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SIMULTANEOUS EXTENSIONS OF SELBERG AND BUZANO INEQUALITIES

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Abstract. We give a simultaneous extension of Selberg and Buzano inequalities: If y_1, y_2 and nonzero vectors $\{z_i; i = 1, 2, \ldots, n\}$ in a Hilbert space $\mathscr{H}$ satisfy the orthogonality condition $\langle y_k, z_i\rangle = 0$ for $i = 1, 2, \ldots, n$ and $k = 1, 2$, then

$$|\langle x, y_1\rangle \langle x, y_2\rangle| + \mathcal{B}(y_1, y_2) \sum_i \frac{|\langle x, z_i\rangle|^2}{\sum_j |\langle z_i, z_j\rangle|} \le \mathcal{B}(y_1, y_2) \|x\|^2$$

holds for all $x \in \mathscr{H}$, where $\mathcal{B}(y_1, y_2) = \frac{1}{2}(\|y_1\| \|y_2\| + |\langle y_1, y_2\rangle|)$.

As an application, we discuss some refinements of the Heinz-Kato-Furuta inequality and the Bernstein inequality.

1. Introduction

Let $\mathscr{H}$ be a Hilbert space in the below. In [21], K. and F. Kubo sought out the Selberg inequality which is an extension of the Bessel inequality, and they gave it an elegant proof by using Geršgorin's theorem.

Selberg inequality. For given nonzero vectors $\{z_i; i = 1, 2, \ldots, n\}$ in $\mathscr{H}$,

$$\sum_i \frac{|\langle x, z_i\rangle|^2}{\sum_j |\langle z_i, z_j\rangle|} \le \|x\|^2 \tag{SI}$$

holds for all $x \in \mathscr{H}$.

In [11], to give simultaneous extensions of Selberg and Heinz-Kato-Furuta inequalities, the following lemma is prepared:

Lemma A. *If $y \in \mathscr{H}$ satisfies $\langle y, z_i\rangle = 0$ for given nonzero vectors $\{z_i; i = 1, 2, \ldots, n\} \subset \mathscr{H}$, then*

$$|\langle x, y\rangle|^2 + \sum_i \frac{|\langle x, z_i\rangle|^2}{\sum_j |\langle z_i, z_j\rangle|} \|y\|^2 \le \|x\|^2 \|y\|^2 \tag{1.1}$$

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holds for all $x \in \mathscr{H}$.

It is regarded as a simultaneous extension of Schwarz and Selberg inequalities.

On the other hand, Buzano inequality, simply (BI), says in [2] that

$$|\langle x, y_1\rangle \langle x, y_2\rangle| \le \frac{1}{2}(\|y_1\|\,\|y_2\| + |\langle y_1, y_2\rangle|)\,\|x\|^2, \tag{BI}$$

holds for all $x, y_1, y_2 \in \mathscr{H}$, which includes Schwarz inequality as in the case $y_1 = y_2$.

In this note, we propose an extension of Lemma A as a simultaneous extension of Selberg and Buzano inequalities. By using this, we discuss some refinements of the Heinz-Kato-Furuta inequality and the Bernstein inequality.

2. Simultaneous extension of Selberg and Buzano inequalities

First of all, we recall the Buzano inequality and its equality condition. For convenience, we denote by

$$\mathcal{B}(y_1, y_2) := \frac{1}{2}(\|y_1\|\,\|y_2\| + |\langle y_1, y_2\rangle|)$$

for $y_1, y_2 \in \mathscr{H}$.

Lemma 2.1. *Let* $\{y_1, y_2\}$ *be a given pair of vectors in* $\mathscr{H}$. *Then* (BI) *holds for all* $x \in \mathscr{H}$:

$$|\langle x, y_1\rangle \langle x, y_2\rangle| \le \mathcal{B}(y_1, y_2)\,\|x\|^2. \tag{BI}$$

Moreover, if $\{y_1, y_2\}$ *is linearly independent, then the equality in* (BI) *holds for* $x \in \mathscr{H}$ *if and only if* $x = a(\|y_2\|\, y_1 + e^{i\theta}\,\|y_1\|\, y_2)$ *for some scalar* a*, where* $\theta = \arg\langle y_1, y_2\rangle$. *Incidentally, if* $\{y_1, y_2\}$ *is linearly dependent, then the equality in* (BI) *holds for* $x \in \mathscr{H}$ *if and only if* $x = ay_1$ *for some scalar* a.

Proof. We first review a proof of (BI). We may assume that $\|x\| = 1$. Then we have

$$\begin{aligned}
|\langle y_1, x\rangle \langle x, y_2\rangle| &= \left|\left\langle \langle y_1, x\rangle x - \frac{1}{2}y_1, y_2\right\rangle + \frac{1}{2}\langle y_1, y_2\rangle\right| \\
&\le \left|\left\langle \langle y_1, x\rangle x - \frac{1}{2}y_1, y_2\right\rangle\right| + \left|\frac{1}{2}\langle y_1, y_2\rangle\right| \\
&\le \left\|\langle y_1, x\rangle x - \frac{1}{2}y_1\right\|\,\|y_2\| + \frac{1}{2}|\langle y_1, y_2\rangle| \\
&= \frac{1}{2}\,\|y_1\|\,\|y_2\| + \frac{1}{2}|\langle y_1, y_2\rangle| \\
&= \mathcal{B}(y_1, y_2).
\end{aligned}$$

Now we assume that $\{y_1, y_2\}$ is linearly independent. Note that the equality holds for (BI) if and only if the equalities hold in the above inequalities and that the equality holds in the first (resp. second) inequality if and only if

$$\arg \langle 2 \langle y_1, x\rangle x - y_1, y_2\rangle = \arg \langle y_1, y_2\rangle := \theta$$

(resp. there exists a scalar k such that

$$ky_2 = 2 \langle y_1, x\rangle x - y_1).$$

Hence it follows that $\arg k = \theta$ and

$$|k| \|y_2\| = \|2 \langle y_1, x\rangle x - y_1\| = \|y_1\|$$

and so $k = \frac{\|y_1\|}{\|y_2\|} e^{i\theta}$.

Next, to determine the form of a vector x, we may assume that $x = ay_1 + by_2$ for some scalars a, b. Since $ky_2 = 2 \langle y_1, x\rangle x - y_1 = 2b \langle y_1, x\rangle y_2 + 2a \langle y_1, x\rangle y_1 - y_1$ and $\{y_1, y_2\}$ is linearly independent, we have

$$2a \langle y_1, x\rangle = 1 \quad \text{and} \quad 2b \langle y_1, x\rangle = k,$$

so that $b = ak$. Therefore it implies that $x = a(y_1 + ky_2)$, that is, $x = c(\|y_2\| y_1 + e^{i\theta} \|y_1\| y_2)$ for some c and θ with $\theta = \arg \langle y_1, y_2\rangle$.

Conversely it is easily checked that if $x = \|y_2\| y_1 + e^{i\theta} \|y_1\| y_2$ where $\theta = \arg \langle y_1, y_2\rangle$, then the equality holds in (BI). As a matter of fact, we have

$$| \langle y_1, x\rangle \langle x, y_2\rangle | = 4\mathcal{B} (y_1, y_2)^2 \|y_1\| \|y_2\| ,$$

and

$$\|x\|^2 = 4\mathcal{B} (y_1, y_2) \|y_1\| \|y_2\| .$$

The latter case where they are linearly dependent is obvious. □

Next we recall the equality condition for the Selberg inequality, see [9, Theorem 2].

Lemma 2.2. *Let $\{z_i; i = 1, 2, \ldots, n\}$ be a given family of nonzero vectors in $\mathscr{H}$ which are not mutually orthogonal. Then the equality*

$$\sum_i \frac{| \langle x, z_i\rangle |^2}{\sum_j | \langle z_i, z_j\rangle |} = \|x\|^2$$

holds for $x \in \mathscr{H}$ if and only if $x = \sum_i a_i z_i$ for some scalars $a_1, \cdots, a_n$ such that $\langle a_i z_i, a_j z_j\rangle \geq 0$ and $|a_i| = |a_j|$ for all i, j.

Now we propose a simultaneous extension of Selberg and Buzano inequalities.

Theorem 2.3. *If $y_1, y_2 \in \mathscr{H}$ satisfy $\langle y_k, z_i\rangle = 0$ for $k = 1, 2$ and given nonzero vectors $\{z_i; i = 1, 2, \ldots, n\} \subset \mathscr{H}$, then*

$$|\langle x, y_1\rangle \langle x, y_2\rangle| + \mathcal{B}(y_1, y_2) \sum_i \frac{|\langle x, z_i\rangle|^2}{\sum_j |\langle z_i, z_j\rangle|} \leq \mathcal{B}(y_1, y_2) \|x\|^2 \tag{2.1}$$

holds for all $x \in \mathscr{H}$.

