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APPROXIMATION OF COMMON SOLUTIONS FOR MONOTONE INCLUSION PROBLEMS AND EQUILIBRIUM PROBLEMS IN HILBERT SPACES

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Abstract. Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $\alpha > 0$ and let A be an α-inverse-strongly monotone mapping of C into H. Let B be a maximal monotone operator on H and let F be a maximal monotone operator on H such that the domain of F is included in C. Let $(A+B)^{-1}0$ and $F^{-1}0$ be the sets of zero points of $A+B$ and F, respectively. In this paper, we prove a strong convergence theorem for finding a point $z_0 \in (A+B)^{-1}0 \cap F^{-1}0$ which is a unique fixed point of a nonlinear operator and also a unique solution of a variational inequality. Using this result, we obtain new and well-known strong convergence theorems in a Hilbert space which are useful in Nonlinear Analysis and Optimization.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let T be a mapping of C into H. We denote by $F(T)$ the set of fixed points of T. A mapping $g : H \to H$ is a contraction if there exists $k \in (0,1)$ such that $\|g(x) - g(y)\| \leq k\|x - y\|$ for all $x, y \in H$. We call such g a k-contraction. A linear bounded operator $G : H \to H$ is called strongly positive if there exists $\overline{\gamma} > 0$ such that $\langle Gx, x\rangle \geq \overline{\gamma}\|x\|^2$ for all $x \in H$. We call such G a strongly positive operator with coefficient $\overline{\gamma} > 0$. Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of positive integers and real numbers, respectively. A mapping $U : C \to H$ is a strict pseudo-contraction [6] if there exists $r \in \mathbb{R}$ with $0 \leq r < 1$ such that

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + r\|(I - U)x - (I - U)y\|^2, \quad \forall x, y \in C.$$

We call such U an r-strict pseudo-contraction. For $\alpha > 0$, a mapping $A : C \to H$ is called α-inverse-strongly monotone if

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

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Let $f : C \times C \to \mathbb{R}$ be a bifunction and let A be a mapping of C into H. A generalized equilibrium problem (with respect to C) is to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) + \langle A\hat{x}, y - \hat{x} \rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of such solutions $\hat{x}$ is denoted by $EP(f, A)$, i.e.,

$$EP(f, A) = \{\hat{x} \in C : f(\hat{x}, y) + \langle A\hat{x}, y - \hat{x} \rangle \geq 0, \ \forall y \in C\}.$$

In the case of $A = 0$, $EP(f, A)$ is denoted by $EP(f)$. In the case of $f = 0$, $EP(f, A)$ is also denoted by $VI(C, A)$. This is the set of solutions of the variational inequality for A; see [14] and [18]. For solving the equilibrium problem, let us assume that the bifunction $f : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
(A3) for all $x, y, z \in C$,

$$\limsup_{t \downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y);$$

(A4) for all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.

Recently, Liu [10] proved the following theorem.

Theorem 1. *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $r \in \mathbb{R}$ with $0 \leq r < 1$ and let U be an r-strict pseudo-contraction of C into H. Let f be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1) − (A4). Let $0 < k < 1$ and let g be a k-contraction of H into itself. Let G be a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma} > 0$. Let $0 < \gamma < \frac{\overline{\gamma}}{k}$ and suppose $F(U) \cap EP(f) \neq \emptyset$. Let $x_1 = x \in H$ and let $\{x_n\} \subset H$ be a sequence generated by*

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C,$$

$$x_{n+1} = \alpha_n \gamma g(x_n) + (I - \alpha_n G)\{(1 - t_n)U + t_n I\}u_n$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0, 1)$, $\{t_n\} \subset [0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfy

$$\alpha_n \to 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty,$$

$$r \leq t_n \leq b < 1, \quad \lim_{n\to\infty} t_n = b, \quad \sum_{n=1}^{\infty} |t_n - t_{n+1}| < \infty,$$

$$\liminf_{n\to\infty} r_n > 0, \quad \textit{and} \quad \sum_{n=1}^{\infty} |r_n - r_{n+1}| < \infty.$$

Then the sequence $\{x_n\}$ *converges strongly to a point* z_0 *of* $F(U) \cap EP(f)$*, where* $z_0 \in F(U) \cap EP(f)$ *is a unique fixed point of* $P_{F(U)\cap EP(f)}(I - G + \gamma g)$*. This point* $z_0 \in F(U) \cap EP(f)$ *is also a unique solution of the variational inequality*

$$\langle (G - \gamma g)z_0, q - z_0 \rangle \geq 0, \quad \forall q \in F(U) \cap EP(f).$$

Let f be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A1)-(A4)$. Defining a set-valued mapping $A_f \subset H \times H$ by

$$A_f x = \begin{cases} \{z \in H : f(x,y) \geq \langle y - x, z \rangle, \ \forall y \in C\}, & \forall x \in C, \\ \emptyset, \quad \forall x \notin C, \end{cases}$$

we have from [19] that A_f is a maximal monotone operator such that the domain is included in C; see Lemma 12 in Section 4 for more details. On the other hand, putting $A = I - U$ for an r-strict pseudo-contraction $U : C \to H$ with $0 \leq r < 1$, we have that $A : C \to H$ is $\frac{1-r}{2}$ -inverse-strongly monotone; see, for example, [13].

In this paper, motivated by these results, we prove a strong convergence theorem for finding a point $z_0 \in (A + B)^{-1}0 \cap F^{-1}0$ which is a unique fixed point of $P_{(A+B)^{-1}0\cap F^{-1}0}(I - G + \gamma g)$, where A is an α-inverse-strongly monotone mapping of C into H with $\alpha > 0$, B is a maximal monotone operator on H, F is a maximal monotone operator on H such that the domain of F is included in C, g is a k-contraction of H into itself with $0 < k < 1$, G is a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma} > 0$ and γ is a real number with $0 < \gamma < \frac{\overline{\gamma}}{k}$. Using this result, we obtain new and well-known strong convergence theorems in a Hilbert space which are useful in Nonlinear Analysis and Optimization.

