Nihonkai Math. J.
Vol.24(2013), 121–134

BI-UNIQUE RANGE SETS FOR MEROMORPHIC FUNCTIONS

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The Author dedicates the paper to his metarnal grandfather late Prof. Dhirananda Roy, who inspired the author a lot during his early research work.

Abstract. In this paper we introduce a new kind of pair of finite range sets in $\mathbb{C}$ for meromorphic functions corresponding to their uniqueness.

1. Introduction

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in [11]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a non-constant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying $S(r,h) = o\{T(r,h)\}$, as $r \longrightarrow \infty$ and $r \notin E$.

Let f and g be two non-constant meromorphic functions and S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and $E_f(S) = \bigcup_{a \in S}\{z : f(z) = a\}$, where each point is counted according to its multiplicity. Denote by $\overline{E}_f(S)$ the reduced form of $E_f(S)$. If $E_f(S) = E_g(S)$, we say that f and g share the set S CM. If $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM.

Every body will admit that new concept or definition always encourage researchers to contemplate that matter seriously. The theory of uniqueness of meromorphic function is no way an exception. The introduction of the novel idea of unique range set for meromophic function (URSM in brief) by Gross and Yang [10] (see also [16]) influenced many mathematicians to pursue their investigations meticulously to find finite URSM's {see [1], [5]-[8], [15], [16]}. The advent of the notion of weighted

2010 *Mathematics Subject Classification.* Primary 30D35.
Key words and phrases. Meromorphic functions, uniqueness, weighted sharing, shared set.
The author is thankful to DST-PURSE programme for financial assistance.

sharing of values and sets by Lahiri [13, 14] further add essence in this context. Bellow we are recalling the same.

Definition 1.1. [13, 14] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2. [13] Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a nonnegative integer or ∞. We denote by $E_f(S, k)$ the set $\bigcup_{a \in S} E_k(a; f)$.

Clearly $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$.

Recently the existing definitions on URSM's have been streamlined in [4] with the help of weighted sharing. Till date the URSM with 11 elements is the smallest available URSM obtained by G. Frank and M. Reinders [5].

In continuation with the famous "Gross question" [9], in 2003, the following question was asked by Lin and Yi in [17].

Question A. *Can one find two finite sets S_j $(j = 1, 2)$ such that any two non-constant meromorphic functions f and g satisfying $E_f(S_j, \infty) = E_g(S_j, \infty)$ for $j = 1, 2$ must be identical ?*

In connection with *Question A*, during the last two decades a famous problem in value distribution theory has been to give explicitly a set S with n elements and make n as small as possible such that any two meromorphic functions f and g that share the value ∞ and the set S must be equal. But the possible answer corresponding to two finite sets in $\mathbb{C}$ has not been explored exhaustively. So it would be interesting to investigate the existence of a pair of finite range sets in $\mathbb{C}$ shared by two meromorphic functions which leads them to-wards their uniqueness. In commensurate with the new type of definition of URSMk as ushered in [4], it will be reasonable to introduce the following definition.

A pair of finite sets S_1 and S_2 in $\mathbb{C}$ is called bi unique range sets for meromorphic (entire) functions with weights m, k if for any two non-constant meromorphic (entire) functions f and g, $E_f(S_1, m) = E_g(S_1, m)$, $E_f(S_2, k) = E_g(S_2, k)$ implies $f \equiv g$. We write S_i's $i = 1, 2$ as BURSMm, k (BURSEm, k) in short. As usual if both $m = k = \infty$, we say S_i's $i = 1, 2$ as BURSM (BURSE).

So far we know H.X.Yi [20] is the first to draw the affirmative answer to the above question in the direction of BURSM prior to its announcement. Yi proved the following theorem.

Theorem A. [20] Let $S_1 = \{a+b, a+b\omega, \ldots, a+b\omega^{n-1}\}$, $S_2 = \{c_1, c_2\}$ where $\omega = e^{\frac{2\pi i}{n}}$ and $b \neq 0$, $c_1 \neq a$, $c_2 \neq a$, $(c_1 - a)^n \neq (c_2 - a)^n$, $(c_k - a)^n (c_j - a)^n \neq b^{2n}$ $(k, j = 1, 2)$ are constants. If $n \geq 9$ then Then S_i's $i = 1$, 2 are *BURSM.*

Recently Yi and Li [19], improved *Theorem A* to a large extent by significantly reducing the cardinality of one of the range sets. In [19] the following result have been proved.

Theorem B. [19] Let $S_1 = \{0, 1\}$, $S_2 = \Big\{z : \frac{(n-1)(n-2)}{2} z^n - n(n-2) z^{n-1} + \frac{n(n-1)}{2} z^{n-2} +1 = 0\}$, where $n(\geq 5)$ is an integer. Then S_i's $i = 1$, 2 are *BURSM.*

The purpose of the paper is to radically improve the result as stated in *Theorem B*. In fact, we shall relax the nature of sharing of both the range sets to a large extend.

The following theorems are the main results of the paper.

Theorem 1.1. *Let* $S_1 = \{0, 1\}$, $S_2 = \Big\{z : \frac{(n-1)(n-2)}{2} z^n - n(n-2) z^{n-1} + \frac{n(n-1)}{2} z^{n-2} -c = 0\}$, *where* $n(\geq 5)$ *is an integer and* $c \neq 0,\ 1,\ \frac{1}{2}$ *is a complex number such that* $c^2 - c + 1 \neq 0$. *Then* S_i*'s* $i = 1$, 2 *are* BURSM1, 3.