Moreover, the equality holds in the above if and only if $x = x_1 \oplus x_2$ where x_1 (resp. x_2) is in the subspace spanned by y_1, y_2 (resp. $z_1, \ldots, z_n$) and satisfies the equality condition in (BI) *(resp.*(SI)*) as in Lemma 2.1 (resp. Lemma 2.2).*

Proof. We put $a_i = \frac{\langle x, z_i\rangle}{\sum_j |\langle z_i, z_j\rangle|}$ and $u = x - \sum_i \frac{\langle x, z_i\rangle}{\sum_j |\langle z_i, z_j\rangle|} z_i = x - \sum_i a_i z_i$. Then we have

$$\begin{aligned}
\|u\|^2 &= \left\| x - \sum_i a_i z_i \right\|^2 \\
&= \|x\|^2 - 2\,\mathrm{Re} \sum_i \bar{a}_i \langle x, z_i\rangle + \left\| \sum_i a_i z_i \right\|^2 \\
&\leq \|x\|^2 - 2 \sum_i \frac{|\langle x, z_i\rangle|^2}{\sum_j |\langle z_i, z_j\rangle|} + \sum_{i,j} |a_i||a_j| |\langle z_i, z_j\rangle| \\
&\leq \|x\|^2 - 2 \sum_i \frac{|\langle x, z_i\rangle|^2}{\sum_j |\langle z_i, z_j\rangle|} + \sum_i |a_i|^2 \sum_j |\langle z_i, z_j\rangle| \\
&= \|x\|^2 - \sum_i \frac{|\langle x, z_i\rangle|^2}{\sum_j |\langle z_i, z_j\rangle|},
\end{aligned}$$

because the second inequality in the above is ensured by

$$\sum_i |a_i|^2 \sum_j |\langle z_i, z_j\rangle| - \sum_{i,j} |a_i||a_j| |\langle z_i, z_j\rangle| = \frac{1}{2} \sum_{i,j} (|a_i| - |a_j|)^2 |\langle z_i, z_j\rangle| \geq 0.$$

Multiplying $\mathcal{B}(y_1, y_2)$ on both sides, it follows from the Buzano inequality that

$$\begin{aligned}
\mathcal{B}(y_1, y_2) \left(\|x\|^2 - \sum_i \frac{|\langle x, z_i\rangle|^2}{\sum_j |\langle z_i, z_j\rangle|} \right) &\geq \mathcal{B}(y_1, y_2) \|u\|^2 \\
&\geq |\langle u, y_1\rangle \langle u, y_2\rangle| = |\langle x, y_1\rangle \langle x, y_2\rangle|,
\end{aligned}$$

and hence we have the desired inequality.

The equality condition is easily obtained by Lemmas 2.1 and 2.2. □

3. Generalizations of Theorem 2.3

Now Furuta [18, Theorem 2] showed the following extension of the Selberg inequality: Let T be an operator on $\mathscr{H}$ with the kernel $\ker(T)$. For given $z_i \notin \ker(T^*)$ for

$i = 1, 2, \ldots, n$,

$$\sum_i \frac{|\langle Tx, z_i\rangle|}{\sum_j |\langle |T^*|^{2(1-\alpha)} z_i, z_j\rangle|} \le \||T|^\alpha x\|^2 \tag{3.1}$$

holds for all $x \in \mathscr{H}$ and $\alpha \in [0, 1]$.

Corollary 3.1. *Let $T = U|T|$ be the polar decomposition of an operator T on $\mathscr{H}$, $z_i \notin \ker(T^*)$ for $i = 1, 2, \ldots, n$ and $\alpha \in [0, 1]$. If $y_1, y_2 \in \mathscr{H}$ satisfy $\langle U|T|^{1-\alpha} y_k, z_i\rangle = 0$ for $i = 1, 2, \ldots, n$ and $k = 1, 2$, then*

$$|\langle |T|^\alpha x, y_1\rangle \langle |T|^\alpha x, y_2\rangle| + \mathcal{B}(y_1, y_2) \sum_i \frac{|\langle Tx, z_i\rangle|^2}{\sum_j |\langle |T^*|^{2(1-\alpha)} z_i, z_j\rangle|} \le \mathcal{B}(y_1, y_2) \||T|^\alpha x\|^2 \tag{3.2}$$

holds for all $x \in \mathscr{H}$.

Proof. We apply Theorem 2.3 by replacing x, z_i to $|T|^\alpha x, |T|^{1-\alpha} U^* z_i$, respectively. Incidentally the orthogonality condition is satisfied. □

Next we propose another refinement of (3.1):

Corollary 3.2. *Let $T = U|T|$ be the polar decomposition of an operator T on $\mathscr{H}$. Suppose that $z_i \notin \ker(T^*)$ for $i = 1, 2, \ldots, n$ and $\alpha, \beta \ge 0$ with $\alpha + \beta \ge 1 \ge \alpha$. If $y_1, y_2 \in \mathscr{H}$ satisfy $\left\langle |T^*|^{\beta+1-\alpha} y_k, z_i\right\rangle = 0$ for $i = 1, 2, \ldots, n$ and $k = 1, 2$, then*

$$|\left\langle T|T|^{\alpha+\beta-1} x, y_1\right\rangle \left\langle T|T|^{\alpha+\beta-1} x, y_2\right\rangle| + \mathcal{B}\left(|T^*|^\beta y_1, |T^*|^\beta y_2\right) \sum_i \frac{|\langle Tx, z_i\rangle|^2}{\sum_j |\langle |T^*|^{2(1-\alpha)} z_i, z_j\rangle|}$$
$$\le \mathcal{B}\left(|T^*|^\beta y_1, |T^*|^\beta y_2\right) \||T|^\alpha x\|^2$$

holds for all $x \in \mathscr{H}$. In particular, if $\left\langle |T^*|^{2(1-\alpha)} y_k, z_i\right\rangle = 0$ for $\alpha \in [0, 1]$, $i = 1, 2, \ldots, n$ and $k = 1, 2$, then

$$|\langle Tx, y_1\rangle \langle Tx, y_2\rangle| + \mathcal{B}\left(|T^*|^{(1-\alpha)} y_1, |T^*|^{1-\alpha} y_2\right) \sum_i \frac{|\langle Tx, z_i\rangle|^2}{\sum_j |\langle |T^*|^{2(1-\alpha)} z_i, z_j\rangle|}$$
$$\le \mathcal{B}\left(|T^*|^{1-\alpha} y_1, |T^*|^{1-\alpha} y_2\right) \||T|^\alpha x\|^2$$

holds for all $x \in \mathscr{H}$.

Proof. We apply Theorem 2.3 by replacing x, z_i, y_k to $|T|^\alpha x, |T|^{1-\alpha} U^* z_i, U^*|T^*|^\beta y_k$. Incidentally the orthogonality condition is satisfied by

$$\left\langle |T^*|^\beta y_k, |T^*|^{1-\alpha} z_i\right\rangle = \left\langle |T^*|^{\beta+1-\alpha} y_k, z_i\right\rangle = 0.$$

□

Corollary 3.3. *Let $T = U|T|$ be the polar decomposition of an operator T on $\mathscr{H}$. Suppose that $z_i \notin \ker(T)$ for $i = 1, 2, \ldots, n$ and $\alpha, \beta \geq 0$ with $\alpha + \beta \geq 1$. If $y_1, y_2 \in \mathscr{H}$ satisfy $\langle T|T|^{\alpha+\beta-1} z_i, y_k \rangle = 0$ for $i = 1, 2, \ldots, n$ and $k = 1, 2$, then*

$$| \langle T|T|^{\alpha+\beta-1}x, y_1 \rangle \langle T|T|^{\alpha+\beta-1}x, y_2 \rangle | + \mathcal{B}\left(|T^*|^\beta y_1, |T^*|^\beta y_2\right) \sum_i \frac{| \langle |T|^{2\alpha}x, z_i \rangle |^2}{\sum_j | \langle |T|^{2\alpha} z_i, z_j \rangle |}$$

$$\leq \mathcal{B}\left(|T^*|^\beta y_1, |T^*|^\beta y_2\right) \||T|^\alpha x\|^2$$

holds for all $x \in \mathscr{H}$.