2. Preliminaries

Throughout this paper, let $\mathbb{N}$ be the set of positive integers, let H be a real Hilbert space with inner product $\langle \cdot , \cdot \rangle$ and norm $\| \cdot \|$. When $\{x_n\}$ is a sequence in H, we denote the strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. We have from [22] that for any $x, y \in H$ and $\lambda \in \mathbb{R}$,

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \tag{2.1}$$

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \tag{2.2}$$

Furthermore we have that for $x, y, u, v \in H$,

$$2\langle x - y, u - v \rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{2.3}$$

All Hilbert spaces satisfy Opial's condition, that is,

$$\liminf_{n\to\infty} \|x_n - u\| < \liminf_{n\to\infty} \|x_n - v\| \tag{2.4}$$

if $x_n \rightharpoonup u$ and $u \neq v$; see [16]. Let C be a nonempty closed convex subset of a Hilbert space H. A mapping $T : C \to H$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. If $T : C \to H$ is nonexpansive, then $F(T)$ is closed and convex; see [22]. For a nonempty closed convex subset D of H, the nearest point projection of H onto D is denoted by P_D, that is, $\|x - P_D x\| \leq \|x - y\|$ for all $x \in H$ and $y \in D$. Such P_D is called the metric projection of H onto D. We know that the metric projection P_D is firmly nonexpansive; $\|P_D x - P_D y\|^2 \leq \langle P_D x - P_D y, x - y\rangle$ for all $x, y \in H$. Further $\langle x - P_D x, y - P_D x\rangle \leq 0$ holds for all $x \in H$ and $y \in D$; see [20].

If A is α-inverse-strongly monotone, then we have that $\langle x - y, Ax - Ay\rangle \geq 0$ and $\|Ax - Ay\| \leq (1/\alpha)\|x - y\|$ for all $x, y \in C$; see, for example, [15, 24] for inverse-strongly monotone mappings. Let B be a mapping of H into 2^H. The effective domain of B is denoted by $\operatorname{dom}(B)$, that is, $\operatorname{dom}(B) = \{x \in H : Bx \neq \emptyset\}$. A multi-valued mapping B is said to be a monotone operator on H if $\langle x - y, u - v\rangle \geq 0$ for all $x, y \in \operatorname{dom}(B)$, $u \in Bx$, and $v \in By$. A monotone operator B on H is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on H. For a maximal monotone operator B on H and $r > 0$, we may define a single-valued operator $J_r = (I + rB)^{-1} \colon H \to \operatorname{dom}(B)$, which is called the resolvent of B for r. We denote by $A_r = \frac{1}{r}(I - J_r)$ the Yosida approximation of B for $r > 0$. We know from [21] that

$$A_r x \in BJ_r x, \quad \forall x \in H,\ r > 0. \tag{2.5}$$

Let B be a maximal monotone operator on H and let $B^{-1}0 = \{x \in H : 0 \in Bx\}$. It is known that $B^{-1}0 = F(J_r)$ for all $r > 0$ and the resolvent J_r is firmly nonexpansive, i.e.,

$$\|J_r x - J_r y\|^2 \leq \langle x - y, J_r x - J_r y\rangle, \quad \forall x, y \in H. \tag{2.6}$$

We also know the following lemma from [19].

Lemma 2. *Let H be a real Hilbert space and let B be a maximal monotone operator on H. For $r > 0$ and $x \in H$, define the resolvent $J_r x$. Then the following holds:*

$$\frac{s-t}{s}\langle J_s x - J_t x, J_s x - x\rangle \geq \|J_s x - J_t x\|^2$$

for all $s, t > 0$ and $x \in H$.

From Lemma 2, we have that

$$\|J_\lambda x - J_\mu x\| \leq (|\lambda - \mu| / \lambda)\|x - J_\lambda x\|$$

for all $\lambda, \mu > 0$ and $x \in H$; see also [8, 20].

To prove our main result, we need the following lemmas:

Lemma 3 ([2]; see also [27]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$, and let $\{\gamma_n\}$ be a sequence of real numbers with* $\limsup_{n\to\infty} \gamma_n \leq 0$. *Suppose that*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n\gamma_n + \beta_n$$

for all $n = 1, 2, \ldots$. Then $\lim_{n\to\infty} s_n = 0$.

Lemma 4 ([11]). *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n\geq n_0}$ of integers as follows:*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ such that $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:

(i) $\tau(n_0) \leq \tau(n_0+1) \leq \ldots$ and $\tau(n) \to \infty$;

(ii) $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ and $\Gamma_n \leq \Gamma_{\tau(n)+1},\ \forall n \geq n_0$.

3. Strong Convergence Theorem

Let C be a nonempty closed and convex subset of a Hilbert space H. Let $\alpha > 0$ and let A be an α-inverse-strongly monotone mapping of C into H. If $0 < \lambda \leq 2\alpha$, then $I - \lambda A : C \to H$ is nonexpansive. In fact, we have that for all $x, y \in C$,

$$\begin{aligned}
\|(I-\lambda A)x - (I-\lambda A)y\|^2 &= \|x - y - \lambda(Ax - Ay)\|^2 \\
&= \|x-y\|^2 - 2\lambda\langle x-y, Ax-Ay\rangle + (\lambda)^2\|Ax-Ay\|^2 \\
&\leq \|x-y\|^2 - 2\lambda\alpha\|Ax-Ay\|^2 + (\lambda)^2\|Ax-Ay\|^2 \\
&= \|x-y\|^2 + \lambda(\lambda-2\alpha)\|Ax-Ay\|^2 \\
&\leq \|x-y\|^2.
\end{aligned}$$

Thus, $I - \lambda A : C \to H$ is nonexpansive. A mapping $g : H \to H$ is a contraction if there exists $k \in (0,1)$ such that $\|g(x) - g(y)\| \leq k\|x - y\|$ for all $x, y \in H$. We also call such a mapping g a k-contraction. A linear bounded self-adjoint operator $G : H \to H$ is called strongly positive if there exists $\overline{\gamma} > 0$ such that $\langle Gx, x\rangle \geq \overline{\gamma}\|x\|^2$ for all $x \in H$. In general, a nonlinear operator $T : H \to H$ is called strongly monotone if there exists $\overline{\gamma} > 0$ such that $\langle x - y, Tx - Ty\rangle \geq \overline{\gamma}\|x - y\|^2$ for all $x, y \in H$. Such T is also called $\overline{\gamma}$-strongly monotone. We know the following result from Marino and Xu [12].

Lemma 5. *Let H be a Hilbert space and let G be a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma} > 0$. If $0 < \gamma \leq \|G\|^{-1}$, then $\|I - \gamma G\| \leq 1 - \gamma\overline{\gamma}$.*

For proving the main theorem, we also need the following lemma which is proved simply by Takahashi [23].

Lemma 6. *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $\alpha > 0$ and let A be an α-inverse-strongly monotone mapping of C into H and let B be a maximal monotone operator on H. Let F be a maximal monotone operator on H such that the domain of F is included in C. Let $0 < k < 1$ and let g be a k-contraction of H into itself. Let G be a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma} > 0$. Let γ be a real number with $0 < \gamma < \frac{\overline{\gamma}}{k}$. Then for any nonempty closed convex subset C of H, $P_C(I - G + \gamma g)$ has a unique fixed point z_0 in C. This point $z_0 \in C$ is also a unique solution of the variational inequality*

$$\langle (G - \gamma g)z_0, q - z_0 \rangle \geq 0, \quad \forall q \in C.$$

In particular, the set $(A + B)^{-1}0 \cap F^{-1}0$ is a nonempty closed and convex subset of H and $P_{(A+B)^{-1}0\cap F^{-1}0}(I - G + \gamma g)$ has a unique fixed point z_0 in $(A+B)^{-1}0 \cap F^{-1}0$.

