1D_7 = 1$ and $E_1D_6 = E_1D_8 = 0$. Hence the dual graph of D (resp. the configuration of $C+D$ and all singular fibers of Φ) is given as (2) in Fig. 1.1 (resp. Fig. 1.2).

Subcase 2. $i = 6$ or 7. By using the same argument as in Subcase 2 of Case 2-3, we know that this subcase does not take place. Here we note that D_0 is not a (-2)-curve.

Subcase 3. $i = 5$. Then the divisor $F_0 := 2(C+D_5)+D_4+D_6$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_0|} : V \to \mathbb{P}^1$, D_0 and D_7 become sections of Φ and $D-(D_0+D_7)$ is contained in fibers of Φ. Let F_1 and F_2 be the fiber of Φ containing D_1 and D_2+D_3, respectively. Note that F_0, F_1 and F_2 are mutually distinct. We infer from [14, Lemma 2.2 (3)] that F_1 and F_2 consist only of (-1)-curves and (-2)-curves. So we see that $D_8 \not\subset \mathrm{Supp}F_2$. If $D_8 \subset \mathrm{Supp}F_1$, then we know that $F_1 = 2E_1 + D_1 + D_8$, where E_1 is a (-1)-curve with $E_1D_1 = E_1D_8 = 1$. Then $-E_1(D^{\#}+K_V) = 1 - (\frac{1}{2}+\frac{1}{6}) = \frac{1}{3} = -C(D^{\#}+K_V)$ and so $E_1 \in \mathrm{MV}(V,D)$. However this is a contradiction because $|E_1+D+K_V| \neq \emptyset$ and (V,D) is not of the first kind by [14, Lemma 2.5]. Suppose that $D_8 \not\subset \mathrm{Supp}F_1$. Then $F_1 = E_1 + D_1 + E_1'$, where E_1 and E_1' are (-1)-curves,

$E_1D_1 = E'_1D_1 = 1$ and $E_1E'_1 = 0$. Then at least one of $E_1 + D$ and $E'_1 + D$ has negative definite intersection matrix, which is a contradiction.

Subcase 4. $i = 4$. Since the intersection matrix of $C + D$ is not negative definite, we know that $3 \leq a_0 \leq 6$.

Suppose that $a_0 = 6$. Then the divisor $F_0 := 10(C+D_4)+8D_5+6D_6+4D_7+2(D_0+D_8) + D_1 + D_2$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_0|} : V \to \mathbb{P}^1$, D_3 becomes a section of Φ and $D - D_3$ is contained in fibers of Φ. Since $\rho(V) = 10 = 2 + (\#F_0 - 1)$, F_0 is the unique singular fiber of Φ. Hence the dual graph of D (resp. the configuration of $C + D$ and all singular fibers of Φ) is given as (3) in Fig. 1.1 (resp. Fig. 1.2).

Suppose that $a_0 = 5$. Then the divisor $F_0 := 5(C+D_4)+4D_5+3D_6+2D_7+D_0+D_8$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_0|} : V \to \mathbb{P}^1$, D_1 and D_2 become sections of Φ and $D - (D_1 + D_2)$ is contained in fibers of Φ. Let F_1 be the fiber of Φ containing D_3. By [14, Lemma 2.2 (3)], SuppF_1 consists of D_3 and some (-1)-curves. Hence we know that $F_1 = E_1 + D_3 + E'_1$, where E_1 and E'_1 are (-1)-curves, $E_1D_3 = E'_1D_3 = 1$ and $E_1E'_1 = 0$. We may assume that $E'_1D_1 = 1$ because D_1 is a section of Φ. Then $E_1D = E_1D_3 = 1$ and so the intersection matrix of $E_1 + D$ is negative definite. This is a contradiction.