Proof. We apply Theorem 2.3 by replacing x, z_i, y_k to $U|T|^\alpha x, U|T|^\alpha z_i, |T^*|^\beta y_k$. Incidentally the orthogonality condition is satisfied, and so the conclusion is obtained. □

4. Extensions via Heinz-Kato-Furuta inequality

In [18], Furuta extended the Heinz-Kato inequality, which is called the Heinz-Kato-Furuta inequality:

The Heiz-Kato-Furuta inequality. Let A and B be positive operators on $\mathscr{H}$. If T satisfies $T^*T \leq A^2$ and $TT^* \leq B^2$, then

$$| \langle T|T|^{\alpha+\beta-1}x, y \rangle | \leq \|A^\alpha x\| \left\|B^\beta y\right\|$$

holds for all $x, y \in \mathscr{H}$ and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \geq 1$. In addition, if A and B are invertible, then $\alpha + \beta \geq 1$ is unnecessary.

Afterwards, several authors have generalized it, e.g. [11], [12], [13], [19].

In this section, we apply the results in the preceding section to extend the Heinz-Kato-Furuta inequality.

To do this, the point is the following lemma:

Lemma 4.1. *If $TT^* \leq B^2$ for some $B \geq 0$, then for $\beta \in [0, 1]$*

$$\mathcal{B}\left(|T^*|^\beta y_1, |T^*|^\beta y_2\right) \leq \left\|B^\beta y_1\right\| \left\|B^\beta y_2\right\|$$

holds for all $y_1, y_2 \in \mathscr{H}$.

Proof. Note that the Löwner-Heinz inequality [23] ensures that

$$|T^*|^{2\beta} \leq B^{2\beta} \quad \text{for } \beta \in [0, 1],$$

so that $\left\||T^*|^\beta y\right\| \leq \left\|B^\beta y\right\|$ for all $y \in \mathscr{H}$. Hence we have

$$\mathcal{B}\left(|T^*|^\beta y_1, |T^*|^\beta y_2\right) \leq \left\||T^*|^\beta y_1\right\| \left\||T^*|^\beta y_2\right\| \leq \left\|B^\beta y_1\right\| \left\|B^\beta y_2\right\|.$$

□

First of all, the following inequality follows from Corollary 3.1 and the Löwner-Heinz inequality.

Corollary 4.2. *Let $T = U|T|$ be the polar decomposition of an operator T on $\mathscr{H}$, $z_i \notin \ker(T^*)$ for $i = 1, 2, \ldots, n$ and $\alpha \in [0, 1]$. If $T^*T \le A^2$ for some positive operator A, and $y_1, y_2 \in \mathscr{H}$ satisfy $\langle U|T|^{1-\alpha}y_k, z_i\rangle = 0$ for $i = 1, 2, \ldots, n$ and $k = 1, 2$, then*

$$| \langle |T|^\alpha x, y_1\rangle \langle |T|^\alpha x, y_2\rangle | + \mathcal{B}(y_1, y_2) \sum_i \frac{| \langle Tx, z_i\rangle |^2}{\sum_j | \langle |T^*|^{2(1-\alpha)} z_i, z_j\rangle |} \le \mathcal{B}(y_1, y_2) \|A^\alpha x\|^2 \tag{4.1}$$

holds for all $x \in \mathscr{H}$.

Now the following inequalities follow from Corollary 3.2 and Lemma 4.1.

Corollary 4.3. *Let $T = U|T|$ be the polar decomposition of an operator T on $\mathscr{H}$. Suppose that $z_i \notin \ker(T^*)$ for $i = 1, 2, \ldots, n$ and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \ge 1$. If $T^*T \le A^2$ and $TT^* \le B^2$ for some $A, B \ge 0$, and $y_1, y_2 \in \mathscr{H}$ satisfy $\left\langle |T^*|^{\beta+1-\alpha} y_k, z_i\right\rangle = 0$ for $i = 1, 2, \ldots, n$ and $k = 1, 2$, then*

$$| \left\langle T|T|^{\alpha+\beta-1}x, y_1\right\rangle \left\langle T|T|^{\alpha+\beta-1}x, y_2\right\rangle | + \mathcal{B}\left(|T^*|^\beta y_1, |T^*|^\beta y_2\right) \sum_i \frac{| \langle Tx, z_i\rangle |^2}{\sum_j | \langle |T^*|^{2(1-\alpha)} z_i, z_j\rangle |}$$
$$\le \left\|B^\beta y_1\right\| \left\|B^\beta y_2\right\| \|A^\alpha x\|^2$$

holds for all $x \in \mathscr{H}$. In particular, if $\left\langle |T^*|^{2(1-\alpha)} y_k, z_i\right\rangle = 0$ for $\alpha \in [0, 1]$, $i = 1, 2, \ldots, n$ and $k = 1, 2$, then

$$| \langle Tx, y_1\rangle \langle Tx, y_2\rangle | + \mathcal{B}\left(|T^*|^{(1-\alpha)} y_1, |T^*|^{1-\alpha} y_2\right) \sum_i \frac{| \langle Tx, z_i\rangle |^2}{\sum_j | \langle |T^*|^{2(1-\alpha)} z_i, z_j\rangle |}$$
$$\le \left\|B^{1-\alpha} y_1\right\| \left\|B^{1-\alpha} y_2\right\| \|A^\alpha x\|^2$$

holds for all $x \in \mathscr{H}$.

Next the following inequality follows from Corollary 3.3 and Lemma 4.1.

Corollary 4.4. *Let $T = U|T|$ be the polar decomposition of an operator T on $\mathscr{H}$. Suppose that $z_i \notin \ker(T^*)$ for $i = 1, 2, \ldots, n$ and $\alpha, \beta \ge 0$ with $\alpha + \beta \ge 1$. If $T^*T \le A^2$ and $TT^* \le B^2$ for some $A, B \ge 0$, and $y_1, y_2 \in \mathscr{H}$ satisfy $\left\langle T|T|^{\alpha+\beta-1} z_i, y_k\right\rangle = 0$ for $i = 1, 2, \ldots, n$ and $k = 1, 2$, then*

$$| \left\langle T|T|^{\alpha+\beta-1}x, y_1\right\rangle \left\langle T|T|^{\alpha+\beta-1}x, y_2\right\rangle | + \mathcal{B}\left(|T^*|^\beta y_1, |T^*|^\beta y_2\right) \sum_i \frac{| \langle |T|^{2\alpha} x, z_i\rangle |^2}{\sum_j | \langle |T|^{2\alpha} z_i, z_j\rangle |}$$
$$\le \left\|B^\beta y_1\right\| \left\|B^\beta y_2\right\| \|A^\alpha x\|^2$$

holds for all $x \in \mathscr{H}$.

5. Extensions of Heinz-Kato-Furuta and Bernstein inequalities.

In [12, Theorem 2], we proposed the following improvement of the Heinz-Kato-Furuta inequality and gave conditions under which the equality holds:

Theorem A. *Let T be an operator on $\mathscr{H}$. If A and B are positive operators on $\mathscr{H}$ such that $T^*T \le A^2$ and $TT^* \le B^2$, then for each $x \in \mathscr{H}$*

$$\left|\left\langle T|T|^{\alpha+\beta-1}x, y\right\rangle\right|^2 + \frac{|\langle |T|^{2\alpha}x, z\rangle|^2 \left\||T^*|^{\beta}y\right\|^2}{\||T|^{\alpha}z\|^2} \le \|A^{\alpha}x\|^2 \left\|B^{\beta}y\right\|^2 \tag{5.1}$$

for all $\alpha, \beta \in [0,1]$ with $\alpha+\beta \ge 1$ and $y, z \in \mathscr{H}$ such that $y \ne 0$, $T|T|^{\alpha+\beta-1}z \ne 0$ and $\left\langle T|T|^{\alpha+\beta-1}z, y\right\rangle = 0$. In the case $\alpha, \beta > 0$, the equality in (5.1) holds if and only if $A^{2\alpha}x = |T|^{2\alpha}x$, $B^{2\beta}y = |T^*|^{2\beta}y$, and $|T|^{\alpha+\beta-1}T^*y$ and $|T|^{2\alpha}(x - \frac{\langle |T|^{2\alpha}x, z\rangle}{\||T|^{\alpha}z\|^2}z)$ are linearly dependent.

It is obvious that (5.1) is just Lin's result [22] in the case of $\alpha+\beta = 1$.

Next we recall the Bernstein inequality [1, p.319].

The Bernstein inequality. *Let S be a selfadjoint operator on $\mathscr{H}$. If e is a unit eigenvector corresponding to an eigenvalue λ of S, then*

$$|\langle x, e\rangle|^2 \le \frac{\|x\|^2 \|Sx\|^2 - \langle x, Sx\rangle^2}{\|(S-\lambda)x\|^2} \tag{5.2}$$

for all $x \in \mathscr{H}$ for which $Sx \ne \lambda x$.