Using Lemmas 5 and 6, we prove the following strong convergence theorem of Halpern's type [9] in a Hilbert space.

Theorem 7. *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $\alpha > 0$ and let A be an α-inverse-strongly monotone mapping of C into H. Let B be a maximal monotone operator on H and let F be a maximal monotone operator on H such that the domain of F is included in C. Let $J_\lambda = (I + \lambda B)^{-1}$ and let $T_r = (I + rF)^{-1}$ be the resolvents of B and F for $\lambda > 0$ and $r > 0$, respectively. Let $0 < k < 1$ and let g be a k-contraction of H into itself. Let G be a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma} > 0$. Let $0 < \gamma < \frac{\overline{\gamma}}{k}$ and suppose $(A + B)^{-1}0 \cap F^{-1}0 \neq \emptyset$. Let $x_1 = x \in H$ and let $\{x_n\} \subset H$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)\{\alpha_n \gamma g(x_n) + (I - \alpha_n G)J_{\lambda_n}(I - \lambda_n A)T_{r_n} x_n\}$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0, \infty)$, $\{\beta_n\} \subset (0, 1)$, $\{\alpha_n\} \subset (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfy

$$0 < a \leq \lambda_n \leq 2\alpha, \quad 0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1,$$

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \text{and} \quad \liminf_{n\to\infty} r_n > 0.$$

Then, the sequence $\{x_n\}$ converges strongly to a point z_0 of $(A + B)^{-1}0 \cap F^{-1}0$, where $z_0 = P_{(A+B)^{-1}0\cap F^{-1}0}(I - G + \gamma g)z_0$.

Proof. Let $z \in (A+B)^{-1}0 \cap F^{-1}0$. Then, $z = J_{\lambda_n}(I - \lambda_n A)z$ and $z = T_{r_n} z$. Putting $z_n = J_{\lambda_n}(I - \lambda_n A)T_{r_n} x_n$ and $u_n = T_{r_n} x_n$, we obtain that

$$\begin{aligned} \|z_n - z\| &= \|J_{\lambda_n}(I - \lambda_n A)T_{r_n} x_n - z\| \\ &= \|J_{\lambda_n}(I - \lambda_n A)T_{r_n} x_n - J_{\lambda_n}(I - \lambda_n A)T_{r_n} z\| \\ &\leq \|x_n - z\| . \end{aligned} \tag{3.1}$$

Putting $y_n = \alpha_n \gamma g(x_n) + (I - \alpha_n G)J_{\lambda_n}(I - \lambda_n A)T_{r_n} x_n$, from $z = \alpha_n Gz + z - \alpha_n Gz$ and Lemma 5 we have that

$$\begin{aligned} \|y_n - z\| &= \|\alpha_n(\gamma g(x_n) - Gz) + (I - \alpha_n G)(z_n - z)\| \\ &\leq \alpha_n \gamma\, k \|x_n - z\| + \alpha_n \|\gamma g(z) - Gz\| + (1 - \alpha_n \overline{\gamma}) \|z_n - z\| \\ &\leq \{1 - \alpha_n(\overline{\gamma} - \gamma\, k)\} \|x_n - z\| + \alpha_n \|\gamma g(z) - Gz\|. \end{aligned}$$

Using this, we get

$$\begin{aligned} \|x_{n+1} - z\| &= \|\beta_n(x_n - z) + (1 - \beta_n)(y_n - z)\| \\ &\leq \beta_n \|x_n - z\| + (1 - \beta_n) \|y_n - z\| \\ &\leq \beta_n \|x_n - z\| \\ &\qquad + (1 - \beta_n)(\{1 - \alpha_n(\overline{\gamma} - \gamma\, k)\} \|x_n - z\| + \alpha_n \|\gamma g(z) - Gz\|) \\ &= \{1 - (1 - \beta_n)\alpha_n(\overline{\gamma} - \gamma\, k)\} \|x_n - z\| \\ &\qquad + (1 - \beta_n)\alpha_n(\overline{\gamma} - \gamma\, k) \frac{\|\gamma g(z) - Gz\|}{\overline{\gamma} - \gamma\, k}. \end{aligned}$$

Putting $K = \max\{\|x_1 - z\|, \frac{\|\gamma g(z) - Gz\|}{\overline{\gamma} - \gamma\, k}\}$, we have that $\|x_n - z\| \leq K$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is bounded. Furthermore, $\{u_n\}$, $\{z_n\}$ and $\{y_n\}$ are bounded.

Using Lemma 6, we can take $z_0 \in (A+B)^{-1}0 \cap F^{-1}0$ such that

$$z_0 = P_{(A+B)^{-1}0 \cap F^{-1}0}(I - G + \gamma g)z_0.$$

From the definition of $\{x_n\}$, we have that

$$x_{n+1} - x_n = \beta_n x_n + (1 - \beta_n)\{\alpha_n \gamma g(x_n) + (I - \alpha_n G)z_n\} - x_n$$

and hence

$$\begin{aligned} x_{n+1} - x_n - (1-\beta_n)\alpha_n \gamma g(x_n) &= \beta_n x_n + (1 - \beta_n)(I - \alpha_n G)z_n - x_n \\ &= (1 - \beta_n)\{(I - \alpha_n G)z_n - x_n\} \\ &= (1 - \beta_n)\{z_n - x_n - \alpha_n G z_n\}. \end{aligned}$$

Thus we have that

$$\begin{aligned} &\langle x_{n+1} - x_n - (1 - \beta_n)\alpha_n \gamma g(x_n), x_n - z_0\rangle \\ &\qquad = (1 - \beta_n)\langle z_n - x_n, x_n - z_0\rangle - (1 - \beta_n)\langle \alpha_n G z_n, x_n - z_0\rangle \end{aligned} \tag{3.2}$$

$$= -(1-\beta_n)\langle x_n - z_n, x_n - z_0\rangle - (1-\beta_n)\alpha_n\langle Gz_n, x_n - z_0\rangle.$$

From (2.3) and (3.1), we have that

$$\begin{aligned} 2\langle x_n - z_n, x_n - z_0\rangle &= \|x_n - z_0\|^2 + \|z_n - x_n\|^2 - \|z_n - z_0\|^2 \\ &\geq \|x_n - z_0\|^2 + \|z_n - x_n\|^2 - \|x_n - z_0\|^2 \\ &= \|z_n - x_n\|^2. \end{aligned} \tag{3.3}$$