Suppose that $a_0 = 4$. Then the divisor $F_0 := 4(C + D_4) + 3D_5 + 2D_6 + D_0 + D_7$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_0|} : V \to \mathbb{P}^1$, D_1, D_2 and D_8 become sections of Φ and $D - (D_1 + D_2 + D_8)$ is contained in fibers of Φ. Let F_1 be the fiber of Φ containing D_3. Then, by using the same argument as in the subcase $a_0 = 5$, we know that $F_1 = E_1 + D_3 + E'_1$, where E_1 and E'_1 are (-1)-curves, $E_1D_3 = E'_1D_3 = 1$ and $E_1E'_1 = 0$. We may assume that $E_1D_1 = 1$ because D_1 is a section of Φ. Then $-E_1(D^\# + K_V) \leq 1 - (\frac{1}{2} + \frac{1}{3}) = \frac{1}{6} = -C(D^\# + K_V)$ and so $E_1 \in \mathrm{MV}(V, D)$. This is a contradiction because $|E_1 + D + K_V| \neq \emptyset$ and (V, D) is not of the first kind by [14, Lemma 2.5].

Suppose that $a_0 = 3$. Then the divisor $F_0 := 3(C + D_4) + 2D_5 + D_0 + D_6$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_0|} : V \to \mathbb{P}^1$, D_1, D_2 and D_7 become sections of Φ and $D - (D_1 + D_2 + D_7)$ is contained in fibers of Φ. Let F_1 be the fiber of Φ containing D_3. If SuppF_1 contains D_8, then it consists of D_3, D_8 and some (-1)-curves by [14, Lemma 2.2 (3)]. Hence we know that $F_1 = 2E_1 + D_3 + D_8$, where E_1 is a (-1)-curve with $E_1D_3 = E_1D_8 = 1$. However this is a contradiction because $1 = F_1D_1 = 2E_1D_1$. So SuppF_1 does not contain D_8. The argument as in the subcase $a_0 = 5$ implies that $F_1 = E_1 + D_3 + E'_1$, where E_1 and E'_1 are (-1)-curves, $E_1D_3 = E'_1D_3 = 1$ and $E_1E'_1 = 0$. Since D_1 is a section of Φ, we may assume that $E_1D_1 = 1$. Then $-E_1(D^\# + K_V) \leq 1 - (\frac{1}{2} + \frac{1}{3}) = \frac{1}{6} = -C(D^\# + K_V)$ and so $E_1 \in \mathrm{MV}(V, D)$. This is a contradiction because $|E_1 + D + K_V| \neq \emptyset$ and (V, D) is not of the first kind by [14, Lemma 2.5].

The assertions (1) and (2) of Theorem 1.1 are thus proved.

We calculate $\pi_1(V\setminus\mathrm{Supp}D) = \pi_1(X_{\mathrm{reg}})$, which proves the assertion (3) of Theorem 1.1. Let $\Phi : V \to \mathbb{P}^1$ be the $\mathbb{P}^1$-fibration as in the assertion (2) of Theorem 1.1. If the dual graph of D is given as (3) in Fig. 1.1, then Φ has a unique singular fiber and $V\setminus\mathrm{Supp}(D+C)$, where C is a (-1)-curve and is a component of the unique singular fiber of Φ (the dotted line in the configuration (3) in Fig. 1.2), is isomorphic to the affine plane $\mathbb{A}^2$. Hence $\pi_1(V \setminus \mathrm{Supp}D) = (1)$. If the dual graph of D is given as (1) (resp. (2)) in Fig. 1.1, then $\Phi|_{V\setminus\mathrm{Supp}D} : V \setminus \mathrm{Supp}D \to \mathbb{P}^1$ becomes an $\mathbb{A}^1$-fibration onto $\mathbb{P}^1$, every fiber of $\Phi|_{V\setminus\mathrm{Supp}D}$ is isomorphic to $\mathbb{A}^1$ and $\Phi|_{V\setminus\mathrm{Supp}D}$ has two multiple fibers whose multiplicities are $(2,2)$ (resp. $(4,2)$). Hence, we infer from the results in [7, §5] that $\pi_1(V \setminus \mathrm{Supp}D) = \mathbb{Z}/2\mathbb{Z}$.