Theorem 1.2. *Let* S_i, $i = 1$, 2 *be given as in* Theorem 1.1. *Then* S_i*'s* $i = 1$, 2 *are* BURSM3, 2.

It is assumed that the readers are familiar with the standard definitions and notations of the value distribution theory as those are available in [11]. We are still going to explain some notations as these are used in the paper.

Definition 1.3. [12] For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f \mid= 1)$ the counting function of simple a points of f. For a positive integer m we denote by $N(r, a; f \mid\leq m) (N(r, a; f \mid\geq m))$ the counting function of those a points of f whose multiplicities are not greater(less) than m, where each a point is counted according to its multiplicity. We denote by $N(r, a; f \mid< m)$, $(N(r, a; f \mid> m))$ the counting function of those a-points of f whose multiplicities are less (greater) than m, where each point is counted according to its multiplicity. We denote by $\overline{N}(r, a; f \mid\leq m)$, $\overline{N}(r, a; f \mid\geq m)$, $\overline{N}(r, a; f \mid< m)$ *and* $\overline{N}(r, a; f \mid> m)$ the reduced forms of $N(r, a; f \mid\leq m)$, $N(r, a; f \mid\geq m)$, $N(r, a; f \mid< m)$ *and* $N(r, a; f \mid> m)$ respectively.

Definition 1.4. [2] Let f and g be two non-constant meromorphic functions such that f and g share $(1, 0)$. Let z_0 be a 1-point of f with multiplicity p, a 1-point of g with multiplicity q. We denote by $\overline{N}_L(r, 1; f)$ the reduced counting function of those 1-points of f and g where $p > q$, by $N_E^{1)}(r, 1; f)$ the counting function of those 1-points of f and g where $p = q = 1$, by $\overline{N}_E^{(2}(r, 1; f)$ the reduced counting function of those 1-points of f and g where $p = q \geq 2$. In the same way we can define $\overline{N}_L(r,1;g)$, $N_E^{1)}(r,1;g)$, $\overline{N}_E^{(2}(r,1;g)$. In a similar manner we can define $\overline{N}_L(r,a;f)$ and $\overline{N}_L(r,a;g)$ for $a \in \mathbb{C} \cup \{\infty\}$. When f and g share $(1,m)$, $m \geq 1$ then $N_E^{1)}(r,1;f) = N(r,1;f \mid= 1)$.

Definition 1.5. [13, 14] Let f, g share $(a,0)$. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Clearly $\overline{N}_*(r,a;f,g) = \overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g) = \overline{N}_L(r,a;f) + \overline{N}_L(r,a;g)$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let f and g be two non-constant meromorphic function and for an integer $n \geq 3$

$$F = \frac{P(f)}{c} = \frac{\frac{(n-1)(n-2)}{2}f^n - n(n-2)f^{n-1} + \frac{n(n-1)}{2}f^{n-2}}{c}, \tag{1}$$

$$G = \frac{P(g)}{c} = \frac{\frac{(n-1)(n-2)}{2}g^n - n(n-2)g^{n-1} + \frac{n(n-1)}{2}g^{n-2}}{c}, \tag{2}$$

where $P(z) = z^{n-2}Q(z)$ and $Q(z) = \{\frac{(n-1)(n-2)}{2}z^2 - n(n-2)z + \frac{n(n-1)}{2}\}$. Henceforth we shall denote by H and Φ the following two functions

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right).$$

$$\Phi = \frac{F'}{F-1} - \frac{G'}{G-1}.$$

Lemma 2.1. [22] *If F, G be two non-constant meromorphic functions such that they share (1,0) and $H \not\equiv 0$, then*

$$N_E^{1)}(r,1;F \mid= 1) = N_E^{1)}(r,1;G \mid= 1) \leq N(r,H) + S(r,F) + S(r,G).$$

Lemma 2.2. *Let F, G be given by (1) and (2). Also let $E_f(S_1,p) = E_g(S_1,p)$ and $E_f(S_2,0) = E_g(S_2,0)$, where S_i's $i = 1, 2$ be given as in* Theorem 1.1. *If $H \not\equiv 0$, then*

$$\begin{aligned} N(r,H) \leq\ & \overline{N}(r,0;f \mid\geq p+1) + \overline{N}(r,1;f \mid\geq p+1) + \overline{N}_*(r,1;F,G) \\ & + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + \overline{N}_0(r,0;f') + \overline{N}_0(r,0;g'), \end{aligned}$$

where $\overline{N}_0(r,0;f')$ is the reduced counting function of those zeros of f' which are not the zeros of $f(f-1)$ and $F-1$, $\overline{N}_0(r,0;g')$ is similarly defined.