From (3.2) and (3.3), we have that

$$\begin{aligned} -2\langle x_n - x_{n+1}, x_n - z_0\rangle &= 2(1-\beta_n)\alpha_n\langle \gamma g(x_n), x_n - z_0\rangle \\ &\quad - 2(1-\beta_n)\langle x_n - z_n, x_n - z_0\rangle - 2(1-\beta_n)\alpha_n\langle Gz_n, x_n - z_0\rangle \\ &\leq 2(1-\beta_n)\alpha_n\langle \gamma g(x_n), x_n - z_0\rangle \\ &\quad - (1-\beta_n)\|z_n - x_n\|^2 - 2(1-\beta_n)\alpha_n\langle Gz_n, x_n - z_0\rangle. \end{aligned} \tag{3.4}$$

Furthermore using (2.3) and (3.4), we have that

$$\begin{aligned} \|x_{n+1} - z_0\|^2 - \|x_n - x_{n+1}\|^2 - \|x_n - z_0\|^2 \\ \leq 2(1-\beta_n)\alpha_n\langle \gamma g(x_n), x_n - z_0\rangle \\ - (1-\beta_n)\|z_n - x_n\|^2 - 2(1-\beta_n)\alpha_n\langle Gz_n, x_n - z_0\rangle. \end{aligned}$$

Setting $\Gamma_n = \|x_n - z_0\|^2$, we have that

$$\begin{aligned} \Gamma_{n+1} - \Gamma_n - \|x_n - x_{n+1}\|^2 \\ \leq 2(1-\beta_n)\alpha_n\langle \gamma g(x_n), x_n - z_0\rangle \\ - (1-\beta_n)\|z_n - x_n\|^2 - 2(1-\beta_n)\alpha_n\langle Gz_n, x_n - z_0\rangle. \end{aligned} \tag{3.5}$$

Noting that

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|(1-\beta_n)\alpha_n(\gamma g(x_n) - Gz_n) + (1-\beta_n)(z_n - x_n)\| \\ &\leq (1-\beta_n)\big(\|z_n - x_n\| + \alpha_n\|\gamma g(x_n) - Gz_n\|\big) \end{aligned} \tag{3.6}$$

and hence

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &\leq (1-\beta_n)^2\big(\|z_n - x_n\| + \alpha_n\|\gamma g(x_n) - Gz_n\|\big)^2 \\ &= (1-\beta_n)^2\|z_n - x_n\|^2 \\ &\quad + (1-\beta_n)^2\big(2\alpha_n\|z_n - x_n\|\|\gamma g(x_n) - Gz_n\| + \alpha_n^2\|\gamma g(x_n) - Gz_n\|^2\big). \end{aligned} \tag{3.7}$$

Thus we have from (3.5) and (3.7) that

$$\begin{aligned} \Gamma_{n+1} - \Gamma_n &\leq \|x_n - x_{n+1}\|^2 + 2(1-\beta_n)\alpha_n\langle \gamma g(x_n), x_n - z_0\rangle \\ &\quad - (1-\beta_n)\|z_n - x_n\|^2 - 2(1-\beta_n)\alpha_n\langle Gz_n, x_n - z_0\rangle \\ &\leq (1-\beta_n)^2\|z_n - x_n\|^2 \\ &\quad + (1-\beta_n)^2\big(2\alpha_n\|z_n - x_n\|\|\gamma g(x_n) - Gz_n\| + \alpha_n^2\|\gamma g(x_n) - Gz_n\|^2\big) \end{aligned}$$

$$
\begin{aligned}
&+ 2(1-\beta_n)\alpha_n\langle \gamma g(x_n), x_n - z_0\rangle - (1-\beta_n)\|z_n - x_n\|^2 \\
&- 2(1-\beta_n)\alpha_n\langle Gz_n, x_n - z_0\rangle
\end{aligned}
$$

and hence

$$
\begin{aligned}
\Gamma_{n+1} - \Gamma_n &+ \beta_n(1-\beta_n)\|z_n - x_n\|^2 \\
&\le (1-\beta_n)^2\big(2\alpha_n\|z_n - x_n\|\|\gamma g(x_n) - Gz_n\| + \alpha_n^2\|\gamma g(x_n) - Gz_n\|^2\big) \quad (3.8)\\
&\quad + 2(1-\beta_n)\alpha_n\langle \gamma g(x_n), x_n - z_0\rangle - 2(1-\beta_n)\alpha_n\langle Gz_n, x_n - z_0\rangle.
\end{aligned}
$$

We will divide the proof into two cases.

Case 1: Suppose that $\Gamma_{n+1} \le \Gamma_n$ for all $n \in \mathbb{N}$. In this case, $\lim_{n\to\infty}\Gamma_n$ exists and then $\lim_{n\to\infty}(\Gamma_{n+1} - \Gamma_n) = 0$. Using $\lim_{n\to\infty}\alpha_n = 0$ and $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$, we have from (3.8) that

$$
\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{3.9}
$$

From (3.6), we have that

$$
\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.10}
$$

We also have that

$$
\begin{aligned}
\|y_n - z_n\| &= \|\alpha_n\gamma g(x_n) + (I - \alpha_n G)z_n - z_n\| \qquad (3.11)\\
&= \alpha_n\|\gamma g(x_n) - Gz_n\| \to 0.
\end{aligned}
$$

Furthermore, from $\|y_n - x_n\| \le \|y_n - z_n\| + \|z_n - x_n\|$, we have that

$$
\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.12}
$$

For $z_0 \in (A+B)^{-1}0 \cap F^{-1}0$, we have from (2.6) that

$$
\begin{aligned}
2\|u_n - z_0\|^2 &= 2\|T_{r_n}x_n - T_{r_n}z_0\|^2 \\
&\le 2\langle x_n - z_0, u_n - z_0\rangle \\
&= \|x_n - z_0\|^2 + \|u_n - z_0\|^2 - \|u_n - x_n\|^2
\end{aligned}
$$

and hence

$$
\|u_n - z_0\|^2 \le \|x_n - z_0\|^2 - \|u_n - x_n\|^2.
$$

Then we have that

$$
\|z_n - z_0\|^2 \le \|u_n - z_0\|^2 \le \|x_n - z_0\|^2 - \|u_n - x_n\|^2. \tag{3.13}
$$