It was extended to nonnormal operators, precisely dominant operators by Furuta [16] and moreover operators with normal eigenvalues [5]. Afterwards we pointed out that eigenvalues and its corresponding eigenvectors of adjoint operators are essential in this discussion [14], and Bessel type inequality [10, Theorem 1] showed the following:

Theorem B. *Let S be an operator on $\mathscr{H}$ and e_i be a unit eigenvector corresponding to an eigenvalue $\bar{\lambda}_i$ of S^* for $i = 1, 2, \ldots, n$. Then for each $x \in \mathscr{H}$ with $\prod_{i=1}^n (S - \lambda_i)x \ne 0$*

$$\sum_i |\langle u_{i-1}, e_i\rangle|^2 \le \|x\|^2 - \frac{|\langle x, \prod_{i=1}^n (S-\lambda_i)x\rangle|^2}{\|\prod_{i=1}^n (S-\lambda_i)x\|^2} \tag{5.3}$$

for $u_i = u_{i-1} - \langle u_{i-1}, e_i\rangle e_i$ with $u_0 = x$ for $i = 1, \ldots, n$.

In particular, if $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal set, then

$$\sum_i |\langle x, e_i\rangle|^2 \le \|x\|^2 - \frac{|\langle x, \prod_{i=1}^n (S-\lambda_i)x\rangle|^2}{\|\prod_{i=1}^n (S-\lambda_i)x\|^2}. \tag{5.4}$$

In this section, we give a simultaneous extension of Theorems A and B. For this, the following result is really important.

Theorem 5.1. *Let T be an operator on $\mathscr{H}$. Then for each $x, y_1, y_2 \in \mathscr{H}$*

$$\left|\left\langle T|T|^{\alpha+\beta-1}x, y_1\right\rangle \left\langle T|T|^{\alpha+\beta-1}x, y_2\right\rangle\right| + \mathcal{B}\left(|T^*|^\beta y_1, |T^*|^\beta y_2\right) \sum_i \frac{\left|\langle |T|^{2\alpha}u_{i-1}, z_i\rangle\right|^2}{\||T|^\alpha z_i\|^2}$$
$$\leq \mathcal{B}\left(|T^*|^\beta y_1, |T^*|^\beta y_2\right) \||T|^\alpha x\|^2 \tag{5.5}$$

for all $\alpha, \beta \geq 0$ with $\alpha+\beta \geq 1$ and $z_1, \ldots, z_n \notin \ker T$ such that $\left\langle T|T|^{\alpha+\beta-1}z_i, y_k\right\rangle = 0$, where $u_i = u_{i-1} - \frac{\langle |T|^{2\alpha}u_{i-1}, z_i\rangle}{\||T|^\alpha z_i\|^2} z_i$ with $u_0 = x$ for $i = 1, 2, \ldots, n$ and $k = 1, 2$. In the case $\alpha, \beta > 0$, if $\{|T^*|^\beta y_1, |T^*|^\beta y_2\}$ is linearly independent, then the equality in (5.5) holds for $x \in \mathscr{H}$ if and only if $U|T|^\alpha u_n = a(\left\||T^*|^\beta y_2\right\| |T^*|^\beta y_1 + e^{i\theta} \left\||T^*|^\beta y_1\right\| |T^*|^\beta y_2)$ for some scalar a, where $\theta = \arg\left\langle |T^*|^\beta y_1, |T^*|^\beta y_2\right\rangle$. Incidentally, if $\{|T^*|^\beta y_1, |T^*|^\beta y_2\}$ is linearly dependent, then the equality in (5.5) holds for $x \in \mathscr{H}$ if and only if $U|T|^\alpha u_n = a|T^*|^\beta y_1$ for some scalar a.

Proof. Noting that $U|T|^{\alpha+\beta} = U|T|^\beta U^* U|T|^\alpha = |T^*|^\beta U|T|^\alpha$ even if either $\alpha = 0$ or $\beta = 0$, we have

$$\begin{aligned}
\left\langle U|T|^\alpha u_n, |T^*|^\beta y_k\right\rangle &= \left\langle U|T|^\alpha u_{n-1}, |T^*|^\beta y_k\right\rangle - \frac{\langle |T|^{2\alpha}u_{n-1}, z_n\rangle}{\||T|^\alpha z_n\|^2} \left\langle U|T|^\alpha z_n, |T^*|^\beta y_k\right\rangle \\
&= \left\langle U|T|^\alpha u_{n-1}, |T^*|^\beta y_k\right\rangle = \cdots = \left\langle U|T|^\alpha x, |T^*|^\beta y_k\right\rangle \\
&= \left\langle T|T|^{\alpha+\beta-1}x, y_k\right\rangle,
\end{aligned}$$

and

$$\||T|^\alpha u_n\|^2 = \||T|^\alpha u_{n-1}\|^2 - \frac{|\langle |T|^{2\alpha}u_{n-1}, z_n\rangle|^2}{\||T|^\alpha z_n\|^2} = \cdots = \||T|^\alpha x\|^2 - \sum_i \frac{|\langle |T|^{2\alpha}u_{i-1}, z_i\rangle|^2}{\||T|^\alpha z_i\|^2}$$

by the definition of u_i and $\left\langle T|T|^{\alpha+\beta-1}y_k, z_i\right\rangle = 0$. Hence it follows from Lemma 2.1 that

$$\begin{aligned}
&\left|\left\langle T|T|^{\alpha+\beta-1}x, y_1\right\rangle \left\langle T|T|^{\alpha+\beta-1}x, y_2\right\rangle\right| = \left|\left\langle U|T|^\alpha u_n, |T^*|^\beta y_1\right\rangle \left\langle U|T|^\alpha u_n, |T^*|^\beta y_2\right\rangle\right| \\
&\leq \mathcal{B}\left(|T^*|^\beta y_1, |T^*|^\beta y_2\right) \|U|T|^\alpha u_n\|^2 = \mathcal{B}\left(|T^*|^\beta y_1, |T^*|^\beta y_2\right) \||T|^\alpha u_n\|^2 \\
&= \mathcal{B}\left(|T^*|^\beta y_1, |T^*|^\beta y_2\right) \||T|^\alpha x\|^2 - \mathcal{B}\left(|T^*|^\beta y_1, |T^*|^\beta y_2\right) \sum_i \frac{\left|\langle |T|^{2\alpha}u_{i-1}, z_i\rangle\right|^2}{\||T|^\alpha z_i\|^2},
\end{aligned}$$

so we obtain the desired inequality (5.5). The equality condition is confirmed by its of Lemma 2.1. □

As a consequence, we have the following refinement of Heinz-Kato-Furuta inequality by Lemma 4.1 and the Löwner-Heinz inequality:

Theorem 5.2. *Let T be an operator on $\mathscr{H}$. If A and B are positive operators on $\mathscr{H}$ such that $T^*T \le A^2$ and $TT^* \le B^2$, then for each $x, y_1, y_2 \in \mathscr{H}$*

$$\left|\left\langle T|T|^{\alpha+\beta-1}x, y_1\right\rangle \left\langle T|T|^{\alpha+\beta-1}x, y_2\right\rangle\right| + \mathcal{B}\left(|T^*|^\beta y_1, |T^*|^\beta y_2\right) \sum_i \frac{\left|\langle |T|^{2\alpha}u_{i-1}, z_i\rangle\right|^2}{\left\||T|^\alpha z_i\right\|^2}$$
$$\le \left\|B^\beta y_1\right\| \left\|B^\beta y_2\right\| \left\|A^\alpha x\right\|^2 \tag{5.6}$$

for all $\alpha, \beta \in [0,1]$ with $\alpha+\beta \ge 1$ and $z_1, \dots, z_n \notin \ker T$ such that $\left\langle T|T|^{\alpha+\beta-1}z_i, y_k\right\rangle = 0$, where $u_i = u_{i-1} - \frac{\left\langle |T|^{2\alpha}u_{i-1}, z_i\right\rangle}{\||T|^\alpha z_i\|^2} z_i$ with $u_0 = x$ for $i = 1, 2, \dots, n$ and $k = 1, 2$. In the case $\alpha, \beta > 0$, the equality in (5.6) holds if and only if $A^{2\alpha}x = |T|^{2\alpha}x$, $B^{2\beta}y_k = |T^*|^{2\beta}y_k$ and the equality condition of (5.5) hold.