Finally, we prove the assertion (4) of Theorem 1.1. It is clear that all the configurations (1)–(3) in Fig. 1.2 are realized. Let (V, D) be a pair whose configuration is given as (n) $(n \in \{1, 2, 3\})$ in Fig. 1.2. We can easily see that the divisor D can be contracted to a rational log canonical singular point. Let $\pi : V \to X$ be the contraction of $\mathrm{Supp}D$. Since $\rho(V) = 1 + \#D$, we have $\rho(X) = 1$, here we note that K_X is $\mathbb{Q}$-Cartier since $\pi(\mathrm{Supp}D)$ is a rational singular point on X. Let $\Phi : V \to \mathbb{P}^1$ be the $\mathbb{P}^1$-fibration as in the assertion (2) of Theorem 1.1 and let F be a general fiber of Φ. Then $FD = 1$ (see Fig. 1.2) and so $FD^\# \leq 1$. Since $\pi^*(K_X) \equiv D^\# + K_V$, we have $F\pi^*(K_X) = F(D^\# + K_V) \leq -1$. This implies that K_X is not nef. Since $\rho(X) = 1$ and K_X is $\mathbb{Q}$-Cartier, we know that $-K_X$ is ample. Therefore, the surface X is a normal del Pezzo surface of rank one with unique rational log canonical singular point.

The proof of Theorem 1.1 is thus completed.

4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2.

Let X be as in Theorem 1.2 and set $P := \mathrm{Sing}X$. Let $\pi : (V, D) \to X$ be the minimal resolution of X, where D is the reduced exceptional divisor. Let $D^\#$ be the $\mathbb{Q}$-divisor constructed in §2.

Lemma 4.1. *The singular point P is rational.*

Proof. Since $\overline{\kappa}(V \setminus \mathrm{Supp}D) = \overline{\kappa}(X_{\mathrm{reg}}) = -\infty$ by the hypothesis and V is a rational surface, it follows from [16, Lemma I.2.1.3 (p. 7)] that each irreducible component of D is a smooth rational curve and the dual graph of D is a tree. So the assertion follows (cf. [2]). □

By Lemma 4.1 and [1], we know that X is projective and $\mathbb{Q}$-factorial.

Lemma 4.2. *The canonical divisor K_X on X is not pseudo-effective. Furthermore, it is not nef.*

Proof. Since the singular point P on X is log canonical, we have $D^{\#} \leq D$. Since $\overline{\kappa}(V \setminus \mathrm{Supp} D) = -\infty$ and D is an SNC-divisor, we infer from [16, Lemma I.3.11 (p. 44)] (or [17, Lemma 2.2.6.2 (p. 80)]) that $D + K_V$ is not pseudo-effective. So K_X is not pseudo-effective, neither.

If K_X is nef, then X is minimal in the sense of [6]. However, this contradicts [6, Theorem 3.3] because K_X is not pseudo-effective. Hence K_X is not nef. □

Let R be a K_X-negative extremal ray of $\overline{NE}(X)$, which exists by Lemma 4.2 and [6, Theorem 3.2]. Since X has only log canonical singularities, i.e., the pair $(X, 0)$ is log canonical, we infer from [6, Theorems 3.2 and 3.3] and [20, Theorem 6.2] that there exists a contraction morphism $\varphi_R : X \to Y$ with the following properties:

(i) For an integral curve T on X, $\varphi_R(T)$ is a point if and only if $[T] \in R$.

(ii) $\mathcal{O}_Y \cong (\varphi_R)_*(\mathcal{O}_X)$.

(iii) For a line bundle L on X such that $LT = 0$ for every curve T with $[T] \in R$, there exists a line bundle L_Y on Y such that $L \cong \varphi_R^*(L_Y)$.

(iv) $\rho(Y) = \rho(X) - 1$.

(v) Y is $\mathbb{Q}$-factorial and has only rational log canonical singularities if $\dim Y = 2$.

Here we note that, if $\dim Y = 2$ then Y has only rational singularities because so is X. By [6, Proposition 3.8], there exists a rational curve C on X such that the K_X-negative extremal ray R is spanned by C and $-K_X C \leq 3$. If $C^2 \geq 0$, then it follows that φ_R gives a Mori fiber space structure on X. See the proof of [20, Theorem 6.2] in [20, §6]. If $C^2 < 0$, then φ_R is the contraction of C to a point on Y and C is the unique curve that is contracted by φ_R.