Thus we have

$$
\begin{aligned}
\|u_n - x_n\|^2 &\le \|x_n - z_0\|^2 - \|z_n - z_0\|^2 \\
&\le \|x_n - z_n\|(\|x_n - z_0\| + \|z_n - z_0\|)
\end{aligned}
$$

and hence

$$
\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.14}
$$

Then we have from (3.12) and (3.14) that

$$\|y_n - u_n\| \le \|y_n - x_n\| + \|x_n - u_n\| \to 0. \tag{3.15}$$

Furthermore, we have from (3.9) and (3.14) that

$$\|z_n - u_n\| \le \|z_n - x_n\| + \|x_n - u_n\| \to 0. \tag{3.16}$$

Take $\lambda_0 \in [a, 2\alpha]$ arbitrarily. Put $w_n = u_n - \lambda_n A u_n$, where $u_n = T_{r_n} x_n$. Using

$$z_n = J_{\lambda_n}(I - \lambda_n A)u_n \text{ and } y_n = \alpha_n \gamma g(x_n) + (I - \alpha_n G)J_{\lambda_n}(I - \lambda_n A)u_n,$$

we have from Lemma 2 and $z_n = \alpha_n G z_n + z_n - \alpha_n G z_n$ that

$$\begin{aligned}
&\|\alpha_n \gamma g(x_n) + (I - \alpha_n G)J_{\lambda_0}(I - \lambda_0 A)u_n - z_n\| \\
&\quad = \|\alpha_n(\gamma g(x_n) - G z_n) + (I - \alpha_n G)(J_{\lambda_0}(I - \lambda_0 A)u_n - z_n)\| \\
&\quad \le \alpha_n(\gamma\, k\|x_n - z_n\| + \|\gamma g(z_n) - G z_n\|) \\
&\qquad + (1 - \alpha_n \overline{\gamma})\|J_{\lambda_0}(I - \lambda_0 A)u_n - z_n\| \\
&\quad \le \alpha_n(\gamma\, k\|x_n - z_n\| + \|\gamma g(z_n) - G z_n\|) \\
&\qquad + \|J_{\lambda_0}(I - \lambda_0 A)u_n - J_{\lambda_0}(I - \lambda_n A)u_n + J_{\lambda_0} w_n - J_{\lambda_n} w_n\| \\
&\quad \le \alpha_n(\gamma\, k\|x_n - z_n\| + \|\gamma g(z_n) - G z_n\|) \\
&\qquad + |\lambda_0 - \lambda_n|\|A u_n\| + \frac{|\lambda_0 - \lambda_n|}{\lambda_0}\|w_n - J_{\lambda_0} w_n\|.
\end{aligned} \tag{3.17}$$

We also have

$$\begin{aligned}
&\|u_n - J_{\lambda_0}(I - \lambda_0 A)u_n\| \\
&\quad \le \|u_n - z_n\| + \|z_n - \{\alpha_n \gamma g(x_n) + (I - \alpha_n G)J_{\lambda_0}(I - \lambda_0 A)u_n\}\| \\
&\qquad + \|\alpha_n \gamma g(x_n) + (I - \alpha_n G)J_{\lambda_0}(I - \lambda_0 A)u_n - J_{\lambda_0}(I - \lambda_0 A)u_n\| \\
&\quad = \|u_n - z_n\| + \|z_n - \{\alpha_n \gamma g(x_n) + (I - \alpha_n G)J_{\lambda_0}(I - \lambda_0 A)u_n\}\| \\
&\qquad + \alpha_n\|\gamma g(x_n) - G J_{\lambda_0}(I - \lambda_0 A)u_n\|.
\end{aligned} \tag{3.18}$$

We will use (3.17) and (3.18) later.

For a unique fixed point z_0 of $P_{(A+B)^{-1}0 \cap F^{-1}0}(I - G + \gamma g)$ in $(A+B)^{-1}0 \cap F^{-1}0$, let us show that

$$\limsup_{n\to\infty} \langle (G - \gamma g)z_0, y_n - z_0 \rangle \ge 0.$$

Put $l = \limsup_{n\to\infty} \langle (G - \gamma g)z_0, y_n - z_0 \rangle$. Without loss of generality, there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $l = \lim_{i\to\infty} \langle (G - \gamma g)z_0, y_{n_i} - z_0 \rangle$ and $\{y_{n_i}\}$ converges weakly to some point $w \in H$. From $\|y_n - u_n\| \to 0$, we also have that $\{u_{n_i}\}$ converges weakly to $w \in C$. On the other hand, since $0 < a \le \lambda_{n_i} \le 2\alpha$, there exists a subsequence $\{\lambda_{n_{i_j}}\}$ of $\{\lambda_{n_i}\}$ such that $\{\lambda_{n_{i_j}}\}$ converges to a number $\lambda_0 \in [a, 2\alpha]$. Using (3.17), we have that

$$\|\alpha_{n_{i_j}}\gamma g(x_{n_{i_j}}) + (I - \alpha_{n_{i_j}}G)J_{\lambda_0}(I - \lambda_0 A)u_{n_{i_j}} - z_{n_{i_j}}\| \to 0.$$

Furthermore, using (3.18), we have that

$$\begin{aligned}
&\|u_{n_{i_j}} - J_{\lambda_0}(I - \lambda_0 A)u_{n_{i_j}}\| \\
&\quad \le \|u_{n_{i_j}} - z_{n_{i_j}}\| + \|z_{n_{i_j}} - \{\alpha_{n_{i_j}}\gamma g(x_{n_{i_j}}) + (I - \alpha_{n_{i_j}}G)J_{\lambda_0}(I - \lambda_0 A)u_{n_{i_j}}\}\| \\
&\qquad + \alpha_{n_{i_j}}\|\gamma g(x_{n_{i_j}}) - GJ_{\lambda_0}(I - \lambda_0 A)u_{n_{i_j}}\| \to 0.
\end{aligned}$$

Since $J_{\lambda_0}(I - \lambda_0 A)$ is nonexpansive, we have $w = J_{\lambda_0}(I - \lambda_0 A)w$. This means that $0 \in Aw + Bw$. We show $w \in F^{-1}0$. Since $\frac{x_{n_{i_j}} - T_{r_{n_{i_j}}} x_{n_{i_j}}}{r_{n_{i_j}}} \in FT_{r_{n_{i_j}}} x_{n_{i_j}}$ and F is a monotone operator, we have that for any $(u, v) \in F$,

$$\langle u - u_{n_{i_j}}, v - \frac{x_{n_{i_j}} - u_{n_{i_j}}}{r_{n_{i_j}}}\rangle \ge 0.$$

Since $\liminf_{n\to\infty} r_n > 0$, $u_{n_{i_j}} \rightharpoonup w$ and $x_{n_{i_j}} - u_{n_{i_j}} \to 0$, we have

$$\langle u - w, v\rangle \ge 0.$$

Since F is a maximal monotone operator, we have $0 \in Fw$ and hence $w \in F^{-1}0$. Thus we have $w \in (A + B)^{-1}0 \cap F^{-1}0$. So, we have

$$l = \lim_{j\to\infty} \langle (G - \gamma g)z_0, y_{n_{i_j}} - z_0\rangle = \langle (G - \gamma g)z_0, w - z_0\rangle \ge 0. \tag{3.19}$$