Proof. Clearly $E_f(S_2,0)=E_g(S_2,0)$ implies $F-1=\frac{P(f)-c}{c}$ and $G-1=\frac{P(g)-c}{c}$ and so F and G share $(1,0)$. First we note that

$$F^{'}=\frac{n(n-1)(n-2)f^{n-3}(f-1)^2f^{'}}{2c},\quad G^{'}=\frac{n(n-1)(n-2)g^{n-3}(g-1)^2g^{'}}{2c},$$

$$F^{''}=\frac{n(n-1)(n-2)f^{n-4}(f-1)[(n-3)(f-1)f^{'2}+2ff^{'2}+f(f-1)f^{''}]}{2c}$$

and

$$G^{''}=\frac{n(n-1)(n-2)g^{n-4}(g-1)[(n-3)(g-1)g^{'2}+2gg^{'2}+g(g-1)g^{''}]}{2c}.$$

In view of the above calculation it is easy to see that

$$H=\frac{2f^{'}}{f-1}-\frac{2g^{'}}{g-1}+\frac{(n-3)f^{'}}{f}-\frac{(n-3)g^{'}}{g}+\frac{f^{''}}{f^{'}}-\frac{g^{''}}{g^{'}}-\left(\frac{2F^{'}}{F-1}-\frac{2G^{'}}{G-1}\right).$$

Since $E_f(S_1,p)=E_g(S_1,p)$ we observe that if z_0 is a 0-point of f (g) then either $g(z_0)=1$ $(f(z_0)=1)$ or $g(z_0)=0$ $(f(z_0)=0)$ and $\overline{N}(r,0;f\mid\geq p+1)+\overline{N}(r,1;f\mid\geq p+1)=\overline{N}(r,0;g\mid\geq p+1)+\overline{N}(r,1;g\mid\geq p+1)$. It can also easily be verified that possible poles of H occur at (i) zeros (1-points) of f and g with multiplicity greater than p, (ii) poles of f and g, (iii) those 1-points of F and G whose multiplicities are distinct from the multiplicities of the corresponding 1-points of G and F respectively, (iv) zeros of $f^{'}$ which are not the zeros of $f(f-1)$ and $F-1$, (v) zeros of which are not the zeros of $g(g-1)$ and $G-1$.

Since H has only simple poles, clearly the lemma follows from above explanations. □

Lemma 2.3. [18] *Let f be a non-constant meromorphic function and $P(f)=a_0+a_1f+a_2f^2+\ldots+a_nf^n$, where $a_0,a_1,a_2\ldots,a_n$ are constants and $a_n\neq 0$. Then $T(r,P(f))=nT(r,f)+O(1)$.*

Lemma 2.4. [3] *Let f and g be two meromorphic functions sharing $(1,m)$, where $1\leq m<\infty$. Then*

$$\overline{N}(r,1;f)+\overline{N}(r,1;g)-N(r,1;f\mid=1)+\left(m-\frac{1}{2}\right)\overline{N}_*(r,1;f,g)$$
$$\leq\frac{1}{2}\left[N(r,1;f)+N(r,1;g)\right]$$

Lemma 2.5. *Let f, g be two non-constant meromorphic functions such that E_f $(\{0,1\},$ $0)=E_g(\{0,1\},0)$ and suppose γ and δ be the roots of the equation $Q(z)=\frac{(n-1)(n-2)}{2}z^2$ $-\,n(n-2)z+\frac{n(n-1)}{2}=0$. Then*

$$(n-1)^2(n-2)^2f^{n-2}(f-\gamma)(f-\delta)g^{n-2}(g-\gamma)(g-\delta)\not\equiv 4c^2,$$

where n (≥ 3) be an integer.

Proof. If possible, let us suppose

$$(n-1)^2(n-2)^2 f^{n-2}(f-\gamma)(f-\delta)g^{n-2}(g-\gamma)(g-\delta) \equiv 4c^2. \tag{3}$$

Let z_0 be a zero of f (g). Then z_0 must be either a 0-point or an 1 point of g (f), which is impossible from (3). It follows that f (g) has no zero.

Next let z_0 be a zero of $f-\gamma$ $(f-\delta)$ with multiplicity p. Then z_0 is a pole of g with multiplicity q such that $p = (n-2)q + 2q = nq \geq n$.

Since the poles of f are the zeros of $g-\gamma$ and $g-\delta$, we get

$$\begin{aligned}\overline{N}(r,\infty;f) &\leq \overline{N}(r,\gamma;g) + \overline{N}(r,\delta;g)\\ &\leq \frac{1}{n}\, N(r,\gamma;g) + \frac{1}{n}\, N(r,\delta;g)\\ &\leq \frac{2}{n} T(r,g).\end{aligned}$$

By the second fundamental theorem we get

$$\begin{aligned}2T(r,f) &\leq \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + \overline{N}(r,\gamma;f) + \overline{N}(r,\delta;f) + S(r,f)\\ &\leq \frac{1}{n}\, N(r,\gamma;f) + \frac{1}{n}\, N(r,\delta;f) + \frac{2}{n} T(r,g) + S(r,f)\\ &\leq \frac{2}{n} T(r,f) + \frac{2}{n} T(r,g) + S(r,f).\end{aligned}$$

i.e.,

$$(2-\frac{2}{n})\; T(r,f) \leq \frac{2}{n}\; T(r,g) + S(r,f). \tag{4}$$

Similarly

$$(2-\frac{2}{n})\; T(r,g) \leq \frac{2}{n}\; T(r,f) + S(r,g) \tag{5}$$

Adding (4) and (5) we get

$$(2-\frac{4}{n})\; \{T(r,f) + T(r,g)\} \leq S(r,f) + S(r,g),$$

a contradiction for $n \geq 3$. This proves the lemma. □

Lemma 2.6. [7] *Let* $R(z) = (n-1)^2(z^n-1)(z^{n-2}-1) - n(n-2)(z^{n-1}-1)^2$, *then* $R(z) = (z-1)^4(z-\beta_1)(z-\beta_2)\ldots(z-\beta_{2n-6})$, *where* $\beta_j \in \mathbb{C} - \{0,1\}(j = 1,2,\ldots,2n-6)$ *are distinct.*