Remark 5.3. *If we put $n = 1$ and $y := y_1 = y_2$ in Theorem 5.2, then we obtain Theorem A. Moreover if $n = 1$ the equality condition of (5.6) ensures one of (5.1) by [12, Lemma]. On the other hand, let S, λ_i and e_i for $i = 1, 2, \dots, n$ be as in Theorem B. Moreover if we put $T = I$ (the identity operator), and replace $\frac{z_i}{\|z_i\|}$ and $y(:= y_1 = y_2)$ to e_i and $\prod_{j=1}^n (S - \lambda_j)x$ respectively in Theorem 5.1, then we obtain Theorem B by the following inequality*

$$\left|\left\langle x, \prod_{i=1}^n (S - \lambda_i)x \right\rangle\right|^2 + \sum_i \left|\langle u_{i-1}, e_i\rangle\right|^2 \left\|\prod_{i=1}^n (S - \lambda_i)x\right\|^2 \le \|x\|^2 \left\|\prod_{i=1}^n (S - \lambda_i)x\right\|^2,$$

so Theorem 5.2 is a simultaneous extension of Theorems A and B. Furthermore we can see that the equality condition of Theorem B (5.3) holds if and only if $\prod_{i=1}^n (S - \lambda_i)x$ and u_n are proportional.

Next we obtain the following corollary by Theorem 5.1. For this we recall that λ is a normal eigenvalue of T if there exists a nonzero vector $e \in \mathscr{H}$ such that $Te = \lambda e$ and $T^*e = \overline{\lambda}e$.

Corollary 5.4. *Let T be an operator on $\mathscr{H}$ and let e_i be an eigenvector corresponding to a nonzero normal eigenvalue λ_i of T for $i = 1, 2, \dots, n$. If $y_k \in \mathscr{H}$ satisfies $T^*y_k \ne 0$ and $\langle e_i, y_k\rangle = 0$ for $i = 1, 2, \dots, n$ and $j = 1, 2$, then for each $\beta \ge 0$*

$$\sum_i |\lambda_i|^2 \left|\langle u_{i-1}, e_i\rangle\right|^2 \le \|Tx\|^2 - \frac{\left|\left\langle T|T|^\beta x, T^*y_1\right\rangle \left\langle T|T|^\beta x, T^*y_2\right\rangle\right|}{\mathcal{B}\left(|T^*|^\beta T^*y_1, |T^*|^\beta T^*y_2\right)} \tag{5.7}$$

for all $x \in \mathscr{H}$, where $u_i = u_{i-1} - \langle u_{i-1}, e_i\rangle e_i$ with $u_0 = x$ for $i = 1, 2, \dots, n$. In the case $\beta > 0$, if $\{|T^*|^\beta T^*y_1, |T^*|^\beta T^*y_2\}$ is linearly independent, then the equality in (5.7) holds for $x \in \mathscr{H}$ if and only if $U|T|^\alpha u_n = a(\left\||T^*|^\beta T^*y_2\right\| |T^*|^\beta T^*y_1 + e^{i\theta} \left\||T^*|^\beta T^*y_1\right\| |T^*|^\beta T^*y_2)$ for some scalar a, where $\theta = \arg\left\langle |T^*|^\beta T^*y_1, |T^*|^\beta T^*y_2\right\rangle$. Incidentally, if $\{|T^*|^\beta T^*y_1, |T^*|^\beta T^*y_2\}$ is linearly dependent, then the equality in (5.7) holds for $x \in \mathscr{H}$ if and only if $U|T|^\alpha u_n = a|T^*|^\beta T^*y_1$ for some scalar a.

In particular, if $\{e_1, e_2, \ldots, e_n\}$ *is an orthonormal set, then*

$$\sum_i |\lambda_i|^2 \, |\langle x, e_i \rangle|^2 \le \|Tx\|^2 - \frac{\left|\left\langle T|T|^\beta x, T^* y_1 \right\rangle \left\langle T|T|^\beta x, T^* y_2 \right\rangle\right|}{\mathcal{B}\left(|T^*|^\beta T^* y_1, |T^*|^\beta T^* y_2\right)}. \tag{5.8}$$

The equality condition for (5.8) *is the same as that of* (5.7), *where* $u_n = x - Qx$ *for* $Q = \mathrm{Proj}[e_1, \ldots, e_n]$.

Proof. We put $\alpha = 1$, $z_i = e_i$ and replace y_k to $T^* y_k$ in Theorem 5.1. Since $\left\langle T|T|^\beta e_i, T^* y_k \right\rangle = 0$ by $\langle e_i, y_k \rangle = 0$ for $i = 1, 2, \ldots, n$ and $k = 1, 2$, the assumption of Theorem 5.1 is satisfied and so it follows that

$$\begin{aligned} &|\left\langle T|T|^\beta x, T^* y_1 \right\rangle \left\langle T|T|^\beta x, T^* y_2 \right\rangle| + \mathcal{B}\left(|T^*|^\beta T^* y_1, |T^*|^\beta T^* y_2\right) \sum_i |\lambda_i|^2 |\left\langle u_{i-1}, e_i \right\rangle|^2 \\ &\le \mathcal{B}\left(|T^*|^\beta T^* y_1, |T^*|^\beta T^* y_2\right) \|Tx\|^2 . \end{aligned}$$

Hence we have the desired inequality (5.7).

If $\{e_i, e_2, \ldots, e_n\}$ is an orthonormal set, then the definition of u_i gives $\langle u_{i-1}, e_i \rangle = \langle u_{i-2}, e_i \rangle = \cdots = \langle u_0, e_i \rangle = \langle x, e_i \rangle$ for each $i = 1, 2, \ldots, n$, so the inequality (5.8) holds. □

6. Application of Furuta inequality

The main tool in this section is the Furuta inequality [15]. We now cite it for convenience:

The Furuta inequality.
If $A \ge B \ge 0$, then for each $r \ge 0$,

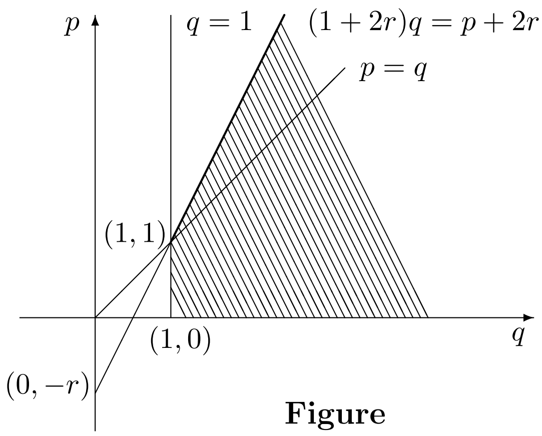
(i) $\quad (B^r A^p B^r)^{\frac{1}{q}} \ge (B^r B^p B^r)^{\frac{1}{q}}$

and

(ii) $\quad (A^r A^p A^r)^{\frac{1}{q}} \ge (A^r B^p A^r)^{\frac{1}{q}}$

hold for $p \ge 0$ and $q \ge 1$ with

$$(1 + 2r)q \ge p + 2r.$$



Figure

We refer [20] and [3] for mean theoretic proofs of it, and [17] for a one-page proof. The best possibility of the domain drawn in the Figure is proved by Tanahashi [24].

The Heinz-Kato-Furuta inequality has been extended by the use of the Furuta inequality in [19]. To give further extensions of the Heinz-Kato-Furuta inequality, we apply the Furuta inequality, too.

First, we have the following extension of Corollary 3.2 by the Furuta inequality:

Theorem 6.1. *Let A and B be positive operators on $\mathscr{H}$ and T an operator such that $T^*T \leq A^2$. Then for each $r, s \geq 0$*

$$\begin{aligned}
&|\left\langle T|T|^{(1+r)\alpha+(1+s)\beta-1}x, y_1\right\rangle \left\langle T|T|^{(1+r)\alpha+(1+s)\beta-1}x, y_2\right\rangle| \\
&+ \mathcal{B}\left(|T^*|^{(1+s)\beta}y_1, |T^*|^{(1+s)\beta}y_2\right) \sum_i \frac{|\langle Tx, z_i\rangle|^2}{\sum_j \langle |T^*|^{2(1-\alpha-r\alpha)}z_i, z_j\rangle} \\
&\leq \mathcal{B}\left(|T^*|^{(1+s)\beta}y_1, |T^*|^{(1+s)\beta}y_2\right) \left\langle (|T|^r A^{2p}|T|^r)^{\frac{(1+r)\alpha}{p+r}} x, x\right\rangle
\end{aligned} \tag{6.1}$$

for all $p, q \geq 1$, $\alpha, \beta \in [0,1]$ with $(1+r)\alpha + (1+s)\beta \geq 1 \geq (1+r)\alpha$ and $x, y_k, z_i \in \mathscr{H}$ such that $z_i \notin \ker(T^*)$ and $\left\langle |T^*|^{(1+s)\beta+1-(1+r)\alpha}y_k, z_i\right\rangle = 0$ for $i = 1, 2, \ldots, n$ and $k = 1, 2$.