Since $y_n - z_0 = \alpha_n(\gamma g(x_n) - Gz_0) + (I - \alpha_n G)(J_{\lambda_n}(I - \lambda_n A)u_n - z_0)$, we have

$$\|y_n - z_0\|^2 \le (1 - \alpha_n\overline{\gamma})^2 \|J_{\lambda_n}(I - \lambda_n A)u_n - z_0\|^2 + 2\alpha_n \langle \gamma g(x_n) - Gz_0, y_n - z_0\rangle.$$

Thus we have

$$\begin{aligned}
\|y_n - z_0\|^2 &\le (1 - \alpha_n\overline{\gamma})^2 \|u_n - z_0\|^2 + 2\alpha_n \langle \gamma g(x_n) - Gz_0, y_n - z_0\rangle \\
&\le (1 - \alpha_n\overline{\gamma})^2 \|x_n - z_0\|^2 + 2\alpha_n \langle \gamma g(x_n) - Gz_0, y_n - z_0\rangle.
\end{aligned}$$

Thus we have that

$$\begin{aligned}
&\|x_{n+1} - z_0\|^2 \le \beta_n \|x_n - z_0\|^2 + (1 - \beta_n) \|y_n - z_0\|^2 \\
&\quad \le \beta_n \|x_n - z_0\|^2 \\
&\qquad + (1 - \beta_n) \big((1 - \alpha_n\overline{\gamma})^2 \|x_n - z_0\|^2 + 2\alpha_n \langle \gamma g(x_n) - Gz_0, y_n - z_0\rangle\big) \\
&\quad = \big(\beta_n + (1 - \beta_n)(1 - \alpha_n\overline{\gamma})^2\big) \|x_n - z_0\|^2 \\
&\qquad + 2(1 - \beta_n)\alpha_n \langle \gamma g(x_n) - Gz_0, y_n - z_0\rangle \\
&\quad \le \big(1 - (1 - \beta_n)(2\alpha_n\overline{\gamma} - (\alpha_n\overline{\gamma})^2)\big) \|x_n - z_0\|^2 \\
&\qquad + 2(1 - \beta_n)\alpha_n\gamma\ k\|x_n - z_0\|^2 + 2(1 - \beta_n)\alpha_n\langle \gamma g(z_0) - Gz_0, y_n - z_0\rangle
\end{aligned}$$

$$
\begin{aligned}
&= (1-2(1-\beta_n)\alpha_n(\overline{\gamma}-\gamma\ k))\,\|x_n-z_0\|^2 \\
&\quad + (1-\beta_n)(\alpha_n\overline{\gamma})^2\|x_n-z_0\|^2 + 2(1-\beta_n)\alpha_n\langle \gamma g(z_0) - Gz_0, y_n - z_0\rangle \\
&\le (1-2(1-\beta_n)\alpha_n(\overline{\gamma}-\gamma\ k))\,\|x_n-z_0\|^2 \\
&\quad + 2(1-\beta_n)\alpha_n(\overline{\gamma}-\gamma\ k)\left(\frac{\alpha_n\overline{\gamma}^2\|x_n-z_0\|^2}{2(\overline{\gamma}-\gamma\ k)} + \frac{\langle \gamma g(z_0) - Gz_0, y_n - z_0\rangle}{\overline{\gamma}-\gamma\ k}\right).
\end{aligned}
$$

Since $\sum_{n=1}^{\infty} 2(1-\beta_n)\alpha_n(\overline{\gamma}-\gamma\ k) = \infty$, from (3.19) and Lemma 3, we obtain that $x_n \to z_0$, where $z_0 = P_{(A+B)^{-1}0\cap F^{-1}0}(I - G + \gamma g)z_0$.

Case 2: Suppose that there exists a subsequence $\{\Gamma_{n_i}\} \subset \{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. In this case, define the sequence $\{\tau(n)\}_{n\ge n_0}$ as follows:

$$\tau(n) = \max\{k \le n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ such that $\{k \le n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then we have from Lemma 4 that $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$. Thus we have from (3.8) that for all $n \in \mathbb{N}$,

$$
\begin{aligned}
&\beta_{\tau(n)}(1-\beta_{\tau(n)})\|z_{\tau(n)} - x_{\tau(n)}\|^2 \\
&\qquad \le (1-\beta_{\tau(n)})^2 2\alpha_{\tau(n)}\|z_{\tau(n)} - x_{\tau(n)}\|\|\gamma g(x_{\tau(n)}) - Gz_{\tau(n)}\| \\
&\qquad\quad + (1-\beta_{\tau(n)})^2\alpha_{\tau(n)}^2\|\gamma g(x_{\tau(n)}) - Gz_{\tau(n)}\|^2 \\
&\qquad\quad + 2(1-\beta_{\tau(n)})\alpha_{\tau(n)}\langle \gamma g(x_{\tau(n)}), x_{\tau(n)} - z_0\rangle \\
&\qquad\quad - 2(1-\beta_{\tau(n)})\alpha_{\tau(n)}\langle Gz_{\tau(n)}, x_{\tau(n)} - z_0\rangle.
\end{aligned} \tag{3.20}
$$

Using $\lim_{n\to\infty}\alpha_n = 0$ and $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$, we have from (3.20) and Lemma 4 that

$$\lim_{n\to\infty} \|z_\tau(n) - x_\tau(n)\| = 0. \tag{3.21}$$

As in the proof of Case 1 we have that

$$\lim_{n\to\infty} \|u_{\tau(n)} - x_{\tau(n)}\| = 0. \tag{3.22}$$

Since $\|y_{\tau(n)} - u_{\tau(n)}\| \le \|y_{\tau(n)} - x_{\tau(n)}\| + \|x_{\tau(n)} - u_{\tau(n)}\|$, we have that

$$\lim_{n\to\infty} \|y_{\tau(n)} - u_{\tau(n)}\| = 0. \tag{3.23}$$

Let us show that

$$\limsup_{n\to\infty} \left\langle (G-\gamma g)z_0, y_{\tau(n)} - z_0\right\rangle \ge 0.$$

Put $l = \limsup_{n\to\infty}\left\langle (G-\gamma g)z_0, y_{\tau(n)} - z_0\right\rangle$. Without loss of generality, there exists a subsequence $\{y_{\tau(n_i)}\}$ of $\{y_{\tau(n)}\}$ such that $l = \lim_{i\to\infty}\left\langle (G-\gamma g)z_0, y_{\tau(n_i)} - z_0\right\rangle$ and $\{y_{\tau(n_i)}\}$ converges weakly some point $w \in H$. From $\|y_n - u_n\| \to 0$, we also have that $\{u_{\tau(n_i)}\}$ converges weakly to $w \in C$. As in the proof of Case 1 we have that