Lemma 2.7. *Let* f, g *be two non-constant meromorphic functions such that* $E_f(\{0,1\}, 0) = E_g(\{0,1\},0)$ *and suppose* n (≥ 4) *be an integer. If*

$$\begin{aligned}&\frac{(n-1)(n-2)}{2} f^n - n(n-2)f^{n-1} + \frac{n(n-1)}{2} f^{n-2}\\ \equiv\; &\frac{(n-1)(n-2)}{2} g^n - n(n-2)g^{n-1} + \frac{n(n-1)}{2} g^{n-2},\end{aligned}$$

then $f \equiv g$.

Proof. From the given condition we can write

$$f^{n-2}(f-\gamma)(f-\delta) \equiv g^{n-2}(g-\gamma)(g-\delta). \tag{6}$$

(6) clearly implies f and g share (∞, ∞). Since $E_f(\{0,1\},0) = E_g(\{0,1\},0)$ it follows that if z_0 is a zero of f (g) then it can not be an 1-point of g (f) as none of γ and δ are zero. So f and g share $(0,\infty)$. Suppose $h = \frac{f}{g}$. Clearly h has no zero and pole. Substituting $f = hg$ in (6) we get

$$\frac{(n-1)(n-2)}{2}(h^n-1)g^2 - n(n-2)(h^{n-1}-1)g + \frac{n(n-1)}{2}(h^{n-2}-1) \equiv 0. \tag{7}$$

Suppose h is not a constant. Then by a simple calculation we have from (7)

$$\left\{(n-1)(n-2)(h^n-1)g - n(n-2)(h^{n-1}-1)\right\}^2 \equiv -n(n-2)R(h), \tag{8}$$

where $R(z)$ is given as in *Lemma 2.6*. So using *Lemma 2.6* we have

$$\begin{aligned} & \left\{(n-1)(n-2)(h^n-1)g - n(n-2)(h^{n-1}-1)\right\}^2 \\ \equiv\ & -n(n-2)(h-1)^4(h-\beta_1)(h-\beta_2)\ldots(h-\beta_{2n-6}), \end{aligned} \tag{9}$$

where $\beta_j \in \mathbb{C} - \{0,1\} (j = 1,2,\ldots,2n-6)$ are distinct. From (8) we see that the zeros of $h - \beta_j$ $(j = 1,2,\ldots,2n-6)$ have multiplicity of order at least 2. So by the second fundamental theorem we get

$$\begin{aligned} (2n-6)T(r,h) &\leq \sum_{j=1}^{2n-6} \overline{N}\,(r,\beta_j;h) + S(r,h) \\ &\leq \frac{1}{2}\sum_{j=1}^{2n-6} N\,(r,\beta_j;h) + S(r,h) \\ &\leq (n-3)T(r,h) + S(r,h), \end{aligned}$$

which is a contradiction for $n \geq 4$. So h is a constant. From (7) we have $h^n - 1 = 0$ and $h^{n-1} - 1 = 0$. It follows that $h \equiv 1$ and so $f \equiv g$. $\square$

Lemma 2.8. *Let S_i, $i = 1, 2$ be defined as in* Theorem 1.1 *and F, G be given by (1) and (2), where $n(\geq 3)$ be an integer. If for two non-constant meromorphic functions f and g $E_f(S_1,p) = E_g(S_1,p)$, $E_f(S_2,m) = E_g(S_2,m)$ and $\Phi \not\equiv 0$, then*

$$\begin{aligned} & \min\{(n-2)p + (n-3),\ 3p+2\}\left\{\overline{N}\,(r,0;f \mid\geq p+1) + \overline{N}\,(r,1;f \mid\geq p+1)\right\} \\ \leq\ & \overline{N}_*(r,1;F,G) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + S(r,f) + S(r,g). \end{aligned}$$

Proof. By the given condition clearly F and G share $(1,m)$. Also we see that

$$\Phi = \frac{n(n-1)(n-2)f^{n-3}(f-1)^2 f'}{2c(F-1)} - \frac{n(n-1)(n-2)g^{n-3}(g-1)^2 g'}{2c(G-1)}.$$

Let z_0 be a zero or a 1-point of f with multiplicity r. Since $E_f(S_1,p) = E_g(S_1,p)$ then that would be a zero of Φ of multiplicity $min\ \{(n-3)r + r - 1,\ 2r + r - 1\}$ i.e., of multiplicity $min\ \{(n-2)r - 1,\ 3r - 1\}$ if $r \leq p$ and a zero of multiplicity at least $min\{(n-3)(p+1) + p,\ 2(p+1) + p\}$ i.e., a zero of multiplicity at least $min\{(n-2)p + (n-3),\ 3p+2\}$ if $r > p$. So by a simple calculation we can write

$$\begin{aligned}
& \min\{(n-2)p + (n-3),\ 3p+2\} \left\{\overline{N}(r,0;f \mid\geq p+1) + \overline{N}(r,1;f \mid\geq p+1)\right\} \\
\leq\ & N(r,0;\Phi) \\
\leq\ & T(r,\Phi) \\
\leq\ & N(r,\infty;\Phi) + S(r,F) + S(r,G) \\
\leq\ & \overline{N}_*(r,1;F,G) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + S(r,f) + S(r,g).
\end{aligned}$$