Proof. By replacing α (resp. β) to $\alpha_1 = (1+r)\alpha$ (resp. $\beta_1 = (1+s)\beta$) in Corollary 3.2, we have

$$\begin{aligned}
&|\left\langle T|T|^{\alpha_1+\beta_1-1}x, y_1\right\rangle \left\langle T|T|^{\alpha_1+\beta_1-1}x, y_2\right\rangle| \\
&+ \mathcal{B}\left(|T^*|^{\beta_1}y_1, |T^*|^{\beta_1}y_2\right) \sum_i \frac{|\langle Tx, z_i\rangle|^2}{\sum_j |\langle |T^*|^{2(1-\alpha_1)}z_i, z_j\rangle|} \\
&\leq \mathcal{B}\left(|T^*|^{\beta_1}y_1, |T^*|^{\beta_1}y_2\right) \left\langle |T|^{2\alpha_1}x, x\right\rangle
\end{aligned}$$

for all $x \in \mathscr{H}$. Next we replace A, B, r and q to A^2, $|T|^2$, $\frac{r}{2}$ and $\frac{p+r}{(1+r)\alpha}$, respectively in the Furuta inequality. Then we have

$$|T|^{2\alpha_1} = |T|^{2(1+r)\alpha} \leq (|T|^r A^{2p}|T|^r)^{\frac{(1+r)\alpha}{p+r}}.$$

Connecting this with the above inequality, we obtain the inequality (6.1). □

Similarly we have the following further extensions by Corollary 3.3:

Theorem 6.2. *Let A and B be positive operators on $\mathscr{H}$ and T an operator such that $T^*T \leq A^2$. Then for each $r, s \geq 0$*

$$\begin{aligned}
&|\left\langle T|T|^{(1+r)\alpha+(1+s)\beta-1}x, y_1\right\rangle \left\langle T|T|^{(1+r)\alpha+(1+s)\beta-1}x, y_2\right\rangle| \\
&+ \mathcal{B}\left(|T^*|^{(1+s)\beta}y_1, |T^*|^{(1+s)\beta}y_2\right) \sum_i \frac{|\left\langle |T|^{2(1+r)\alpha}x, z_i\right\rangle|^2}{\sum_j |\langle |T|^{2(1+r)\alpha}z_i, z_j\rangle|} \\
&\leq \mathcal{B}\left(|T^*|^{(1+s)\beta}y_1, |T^*|^{(1+s)\beta}y_2\right) \left\langle (|T|^r A^{2p}|T|^r)^{\frac{(1+r)\alpha}{p+r}} x, x\right\rangle
\end{aligned} \tag{6.2}$$

for all $p, q \geq 1$, $\alpha, \beta \in [0,1]$ with $(1+r)\alpha + (1+s)\beta \geq 1$ and $x, y_k, z_i \in \mathscr{H}$ such that $z_i \notin \ker(T)$ and $\left\langle T|T|^{(1+r)\alpha+(1+s)\beta-1}z_i, y_k\right\rangle = 0$ for $i = 1, 2, \ldots, n$ and $k = 1, 2$.

Proof. By replacing α (resp. β) to $\alpha_1 = (1+r)\alpha$ (resp. $\beta_1 = (1+s)\beta$) in Corollary 3.3, we have

$$\begin{aligned}
&|\left\langle T|T|^{\alpha_1+\beta_1-1}x, y_1\right\rangle \left\langle T|T|^{\alpha_1+\beta_1-1}x, y_2\right\rangle| \\
&+ \mathcal{B}\left(|T^*|^{\beta_1}y_1, |T^*|^{\beta_1}y_2\right) \sum_i \frac{|\langle |T|^{2\alpha_1}x, z_i\rangle|^2}{\sum_j |\langle |T|^{2\alpha_1}z_i, z_j\rangle|} \\
&\le \mathcal{B}\left(|T^*|^{\beta_1}y_1, |T^*|^{\beta_1}y_2\right) \left\langle |T|^{2\alpha_1}x, x\right\rangle .
\end{aligned}$$

By the use of the Furuta inequality for $|T|^2 \le A^2$, we have

$$|T|^{2\alpha_1} = |T|^{2(1+r)\alpha} \le (|T|^r A^{2p} |T|^r)^{\frac{(1+r)\alpha}{p+r}} .$$

Combining them, we obtain the inequality (6.2). □

Theorem 5.1 also gives us improvement of the Heinz-Kato-Furuta inequality via the Furuta inequality with the same proof as the preceding theorem.

Theorem 6.3. *Let T be an operator on $\mathscr{H}$. If A and B are positive operators on $\mathscr{H}$ such that $T^*T \le A^2$ and $TT^* \le B^2$, then for each $x, y_1, y_2 \in \mathscr{H}$*

$$\begin{aligned}
&\left|\left\langle T|T|^{(1+r)\alpha+(1+s)\beta-1}x, y_1\right\rangle \left\langle T|T|^{(1+r)\alpha+(1+s)\beta-1}x, y_2\right\rangle\right| \\
&+ \mathcal{B}\left(|T^*|^{(1+s)\beta}y_1, |T^*|^{(1+s)\beta}y_2\right) \sum_i \frac{\left|\left\langle |T|^{2(1+r)\alpha}u_{i-1}, z_i\right\rangle\right|^2}{\left\||T|^{(1+r)\alpha}z_i\right\|^2} \\
&\le \mathcal{B}\left(|T^*|^{(1+s)\beta}y_1, |T^*|^{(1+s)\beta}y_2\right) \left\langle (|T|^r A^{2p}|T|^r)^{\frac{(1+r)\alpha}{p+r}} x, x\right\rangle
\end{aligned} \tag{6.3}$$

for all $p, q \ge 1$, $r, s > 0$, $\alpha, \beta \in [0,1]$ with $(1+r)\alpha + (1+s)\beta \ge 1$ and $z_1, \cdots, z_n \notin \ker T$ such that $\left\langle T|T|^{(1+r)\alpha+(1+s)\beta-1}z_i, y_k\right\rangle = 0$, where $u_i = u_{i-1} - \frac{\left\langle |T|^{2(1+r)\alpha}u_{i-1}, z_i\right\rangle}{\left\||T|^{(1+r)\alpha}z_i\right\|^2} z_i$ with $u_0 = x$ for $i = 1, 2, \cdots, n$ and $k = 1, 2$.

In the case $\alpha, \beta > 0$, if $\{|T^*|^{(1+s)\beta}y_1, |T^*|^{(1+s)\beta}y_2\}$ is linearly independent, then the equality in (6.3) holds for $x \in \mathscr{H}$ if and only if $|T|^{2(1+r)\alpha}x = (|T|^r A^{2p}|T|^r)^{\frac{(1+r)\alpha}{p+r}} x$ and $U|T|^{(1+r)\alpha}u_n = a(\left\||T^*|^{(1+s)\beta}y_2\right\| |T^*|^{(1+s)\beta}y_1 + e^{i\theta} \left\||T^*|^{(1+s)\beta}y_1\right\| |T^*|^{(1+s)\beta}y_2)$ for some scalar a, where $\theta = \arg\left\langle |T^*|^{(1+s)\beta}y_1, |T^*|^{(1+s)\beta}y_2\right\rangle$. Incidentally, if $\{|T^*|^{(1+s)\beta}y_1, |T^*|^{(1+s)\beta}y_2\}$ is linearly dependent, then the equality in (6.3) holds for $x \in \mathscr{H}$ if and only if $|T|^{2(1+r)\alpha}x = (|T|^r A^{2p}|T|^r)^{\frac{(1+r)\alpha}{p+r}} x$ and $U|T|^{(1+r)\alpha}u_n = a|T^*|^{(1+s)\beta}y_1$ for some scalar a.

We remark that the condition $(1+r)\alpha + (1+s)\beta \ge 1$ in above is unnecessary if T is either positive or invertible.