$w \in (A+B)^{-1}0$. Since F is a maximal monotone operator, as in the proof of Case 1 we can also show $w \in F^{-1}0$. Thus we have $w \in (A+B)^{-1}0 \cap F^{-1}0$. Then we have

$$l = \lim_{i\to\infty} \langle (G-\gamma g)z_0, y_{\tau(n_i)} - z_0\rangle = \langle (G-\gamma g)z_0, w - z_0\rangle \geq 0.$$

As in the proof of Case 1, we also have that

$$\left\|y_{\tau(n)} - z_0\right\|^2 \leq (1-\alpha_{\tau(n)}\overline{\gamma})^2 \left\|x_{\tau(n)} - z_0\right\|^2 + 2\alpha_{\tau(n)} \left\langle \gamma g(x_{\tau(n)}) - Gz_0, y_{\tau(n)} - z_0\right\rangle$$

and then

$$\begin{aligned}\|x_{\tau(n)+1} - z_0\|^2 &\leq \left(1 - 2(1-\beta_{\tau(n)})\alpha_{\tau(n)}(\overline{\gamma} - \gamma\ k)\right) \left\|x_{\tau(n)} - z_0\right\|^2 \\ &\quad + (1-\beta_{\tau(n)})(\alpha_{\tau(n)}\overline{\gamma})^2 \|x_{\tau(n)} - z_0\|^2 + 2(1-\beta_{\tau(n)})\alpha_{\tau(n)}\langle \gamma g(z_0) - Gz_0, y_{\tau(n)} - z_0\rangle.\end{aligned}$$

From $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$, we have that

$$\begin{aligned}&2(1-\beta_{\tau(n)})\alpha_{\tau(n)}(\overline{\gamma} - \gamma\ k)) \left\|x_{\tau(n)} - z_0\right\|^2 \\ &\quad \leq (1-\beta_{\tau(n)})(\alpha_{\tau(n)}\overline{\gamma})^2 \|x_{\tau(n)} - z_0\|^2 + 2(1-\beta_{\tau(n)})\alpha_{\tau(n)}\langle \gamma g(z_0) - Gz_0, y_{\tau(n)} - z_0\rangle.\end{aligned}$$

Since $(1-\beta_{\tau(n)})\alpha_{\tau(n)} > 0$, we have that

$$\begin{aligned}&2(\overline{\gamma} - \gamma\ k) \left\|x_{\tau(n)} - z_0\right\|^2 \\ &\qquad \leq \alpha_{\tau(n)}\overline{\gamma}^2 \|x_{\tau(n)} - z_0\|^2 + 2\langle \gamma g(z_0) - Gz_0, y_{\tau(n)} - z_0\rangle.\end{aligned}$$

Thus we have that

$$\limsup_{n\to\infty} 2(\overline{\gamma} - \gamma\ k) \left\|x_{\tau(n)} - z_0\right\|^2 \leq 0$$

and hence $\|x_{\tau(n)} - z_0\| \to 0$ as $n \to \infty$. From (3.6), we have also that $x_{\tau(n)} - x_{\tau(n)+1} \to 0$. Thus $\|x_{\tau(n)+1} - z_0\| \to 0$ as $n \to \infty$. Using Lemma 4 again, we obtain that

$$\|x_n - z_0\| \leq \|x_{\tau(n)+1} - z_0\| \to 0$$

as $n \to \infty$. This completes the proof. □

4. Applications

In this section, using Theorem 7, we can obtain well-known and new strong convergence theorems for in a Hilbert space. Let H be a Hilbert space and let f be a proper lower semicontinuous convex function of H into $(-\infty, \infty]$. The subdifferential ∂f of f is defined as follows:

$$\partial f(x) = \{z \in H : f(x) + \langle z, y - x\rangle \leq f(y),\ \forall y \in H\}$$

for all $x \in H$. From Rockafellar [17], we know that ∂f is a maximal monotone operator. Let C be a nonempty closed convex subset of H and let i_C be the indicator function of C, i.e.,

$$i_C(x) = \begin{cases} 0, & x \in C, \\ \infty, & x \notin C. \end{cases}$$

Then i_C is a proper lower semicontinuous convex function on H and then the subdifferential ∂i_C of i_C is a maximal monotone operator. Thus we can define the resolvent J_λ of ∂i_C for $\lambda > 0$, i.e.,

$$J_\lambda x = (I + \lambda \partial i_C)^{-1} x$$

for all $x \in H$. We have that for any $x \in H$ and $u \in C$,

$$\begin{aligned} u = & J_\lambda x \iff x \in u + \lambda \partial i_C u \iff x \in u + \lambda N_C u \\ & \iff x - u \in \lambda N_C u \\ & \iff \frac{1}{\lambda} \langle x - u, v - u \rangle \leq 0, \ \forall v \in C \\ & \iff \langle x - u, v - u \rangle \leq 0, \ \forall v \in C \\ & \iff u = P_C x, \end{aligned}$$

where $N_C u$ is the normal cone to C at u, i.e.,

$$N_C u = \{z \in H : \langle z, v - u \rangle \leq 0, \ \forall v \in C\}.$$

Using Theorem 7, we first prove a strong convergence theorem for inverse-strongly monotone operators in a Hilbert space.

Theorem 8. *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $\alpha > 0$ and let A be an α-inverse-strongly monotone mapping of C into H. Let $0 < k < 1$ and let g be a k-contraction of H into itself. Let G be a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma} > 0$. Let $0 < \gamma < \frac{\overline{\gamma}}{k}$ and suppose $VI(C, A) \neq \emptyset$. Let $x_1 = x \in H$ and let $\{x_n\} \subset H$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)\{\alpha_n \gamma g(x_n) + (I - \alpha_n G) P_C (I - \lambda_n A) P_C x_n\}$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0, \infty)$, $\{\beta_n\} \subset (0, 1)$, and $\{\alpha_n\} \subset (0, 1)$ satisfy

$$0 < a \leq \lambda_n \leq 2\alpha, \quad 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1,$$

$$\lim_{n \to \infty} \alpha_n = 0, \quad \textit{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then the sequence $\{x_n\}$ converges strongly to a point z_0 of $VI(C, A)$, where $z_0 = P_{VI(C,A)}(I - G + \gamma g) z_0$.