□

Lemma 2.9. *Let S_i, $i = 1,\ 2$ be defined as in* Theorem 1.1 *and F, G be given by (1) and (2). If for two non-constant meromorphic functions f and g, $E_f(S_1,p) = E_g(S_1,p)$, $E_f(S_2,m) = E_g(S_2,m)$, where $0 \leq p < \infty$, $2 \leq m < \infty$ and $H \not\equiv 0$, then*

$$\begin{aligned}
& (n+1)\left\{T(r,f) + T(r,g)\right\} \\
\leq\ & 2\left\{\overline{N}(r,0;f) + \overline{N}(r,1;f)\right\} + \overline{N}(r,0;f \mid\geq p+1) + \overline{N}(r,1;f \mid\geq p+1) \\
& + 2\overline{N}(r,\infty;f) + 2\overline{N}(r,\infty;g) + \frac{1}{2}\left[N(r,1;F) + N(r,1;G)\right] \\
& - \left(m - \frac{3}{2}\right)\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g).
\end{aligned}$$

Proof. By the second fundamental theorem we get

$$\begin{aligned}
& (n+1)\{T(r,f) + T(r,g)\} \\
\leq\ & \overline{N}(r,1;F) + \overline{N}(r,0;f) + \overline{N}(r,1;f) + \overline{N}(r,\infty;f) + \overline{N}(r,1;G) + \overline{N}(r,0;g) \\
& + \overline{N}(r,1;g) + \overline{N}(r,\infty;g) - N_0(r,0;f') - N_0(r,0;g') + S(r,f) + S(r,g).
\end{aligned} \tag{10}$$

Using *Lemmas 2.1*, *2.2*, *2.3* and *2.4* we see that

$$\begin{aligned}
& \overline{N}(r,1;F) + \overline{N}(r,1;G) \\
\leq\ & \frac{1}{2}\left[N(r,1;F) + N(r,1;G)\right] + N(r,1;F \mid= 1) - \left(m - \frac{1}{2}\right)\overline{N}_*(r,1;F,G) \\
\leq\ & \frac{1}{2}\left[N(r,1;F) + N(r,1;G)\right] + \overline{N}(r,0;f \mid\geq p+1) + \overline{N}(r,1;f \mid\geq p+1) \\
& + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) - \left(m - \frac{3}{2}\right)\overline{N}_*(r,1;F,G) + \overline{N}_0(r,0;f') \\
& + \overline{N}_0(r,0;g') + S(r,f) + S(r,g).
\end{aligned} \tag{11}$$

Using (11) in (10) and noting that $\overline{N}(r,0;f)+\overline{N}(r,1;f)=\overline{N}(r,0;g)+\overline{N}(r,1;g)$ the lemma follows. □

Lemma 2.10. [22] *If $H\equiv 0$, then F, G share $(1,\infty)$.*

Lemma 2.11. *Let F, G be given by (1) and (2) and they share $(1,m)$. Also let $\alpha_1,\alpha_2,\ldots,\alpha_n$ be the distinct elements of the set $\left\{z:\frac{(n-1)(n-2)}{2}z^n-n(n-2)z^{n-1}+\frac{n(n-1)}{2}z^{n-2}-c=0\right\}$, where $c\neq 0,\ 1,\ \frac{1}{2}$ is a complex number such that $c^2-c+1\neq 0$ and $n\ (\geq 3)$ is an integer. Then*

$$\overline{N}_L(r,1;F)\leq\frac{1}{m+1}\left[\overline{N}(r,0;f)+\overline{N}(r,\infty;f)-N_{\otimes}(r,0;f')\right]+S(r,f),$$

where $N_{\otimes}(r,0;f')=N(r,0;f'\mid f\neq 0,\alpha_1,\alpha_2\ldots\alpha_n)$.

Proof. The proof can be carried out in the line of proof of *Lemma 2.14* [2]. So we omit the detail. □

3. Proofs of the theorems

Proof of Theorem 1.1. Let F, G be given by (1) and (2). Then F and G share $(1,3)$. We consider the following cases.

Case 1. Suppose that $\Phi\not\equiv 0$.

Subcase 1.1. Let $H\not\equiv 0$. Then using *Lemma 2.3* and *Lemma 2.8* with $p=1$ and $m=3$ and again using *Lemma 2.8* for $p=0$ we obtain

$$\begin{aligned}
&(n+1)\left\{T(r,f)+T(r,g)\right\} &(1)\\
\leq\ & 2\left\{\overline{N}(r,0;f)+\overline{N}(r,1;f)\right\}+\frac{1}{5}\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}_*(r,1;F,G)\}\\
&+2\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+\frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]-\frac{3}{2}\overline{N}_*(r,1;F,G)\\
&+S(r,f)+S(r,g)\\
\leq\ & \frac{16}{5}\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+\frac{n}{2}\left[T(r,f)+T(r,g)\right]+S(r,f)+S(r,g)\\
\leq\ & \left\{\frac{n}{2}+\frac{16}{5}\right\}\left[T(r,f)+T(r,g)\right]+S(r,f)+S(r,g).
\end{aligned}$$

(1) gives a contradiction for $n\geq 5$.