7. Heinz-Kato-Furuta inequality under the chaotic order

From the operator monotonicity of the logarithmic function, we introduced the chaotic order among positive invertible operators by $A \gg B$ if $\log A \geq \log B$ in [4], and obtained a characterization of the chaotic order in terms of Furuta's type operator inequality [6], [7] and [8]. Furthermore based on this, in [13, Theorem 4] we gave a chaotic order version of Theorem A by the Furuta inequality. We show a variant of Theorem 5.2 by chaotic order. For this, we use the following characterization of the chaotic order which is an extension of Ando's theorem [4], [6], [7], [8] and [25] for a polished proof.

Theorem C. *For positive invertible operators A and B, $A \gg B$ if and only if*

$$(B^r A^p B^r)^{\frac{1}{q}} \geq (B^r B^p B^r)^{\frac{1}{q}}$$

holds for $q \geq 1$, $p, r \geq 0$ with $2rq \geq p + 2r$.

Now in [11, Theorem 4] we showed the following theorem as a simultaneous extension of the Heinz-Kato-Furuta inequality and the Selberg inequality:

Theorem D. *Let T be an operator on $\mathscr{H}$. Then for each $x, y \in \mathscr{H}$*

$$\left|\left\langle T|T|^{\alpha+\beta-1}x, y\right\rangle\right|^2 + \sum_i \frac{|\langle |T|^{2\alpha}x, z_i\rangle|^2 \left\||T^*|^\beta y\right\|^2}{\sum_i |\langle |T|^{2\alpha} z_i, z_j\rangle|} \leq \left\langle |T|^{2\alpha}x, x\right\rangle \left\langle |T^*|^{2\beta}y, y\right\rangle \quad (7.1)$$

for all $\alpha, \beta \geq 0$ with $\alpha + \beta \geq 1$ and $z_i \notin \ker T$ such that $\left\langle T|T|^{\alpha+\beta-1}z_i, y\right\rangle = 0$ for $i = 1, 2, \ldots, n$.

Moreover Theorem D was extended in [11, Theorem 8] by applying the Furuta inequality. We now show the chaotic version of Corollary 3.2 by applying Theorem C:

Theorem 7.1. *Let T be an invertible operator on $\mathscr{H}$. If A and B are positive invertible operators on $\mathscr{H}$ such that $A^2 \gg T^*T$, then for each $x, y_1, y_2 \in \mathscr{H}$*

$$\begin{aligned} &\left|\left\langle T|T|^{r\alpha+s\beta-1}x, y_1\right\rangle \left\langle T|T|^{r\alpha+s\beta-1}x, y_2\right\rangle\right| \\ &\quad + \mathcal{B}\left(|T^*|^{s\beta}y_1, |T^*|^{s\beta}y_2\right) \sum_i \frac{|\langle Tx, z_i\rangle|^2}{\sum_j |\langle |T^*|^{2(1-r\alpha)}z_i, z_j\rangle|} \\ &\quad \leq \mathcal{B}\left(|T^*|^{s\beta}y_1, |T^*|^{s\beta}y_2\right) \left\langle (|T|^r A^{2p}|T|^r)^{\frac{r\alpha}{p+r}}x, x\right\rangle. \end{aligned} \quad (7.2)$$

for all $p, q \geq 0$, $r, s \geq 0$, $\alpha, \beta \in [0, 1]$ with $r\alpha + s\beta \geq 1 \geq r\alpha$ and $z_i \notin \ker T^*$ such that $\left\langle T|T|^{s\beta+1-r\alpha}z_i, y_k\right\rangle = 0$ for $i = 1, 2, \ldots, n$ and $k = 1, 2$.

Proof. By replacing α and β to $r\alpha$ and $s\beta$ respectively in Corollary 3.2, we have

$$\left|\left\langle T|T|^{r\alpha+s\beta-1}x, y_1\right\rangle \left\langle T|T|^{r\alpha+s\beta-1}x, y_2\right\rangle\right|$$

$$
\begin{aligned}
&+\mathcal{B}\left(|T^*|^{s\beta}y_1, |T^*|^{s\beta}y_2\right)\sum_i \frac{|\langle Tx, z_i\rangle|^2}{\sum_j |\langle |T^*|^{2(1-r\alpha)}z_i, z_j\rangle|} \\
&\leq \mathcal{B}\left(|T^*|^{s\beta}y_1, |T^*|^{s\beta}y_2\right)\left\langle |T|^{2r\alpha}x, x\right\rangle .
\end{aligned}
$$

Moreover we replace A, B, r and q to A^2, $|T|^2$, $\frac{r}{2}$ and $\frac{p+r}{r\alpha}$, respectively in Theorem C. Then we have

$$|T|^{2r\alpha} \leq (|T|^r A^{2p}|T|^r)^{\frac{r\alpha}{p+r}}.$$

Combining inequalities above, we obtain the desired inequality (7.2). □

By the same method as Theorem 7.1, we shall show the following theorem as an extension of Corollary 3.3 under the chaotic order:

Theorem 7.2. *Let T be an invertible operator on $\mathscr{H}$. If A and B are positive invertible operators on $\mathscr{H}$ such that $A^2 \gg T^*T$, then for each $x, y_1, y_2 \in \mathscr{H}$*

$$
\begin{aligned}
&\left|\left\langle T|T|^{r\alpha+s\beta-1}x, y_1\right\rangle \left\langle T|T|^{r\alpha+s\beta-1}x, y_2\right\rangle\right| \\
&+\mathcal{B}\left(|T^*|^{s\beta}y_1, |T^*|^{s\beta}y_2\right)\sum_i \frac{|\langle |T|^{2r\alpha}x, z_i\rangle|^2}{\sum_j |\langle |T|^{2r\alpha}z_i, z_j\rangle|} \qquad (7.3)\\
&\leq \mathcal{B}\left(|T^*|^{s\beta}y_1, |T^*|^{s\beta}y_2\right)\left\langle (|T|^r A^{2p}|T|^r)^{\frac{r\alpha}{p+r}}x, x\right\rangle
\end{aligned}
$$

for all $p, q \geq 0$, $r, s \geq 0$, $\alpha, \beta \in [0,1]$ with $r\alpha + s\beta \geq 1$ and $z_i \notin \ker T$ such that $\left\langle T|T|^{r\alpha+s\beta-1}z_i, y_k\right\rangle = 0$ for $i = 1, 2, \ldots, n$ and $k = 1, 2$.

Proof. By replacing α and β to $r\alpha$ and $s\beta$ respectively in Corollary 3.3, we have

$$
\begin{aligned}
&\left|\left\langle T|T|^{r\alpha+s\beta-1}x, y_1\right\rangle \left\langle T|T|^{r\alpha+s\beta-1}x, y_2\right\rangle\right| \\
&+\mathcal{B}\left(|T^*|^{s\beta}y_1, |T^*|^{s\beta}y_2\right)\sum_i \frac{|\langle |T|^{2r\alpha}x, z_i\rangle|^2}{\sum_j |\langle |T|^{2r\alpha}z_i, z_j\rangle|} \\
&\leq \mathcal{B}\left(|T^*|^{s\beta}y_1, |T^*|^{s\beta}y_2\right)\left\langle |T|^{2r\alpha}x, x\right\rangle .
\end{aligned}
$$

Hence we have the desired inequality from Theorem C. □

The following is a chaotic version of Theorem 5.1:

Theorem 7.3. *Let T be an invertible operator on $\mathscr{H}$. If A and B are positive invertible operators on $\mathscr{H}$ such that $A^2 \gg T^*T$ and $B^2 \gg TT^*$, then for each $x, y \in \mathscr{H}$*

$$
\begin{aligned}
&\left|\left\langle T|T|^{r\alpha+s\beta-1}x, y_1\right\rangle \left\langle T|T|^{r\alpha+s\beta-1}x, y_2\right\rangle\right| \\
&+\mathcal{B}\left(|T^*|^{s\beta}y_1, |T^*|^{s\beta}y_2\right)\sum_i \frac{|\langle |T|^{2r\alpha}u_{i-1}, z_i\rangle|^2}{\||T|^{r\alpha}z_i\|^2} \qquad (7.4)\\
&\leq \mathcal{B}\left(|T^*|^{s\beta}y_1, |T^*|^{s\beta}y_2\right)\left\langle (|T|^r A^{2p}|T|^r)^{\frac{r\alpha}{p+r}}x, x\right\rangle .
\end{aligned}
$$

for all $p, q \geq 0$, $r, s \geq 0$, $\alpha, \beta \in [0,1]$ with $r\alpha + s\beta \geq 1$ and $z_1, \cdots, z_n \notin \ker T$ such that $\left\langle T|T|^{r\alpha+s\beta-1} z_i, y_k \right\rangle = 0$, where $u_i = u_{i-1} - \frac{\left\langle |T|^{2r\alpha} u_{i-1}, z_i \right\rangle}{\| |T|^{r\alpha} z_i \|^2} z_i$ with $u_0 = x$ for $i = 1, 2, \ldots, n$ and $k = 1, 2$.