Lemma 4.3. *With the same notations and assumptions as above, assume that $C^2 < 0$. Then the following assertions hold true.*

(1) *Y has at most one singular point and the singular point on Y is rational and log canonical.*

(2) *$\pi_1(X_{\mathrm{reg}})$ is a residue group of $\pi_1(Y_{\mathrm{reg}})$ and $\overline{\kappa}(Y_{\mathrm{reg}}) = -\infty$.*

Proof. Set $Q := \varphi_R(P)$. If $P \notin C$, then C is a (-1)-curve on X_{reg} because $C^2 < 0$ and $CK_X < 0$. So, Y has only one singular point and $Q = \mathrm{Sing} Y$. Hence $\pi_1(X_{\mathrm{reg}}) = \pi_1(Y_{\mathrm{reg}})$ and $\overline{\kappa}(Y_{\mathrm{reg}}) = \overline{\kappa}(X_{\mathrm{reg}}) = -\infty$. If $P \in C$, then Q is a unique singular point on Y, here we may assume that Y is not smooth. By (v) as above, Q is rational and log canonical. Let C' be the proper transform of C on V. It is clear that $V \setminus \mathrm{Supp}(C' + D) = Y \setminus Q$, $(C')^2 < 0$, and $C'(D^{\#} + K_V) = C'\pi^* K_X < 0$. Then $C' K_V \leq C'(D^{\#} + K_V) < 0$. So C' is a (-1)-curve on V. Moreover, since the divisor $C' + D$ can be contracted to a rational singular point, $C'D = 1$ (see [2]) and then

$\overline{\kappa}(V \setminus \mathrm{Supp}(C' + D)) = \overline{\kappa}(V \setminus \mathrm{Supp}D) = -\infty$. The assertions (1) and (2) are thus verified. $\square$

Since X has only rational log canonical singularities and K_X is not pseudo-effective, we infer from [6, Theorem 3.3] (or [20, Theorem 0.1]) that there exists a sequence of birational morphisms $\varphi_i : X_i \to X_{i+1}$ $(i = 0, 1, \ldots, s-1)$, where $X_0 := X$, with the following properties:

(a) X_i has only rational log canonical singularities for $i = 0, 1, \ldots, s-1$.
(b) X_s is a Mori fiber space.

Moreover, by using Lemma 4.3 repeatedly, we know that:

(c) X_i has at most one singular point for $i = 0, 1, \ldots, s-1$.
(d) $\pi_1((X_i)_{\mathrm{reg}})$ is a residue group of $\pi_1((X_{i+1})_{\mathrm{reg}})$ for $i = 0, 1, \ldots, s-1$.

Therefore, in order to prove Theorem 1.2, we consider only the case where X is a Mori fiber space, i.e., there exists a K_X-negative extremal contraction morphism $\varphi : X \to Z$ onto a normal variety Z of $\dim Z < \dim X = 2$. If $\dim Z = 1$, then general fiber of φ is isomorphic to $\mathbb{P}^1$ and $Z \cong \mathbb{P}^1$ since X is a rational surface. Let F be the fiber of φ containing P. Since P is a unique singular point on X, we see that $X \setminus \mathrm{Supp}F \cong \mathbb{P}^1 \times (Z \setminus \varphi(F)) = \mathbb{P}^1 \times \mathbb{A}^1$ is simply connected. Hence $\pi_1(X_{\mathrm{reg}}) = (1)$.

Finally, we consider the case $\dim Z = 0$. Then X is a normal del Pezzo surface of rank one. If P is not log terminal, then $\pi_1(X_{\mathrm{reg}})$ is (1) or $\mathbb{Z}/2\mathbb{Z}$ by (3) of Theorem 1.1. So we consider the case where P is log terminal. Since X is a rational surface, $H_1(X_{\mathrm{reg}}, \mathbb{Z})$ is finite (see [8, Lemma 1.4]). We infer from [11, Main Theorem] that X_{reg} contains $\mathbb{A}^1_* \times \mathbb{A}^1_*$, where $\mathbb{A}^1_*$ is the affine line minus one point, as a Zariski open subset. Since $\pi_1(X_{\mathrm{reg}})$ is a residue group of $\pi_1(\mathbb{A}^1_* \times \mathbb{A}^1_*) \cong \mathbb{Z} \oplus \mathbb{Z}$, it is abelian. Therefore, the assertions of Theorem 1.2 follow.

Proof of Theorem 1.2 is thus completed.

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