Subcase 1.2. $H\equiv 0$. Then

$$F\equiv\frac{AG+B}{CG+D},\tag{2}$$

where A, B, C, D are constants such that $AD-BC\neq 0$. Also $T(r,F)=T(r,G)+O(1)$. i.e.,

$$nT(r,f)=nT(r,g)+O(1).\tag{3}$$

In view of *lemma 2.10* it follows that F and G share $(1,\infty)$. Since $P(1)=1$, by a simple computation it can be easily seen that 1 is a zero with multiplicity 3 of $F-\frac{1}{c}=\frac{P(f)-1}{c}$ and hence $F-\frac{1}{c}=(f-1)^3\ Q_{n-3}(f)$, where $Q_{n-3}(f)$ is a polynomial in f of degree $n-3$ and thus $\overline{N}\left(r,\frac{1}{c};F\right)\leq\overline{N}(r,1;f)+\overline{N}\left(r,0;Q_{n-3}(f)\right)\leq\overline{N}(r,1;f)+(n-3)T(r,f)+S(r,f)$.

We now consider the following cases.

Subcase 1.2.1. Let $AC\neq 0$. From (2) we get

$$\overline{N}(r,\infty;G)=\overline{N}\left(r,\frac{A}{C};F\right). \tag{4}$$

Since F and G share $(1,\infty)$, it follows that $\frac{A}{C}\neq 1$. Suppose $\frac{A}{C}\neq\frac{1}{c}$. So in view of *Lemma 2.3* and (3), by the second fundamental theorem we get

$$\begin{aligned}(n+1)T(r,f) &\leq \overline{N}(r,0;f)+\overline{N}(r,1;f)+\overline{N}(r,\infty;f)+\overline{N}\left(r,\frac{A}{C};F\right)+S(r,f)\\ &= 2T(r,f)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+S(r,f)\\ &\leq 4T(r,f)+S(r,f),\end{aligned}$$

which gives a contradiction for $n\geq 5$.

Next suppose $\frac{A}{C}=\frac{1}{c}$. In view of *Lemmas 2.3, 2.8* with $p=0$ and (3), by the second fundamental theorem we get

$$\begin{aligned}&(n-1)T(r,f)\\ \leq\ & \overline{N}(r,0;f)+\overline{N}(r,1;f)+\overline{N}(r,\infty;f)+\overline{N}\left(r,\frac{1}{c};F\right)+S(r,f)\\ =\ & \overline{N}(r,0;f)+\overline{N}(r,1;f)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+S(r,f)\\ \leq\ & \frac{3}{2}\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+S(r,f)\\ \leq\ & 3T(r,f)+S(r,f),\end{aligned}$$

which is a contradiction for $n\geq 5$.

Subcase 1.2.2. Let $A\neq 0$ and $C=0$. Then $F=\alpha_0 G+\beta_0$, where $\alpha_0=\frac{A}{D}$ and $\beta_0=\frac{B}{D}$.

We note that 1 can not be an exceptional value Picard (e.v.P.) of F (G). For, if it happens, then f (g) omits $n\geq 5$ values which is a contradiction.

So F and G have some 1-points. Then $\alpha_0+\beta_0=1$ and hence

$$F\equiv\alpha_0 G+1-\alpha_0. \tag{5}$$

Suppose $\alpha_0 \neq 1$. If $1-\alpha_0 \neq \frac{1}{c}$ then using *Lemma 2.8*, (3) and the second fundamental theorem we get

$$\begin{aligned}
& 2T(r,F) \\
\leq\ & \overline{N}(r,0;F) + \overline{N}(r,1-\alpha_0;F) + \overline{N}\left(r,\frac{1}{c};F\right) + \overline{N}(r,\infty;F) + S(r,F) \\
\leq\ & \overline{N}(r,0;f) + 2T(r,f) + \overline{N}(r,0;G) + \overline{N}(r,1;f) + (n-3)T(r,f) + \overline{N}(r,\infty;f) \\
& + S(r,f) \\
\leq\ & (n-1)T(r,f) + 3T(r,g) + \frac{1}{2}\{\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g)\} + \overline{N}(r,\infty;f) + S(r,f) \\
\leq\ & (n+4)T(r,f) + S(r,f),
\end{aligned}$$

which implies a contradiction in view of *Lemma 2.3* and $n \geq 5$. If $1-\alpha_0 = \frac{1}{c}$, then we have from (5)

$$cF \equiv (c-1)G + 1.$$

Noting that $c \neq \frac{1}{2}$ and $\overline{N}(r,0;f)+\overline{N}(r,1;f) = \overline{N}(r,0;g)+\overline{N}(r,1;g)$, using *Lemma 2.8* (3) and (5)we can obtain by the second fundamental theorem

$$\begin{aligned}
& 2T(r,G) \\
\leq\ & \overline{N}(r,0;G) + \overline{N}\left(r,\frac{1}{1-c};G\right) + \overline{N}(r,\frac{1}{c};G) + \overline{N}(r,\infty;G) + S(r,G) \\
\leq\ & \overline{N}(r,0;g) + 2T(r,g) + \overline{N}(r,0;f) + 2T(r,f) + \overline{N}(r,1;g) + (n-3)T(r,g) \\
& + \overline{N}(r,\infty;g) + S(r,g) \\
\leq\ & 3T(r,f) + nT(r,g) + \frac{1}{2}\{\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g)\} + S(r,g) \\
\leq\ & (n+4)T(r,g) + S(r,g),
\end{aligned}$$

which implies a contradiction in view of *Lemma 2.3* and $n \geq 5$. So $\alpha_0 = 1$ and hence $F \equiv G$. So by *Lemma 2.7* we get $f \equiv g$.