Proof. We replace α and β to $r\alpha$ and $s\beta$, respectively in Theorem 5.1. Then we have

$$\begin{aligned}
&\left| \left\langle T|T|^{r\alpha+s\beta-1} x, y_1 \right\rangle \left\langle T|T|^{r\alpha+s\beta-1} x, y_2 \right\rangle \right| + \mathcal{B}\left(|T^*|^{s\beta} y_1, |T^*|^{s\beta} y_2 \right) \sum_i \frac{\left| \left\langle |T|^{2r\alpha} u_{i-1}, z_i \right\rangle \right|^2}{\| |T|^{r\alpha} z_i \|^2} \\
&\quad \leq \mathcal{B}\left(|T^*|^{s\beta} y_1, |T^*|^{s\beta} y_2 \right) \left\langle |T|^{2r\alpha} x, x \right\rangle .
\end{aligned}$$

Hence we have the desired inequality from Theorem C. □

Next we interpolate between Theorems 6.1 and 7.1 by the use of Furuta's type operator inequality which interpolates the Furuta inequality and Theorem C.

Theorem 7.4. *Let T be an operator on $\mathscr{H}$. If A and B are positive operators on $\mathscr{H}$ such that $|T|^\delta \leq A^\delta$ and $|T^*|^\delta \leq B^\delta$ for some $\delta \in [0,1]$, then for each $x, y_1, y_2 \in \mathscr{H}$*

$$\begin{aligned}
&\left| \left\langle T|T|^{(\delta+r)\alpha+(\delta+s)\beta-1} x, y_1 \right\rangle \left\langle T|T|^{(\delta+r)\alpha+(\delta+s)\beta-1} x, y_2 \right\rangle \right| \\
&+ \mathcal{B}\left(|T^*|^{(\delta+s)\beta} y_1, |T^*|^{(\delta+s)\beta} y_2 \right) \sum_i \frac{\left| \left\langle Tx, z_i \right\rangle \right|^2}{\sum_j \left| \left\langle |T^*|^{2(1-(\delta+r)\alpha)} z_i, z_j \right\rangle \right|} \qquad (7.5) \\
&\leq \mathcal{B}\left(|T^*|^{(\delta+s)\beta} y_1, |T^*|^{(\delta+s)\beta} y_2 \right) \left\langle \left(|T|^r A^{2p} |T|^r \right)^{\frac{(\delta+r)\alpha}{p+r}} x, x \right\rangle .
\end{aligned}$$

for all $p \geq \delta$, $q \geq 1$, $r, s \geq 0$, $\alpha, \beta \in [0,1]$ with $(\delta+r)\alpha + (\delta+s)\beta \geq 1 \geq (\delta+r)\alpha$ and $z_i \notin \ker T^*$ such that $\left\langle T|T|^{(\delta+s)\beta+1-(\delta+r)\alpha} z_i, y_k \right\rangle = 0$ for $i = 1, 2, \ldots, n$ and $k = 1, 2$.

Proof. By replacing α (resp. β) to $(\delta + r)\alpha$ (resp. $(\delta + s)\beta$) in Corollary 3.2, we have

$$\begin{aligned}
&\left| \left\langle T|T|^{(\delta+r)\alpha+(\delta+s)\beta-1} x, y_1 \right\rangle \left\langle T|T|^{(\delta+r)\alpha+(\delta+s)\beta-1} x, y_2 \right\rangle \right| \\
&+ \mathcal{B}\left(|T^*|^{(\delta+s)\beta} y_1, |T^*|^{(\delta+s)\beta} y_2 \right) \sum_i \frac{\left| \left\langle Tx, z_i \right\rangle \right|^2}{\sum_j \left| \left\langle |T^*|^{2(1-(\delta+r)\alpha)} z_i, z_j \right\rangle \right|} \\
&\leq \mathcal{B}\left(|T^*|^{(\delta+s)\beta} y_1, |T^*|^{(\delta+s)\beta} y_2 \right) \left\langle |T|^{2(\delta+r)\alpha} x, x \right\rangle
\end{aligned}$$

for all $x \in \mathscr{H}$.

Moreover it is known in [7] that

$$|T|^{2(\delta+r)\alpha} \leq \left(|T|^r A^{2p} |T|^r \right)^{\frac{(\delta+r)\alpha}{p+r}} .$$

Combining above inequalities, we obtain the desired inequality (7.5). □

Now we have the following theorem interpolating between Theorems 6.2 and 7.2.

Theorem 7.5. *Let T be an operator on $\mathscr{H}$. If A and B are positive operators on $\mathscr{H}$ such that $|T|^\delta \le A^\delta$ and $|T^*|^\delta \le B^\delta$ for some $\delta \in [0,1]$, then for each $x, y_1, y_2 \in \mathscr{H}$*

$$\begin{aligned}
&\left|\left\langle T|T|^{(\delta+r)\alpha+(\delta+s)\beta-1}x, y_1\right\rangle \left\langle T|T|^{(\delta+r)\alpha+(\delta+s)\beta-1}x, y_2\right\rangle\right| \\
&\quad + \mathcal{B}\left(|T^*|^{(\delta+s)\beta}y_1, |T^*|^{(\delta+s)\beta}y_2\right) \sum_i \frac{\left|\left\langle |T|^{2(\delta+r)\alpha}x, z_i\right\rangle\right|^2}{\sum_j |\langle |T|^{2(\delta+r)\alpha}z_i, z_j\rangle|} \\
&\le \mathcal{B}\left(|T^*|^{(\delta+s)\beta}y_1, |T^*|^{(\delta+s)\beta}y_2\right) \left\langle \left(|T|^r A^{2p} |T|^r\right)^{\frac{(\delta+r)\alpha}{p+r}} x, x\right\rangle
\end{aligned} \tag{7.6}$$

for all $p \ge \delta$, $q \ge 1$, $r, s > 0$, $\alpha, \beta \in [0,1]$ with $(\delta+r)\alpha + (\delta+s)\beta \ge 1$ and $z_i \notin \ker T$ such that $\left\langle T|T|^{(\delta+r)\alpha+(\delta+s)\beta-1}z_i, y_k\right\rangle = 0$ for $i = 1, 2, \ldots, n$ and $k = 1, 2$.

In addition, we show the following theorem interpolating between Theorems 6.3 and 7.3:

Theorem 7.6. *Let T be an operator on $\mathscr{H}$. If A and B are positive operators on $\mathscr{H}$ such that $|T|^\delta \le A^\delta$ and $|T^*|^\delta \le B^\delta$ for some $\delta \in [0,1]$, then for each $x, y_1, y_2 \in \mathscr{H}$*

$$\begin{aligned}
&\left|\left\langle T|T|^{(\delta+r)\alpha+(\delta+s)\beta-1}x, y_1\right\rangle \left\langle T|T|^{(\delta+r)\alpha+(\delta+s)\beta-1}x, y_2\right\rangle\right| \\
&\quad + \mathcal{B}\left(|T^*|^{(\delta+s)\beta}y_1, |T^*|^{(\delta+s)\beta}y_2\right) \sum_i \frac{\left|\left\langle |T|^{2(\delta+r)\alpha}u_{i-1}, z_i\right\rangle\right|^2}{\left\||T|^{(\delta+r)\alpha}z_i\right\|^2} \\
&\le \mathcal{B}\left(|T^*|^{(\delta+s)\beta}y_1, |T^*|^{(\delta+s)\beta}y_2\right) \left\langle \left(|T|^r A^{2p} |T|^r\right)^{\frac{(\delta+r)\alpha}{p+r}} x, x\right\rangle
\end{aligned} \tag{7.7}$$

for all $p \ge \delta$, $q \ge 1$, $r, s \ge 0$, $\alpha, \beta \in [0,1]$ with $(\delta+r)\alpha+(\delta+s)\beta \ge 1$ and $z_1, \cdots, z_n \notin \ker T$ such that $\left\langle T|T|^{(\delta+r)\alpha+(\delta+s)\beta-1}z_i, y_k\right\rangle = 0$, where $u_i = u_{i-1} - \frac{\left\langle |T|^{2(\delta+r)\alpha}u_{i-1}, z_i\right\rangle}{\left\||T|^{(\delta+r)\alpha}z_i\right\|^2} z_i$ with $u_0 = x$ for $i = 1, 2, \ldots, n$ and $k = 1, 2$.

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