Subcase 1.2.3. Let $A = 0$ and $C \neq 0$. Then $F \equiv \frac{1}{\gamma_0 G + \delta_0}$, where $\gamma_0 = \frac{C}{B}$ and $\delta_0 = \frac{D}{B}$.

Clearly 1 can not be an e.v.P. of F and so of G.

Since F and G have some 1-points we have $\gamma_0 + \delta_0 = 1$ and so

$$F \equiv \frac{1}{\gamma_0 G + 1 - \gamma_0}. \tag{6}$$

Suppose $\gamma_0 \neq 1$. If $\gamma_0 \neq 1-c$, then noting that $\overline{N}(r,0;G) = \overline{N}(r,\frac{1}{1-\gamma_0};F) \neq \overline{N}\left(r,\frac{1}{c};F\right)$, by the second fundamental using *Lemmas 2.3* and *2.8* we can again deduce a contradiction as above when $n \geq 5$.

If $\gamma_{0} = 1 - c$ from (6) we have

$$F \equiv \frac{1}{(1-c)G + c}.$$

We know from the given condition that $\frac{1}{c} \neq \frac{c}{c-1}$. Now in the same way as above using (3), *Lemmas 2.3* and *2.8*, second fundamental theorem yields

$$\begin{aligned}
& 2T(r,G) \\
\leq\ & \overline{N}(r,0;G) + \overline{N}(r,\frac{1}{c};G) + \overline{N}(r,\frac{c}{c-1};G) + \overline{N}(r,\infty;G) + S(r,G) \\
\leq\ & \overline{N}(r,0;g) + \overline{N}(r,1;g) + 2T(r,g) + (n-3)T(r,g) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) \\
& +S(r,g) \\
\leq\ & (n-1)T(r,g) + \frac{3}{2}\{\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g)\} + S(r,g),
\end{aligned}$$

which implies a contradiction for $n \geq 5$. So we must have $\gamma_{0} = 1$ then $FG \equiv 1$, which is impossible by *Lemma 2.5*.

Case 2. Suppose that $\Phi \equiv 0$. On integration we get $(F-1) \equiv A(G-1)$ for some non zero constant A. So in view of *Lemma 2.3* we have $T(r,f) = T(r,g) + O(1)$. Since by the given condition of the theorem $E_f(S_1,0) = E_g(S_1,0)$ we consider the following cases.

Subcase 2.1. Let us first assume f and g share $(0,0)$ and $(1,0)$. If none of 0 and 1 is an e.v.P. of f and g, then we have $A = 1$. Similarly if one of 0 or 1 is an e.v.P. of f and g, then we get $A = 1$ and so in both cases we have $F \equiv G$, which in view of *Lemma 2.7* implies $f \equiv g$. If both 0 and 1 are e.v.P. of f as well as g then noting that here $F \equiv AG + (1-A)$ which is similar to (5), we can handle the situation as done in *Subcase 1.2.2.*. So we omit the detail.

Subcase 2.2. Next suppose that there is at least one point z_0 such that $f(z_0) = 0$ and $g(z_0) = 1$. Note that from $(F-1) \equiv A(G-1)$ we get $P(f) - c(1-A) \equiv AP(g)$. If $A \neq 1$, then $c(1-A) \neq 0$. If $c(1-A) = 1$, then $A = \frac{c-1}{c}$. So $F - \frac{1}{c} \equiv \frac{c-1}{c^2}G$. At the point z_0, we have $F(z_0) = 0$ and $G(z_0) = 1$. Putting this values we obtain $\frac{-1}{c} = \frac{c-1}{c^2}$ which implies $c = \frac{1}{2}$, a contradiction. So $c(1-A) \neq 0, 1$. Hence $P(z) - c(1-A)$ has simple zeros and consequently we have $(f-\omega_1)(f-\omega_2)\ldots(f-\omega_n) = A\frac{(n-1)(n-2)}{2}g^{n-2}(g-\gamma)(g-\delta)$, where ω_i be the distinct zeros of $P(f) - c(1-A)$. So by the second fundamental theorem we get

$$\begin{aligned}
& nT(r,f) \\
\leq\ & \overline{N}(r,0;f) + \overline{N}(r,1;f) + \sum_{j=1}^{n} \overline{N}\,(r,w_j;f) + S(r,f) \\
\leq\ & \overline{N}(r,0;g) + \overline{N}(r,1;g) + \overline{N}(r,\gamma;g) + \overline{N}(r,\delta;g) + S(r,f) \\
\leq\ & 4T(r,f) + S(r,f),
\end{aligned}$$

which is a contradiction for $n \geq 5$. This completes the proof of the theorem. $\square$

Proof of Theorem 1.2. Let F, G be given by (1) and (2). Then F and G share $(1, 2)$. We consider the following cases.
Case 1. Suppose that $\Phi \not\equiv 0$.
Subcase 1.1 Let $H \not\equiv 0$. Then using *Lemma 2.3*, *Lemma 2.11* and *Lemma 2.8* with $p = 3$ and $m = 2$ and using *Lemma 2.8* for $p = 3$ and $p = 0$ we obtain

$$\begin{aligned}
& (n+1)\left\{T(r,f)+T(r,g)\right\} \qquad (7)\\
\leq\ & 2\left\{\overline{N}(r,0;f)+\overline{N}\left(r,1;f\right)\right\}+\frac{1}{11}\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}_*(r,1;F,G)\}\\
& +2\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+\frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]-\frac{1}{2}\overline{N}_*(r,1;F,G)\\
& +S(r,f)+S(r,g)\\
\leq\ & \frac{34}{11}\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+\frac{n}{2}\left[T(r,f)+T(r,g)\right]+\frac{13}{22}\overline{N}_*(r,1;F,G)\\
& +S(r,f)+S(r,g)\\
\leq\ & \left\{\frac{n}{2}+\frac{34}{11}\right\}\left[T(r,f)+T(r,g)\right]+\frac{13}{66}\{\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)\\
& +\overline{N}(r,\infty;g)\}+S(r,f)+S(r,g)\\
\leq\ & \left\{\frac{n}{2}+\frac{115}{33}\right\}\left[T(r,f)+T(r,g)\right]+S(r,f)+S(r,g).
\end{aligned}$$

(7) gives a contradiction for $n \geq 5$.

We omit the rest of the proof as the same can be done in the line of proof of *Theorem 1.1.* $\square$

Acknowledgements. The author is thankful to the referee for some valuable suggestions.

References

[1] X. Bai, Q. Han and A. Chen, *On a result of H. Fujimoto*, J. Math. Kyoto Univ., **49**(3) (2009), 631-643.

[2] A. Banerjee, *Some uniqueness results on meromorphic functions sharing three sets*, Ann. Polon. Math. **92**(3) (2007), 261-274.

[3] A. Banerjee and P. Bhattacharajee, *Uniqueness and set sharing of derivatives of meromorphic functions*, Math. Slovaca, **61**(2) (2011), 197-214.

[4] A. Banerjee and I. Lahiri, *A uniqueness polynomial generating a unique range set and vise versa*, Comput. Methods Funct. Theo., **12**(2) (2012), 527-539.

[5] S. Bartels, *Meromorphic functions sharing a set with 17 elements ignoring multiplicities*, Complex Var. Theory Appl., **39** (1998), 85-92.

[6] M. Fang and H. Guo, *On unique range sets for meromorphic or entire functions*, Acta Math. Sinica (New Ser.) **14**(4) (1998), 569-576.
[7] G. Frank and M. Reinders, *A unique range set for meromorphic functions with 11 elements*, Complex Var. Theory Appl., **37** (1998), 185-193.
[8] H. Fujimoto, *On uniqueness of meromorphic functions sharing finite sets*, Amer. J. Math., **122** (2000), 1175-1203.
[9] F. Gross, *Factorization of meromorphic functions and some open problems*, Proc. Conf. Univ. Kentucky, Leixngton, Ky(1976); Complex Analysis, Lecture Notes in Math., **599** (1977), 51-69, Springer Verlag.
[10] F. Gross and C. C. Yang, *On preimage and range sets of meromorphic functions*, Proc. Japan Acad., **58** (1982), 17-20.
[11] W. K. Hayman, *Meromorphic Functions*, The Clarendon Press, Oxford (1964).
[12] I. Lahiri, *Value distribution of certain differential polynomials*, Int. J. Math. Math. Sci., **28**(2) (2001), 83-91.
[13] I. Lahiri, *Weighted sharing and uniqueness of meromorphic functions*, Nagoya Math. J., **161** (2001), 193-206.
[14] I. Lahiri, *Weighted value sharing and uniqueness of meromorphic functions*, Complex Var. Theory Appl., **46** (2001), 241-253.
[15] P. Li and C. C. Yang, *On the unique range sets for meromorphic functions*, Proc. Amer. Math. Soc., **124** (1996), 177-185.
[16] P. Li and C. C. Yang, *Some further results on the unique range sets for meromorphic functions*, Kodai Math. J., **18** (1995), 437-450.
[17] W. C. Lin and H. X. Yi, *Some further results on meromorphic functions that share two sets*, Kyungpook Math. J., **43** (2003), 73-85.
[18] C. C. Yang, *On deficiencies of differential polynomials II*, Math. Z., **125** (1972), 107-112.
[19] B. Yi and Y. H. Li, *The uniqueness of meromorphic functions that share two sets with CM*, Acta Math. Sinica, Chinese Ser., **55**(2) (2012), 363-368.
[20] H. X. Yi, *Uniqueness of meromorphic functions and a question of Gross*, Sci. China (A), **37**(7) (1994), 802-813.
[21] H. X. Yi, *Unicity theorems for meromorphic or entire functions III*, Bull. Austral. Math. Soc., **53** (1996), 71-82.
[22] H. X. Yi, *Meromorphic functions that share one or two values II*, Kodai Math. J., **22** (1999), 264-272.

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Received July 23, 2013