Proof. Put $B = F = \partial i_C$ in Theorem 7. Then we have that for $\lambda_n > 0$ and $r_n > 0$,

$$J_{\lambda_n} = T_{r_n} = P_C.$$

Furthermore, we have $(\partial i_C)^{-1}0 = C$ and $(A + \partial i_C)^{-1}0 = VI(C, A)$. In fact, we have that for $z \in C$,

$$\begin{aligned} z \in (A + \partial i_C)^{-1}0 &\Longleftrightarrow 0 \in Az + \partial i_C z \\ &\Longleftrightarrow 0 \in Az + N_C z \\ &\Longleftrightarrow -Az \in N_C z \\ &\Longleftrightarrow \langle -Az, v - z\rangle \leq 0, \ \forall v \in C \\ &\Longleftrightarrow \langle Az, v - z\rangle \geq 0, \ \forall v \in C \\ &\Longleftrightarrow z \in VI(C, A). \end{aligned}$$

Thus we obtain the desired result by Theorem 7. □

Let C be a nonempty closed convex subset of H. Then, $U : C \to H$ is called a widely strict pseudo-contraction if there exists $r \in \mathbb{R}$ with $r < 1$ such that

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + r\|(I - U)x - (I - U)y\|^2, \quad \forall x, y \in C.$$

We call such U a widely r-strict pseudo-contraction. If $0 \leq r < 1$, then U is a strict pseudo-contraction. Furthermore, if $r = 0$, then U is nonexpansive. Conversely, let $T : C \to H$ be a nonexpansive mapping and define $U : C \to H$ by $U = \frac{1}{1+n}T + \frac{n}{1+n}I$ for all $x \in C$ and $n \in \mathbb{N}$. Then U is a widely $(-n)$-strict pseudo-contraction. In fact, from the definition of U, it follows that $T = (1+n)U - nI$. Since T is nonexpansive, we have that for any $x, y \in C$,

$$\|(1 + n)Ux - nx - ((1 + n)Uy - ny)\|^2 \leq \|x - y\|^2$$

and hence

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 - n\|(I - U)x - (I - U)y\|^2.$$

Using Theorem 7, we obtain the following strong convergence theorem [28] which is related to Zhou's result [28] in a Hilbert space.

Theorem 9. *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $r \in \mathbb{R}$ with $r < 1$ and let U be a widely r-strict pseudo-contraction of C into H such that $F(U) \neq \emptyset$. Let $x_1 = x \in C$ and let $\{x_n\} \subset C$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)\{\alpha_n u + (1 - \alpha_n)P_C\{(1 - t_n)U + t_n I\}x_n\}$$

for all $n \in \mathbb{N}$, where $\{t_n\} \subset (-\infty, 1)$, $\{\beta_n\} \subset (0, 1)$, and $\{\alpha_n\} \subset (0, 1)$ satisfy

$$r \leq t_n \leq b < 1, \quad 0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1,$$

$$\lim_{n\to\infty} \alpha_n = 0, \quad \textit{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then, the sequence $\{x_n\}$ *converges strongly to a point* z_0 *of* $F(U)$*, where* $z_0 = P_{F(U)}u$*.*

Proof. Put $B = F = \partial i_C$ and $A = I - U$ in Theorem 7. Furthermore, put $g(x) = u$ and $G(x) = x$ for all $x \in H$. Then

$$\langle G(x), x\rangle = \|x\|^2 \geq \frac{1}{2}\|x\|^2$$

Thus we have $\overline{\gamma} = \frac{1}{2}$. Since $\|g(x) - g(y)\| = 0 \leq \frac{1}{3}\|x - y\|$ for all $x, y \in H$, we can take $k = \frac{1}{3}$ and hence set $\gamma = 1$. Putting $a = 1 - b, \lambda_n = 1 - t_n$ and $2\alpha = 1 - r$ in Theorem 7, we get from $r \leq t_n \leq b < 1$ that $0 < a \leq \lambda_n \leq 2\alpha$,

$$\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| = \sum_{n=1}^{\infty} |t_{n+1} - t_n| < \infty$$

and

$$I - \lambda_n A = I - (1 - t_n)(I - U) = (1 - t_n)U + t_n I.$$

Furthermore, we have that for $z \in C$,

$$\begin{aligned} z \in (A + \partial i_C)^{-1}0 &\iff 0 \in Az + \partial i_C z \\ &\iff 0 \in z - Uz + N_C z \\ &\iff Uz - z \in N_C z \\ &\iff \langle Uz - z, v - z\rangle \leq 0, \ \forall v \in C \\ &\iff P_C U z = z. \end{aligned}$$

Since $F(U) \neq \emptyset$, we get, as in the proof of [28, Fact 3], that $F(P_C U) = F(U)$. We also have $z_0 = P_{F(U)}(I - G + \gamma g)z_0 = P_{F(U)}(z_0 - z_0 + 1 \cdot u) = P_{F(U)}u$. Thus we obtain the desired result by Theorem 7. □

Let $f : C \times C \to \mathbb{R}$ be a bifunction which satisfies the conditions $(A1) - (A4)$ in Introduction. Then, we know the following lemma which appears implicitly in Blum and Oettli [4].

Lemma 10 (Blum and Oettli)**.** *Let* C *be a nonempty closed convex subset of* H *and let* f *be a bifunction of* $C \times C$ *into* $\mathbb{R}$ *satisfying* $(A1) - (A4)$*. Let* $r > 0$ *and* $x \in H$*. Then, there exists* $z \in C$ *such that*

$$f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C.$$

The following lemma was also given in Combettes and Hirstoaga [7].

Lemma 11. *Assume that $f : C \times C \to \mathbb{R}$ satisfies $(A1) - (A4)$. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r x = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C \right\}$$

for all $x \in H$. Then, the following hold:

(1) *T_r is single-valued;*

(2) *T_r is a firmly nonexpansive mapping, i.e., for all $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3) *$F(T_r) = EP(f)$;*

(4) *$EP(f)$ is closed and convex.*

We call such T_r the resolvent of f for $r > 0$. Using Lemmas 10 and 11, Takahashi, Takahashi and Toyoda [19] obtained the following lemma. See [1] for a more general result.

Lemma 12. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $f : C \times C \to \mathbb{R}$ satisfy $(A1) - (A4)$. Let A_f be a set-valued mapping of H into itself defined by*

$$A_f x = \begin{cases} \{z \in H : f(x, y) \geq \langle y - x, z\rangle, \ \forall y \in C\}, & \forall x \in C, \\ \emptyset, \quad \forall x \notin C. \end{cases}$$

Then, $EP(f) = A_f^{-1}0$ and A_f is a maximal monotone operator with $\operatorname{dom}(A_f) \subset C$. Furthermore, for any $x \in H$ and $r > 0$, the resolvent T_r of f coincides with the resolvent of A_f, i.e.,

$$T_r x = (I + rA_f)^{-1}x.$$

Using Theorem 7, we obtain the following strong convergence theorem which is related to Liu's result [10] for strict pseudo-contractions in a Hilbert space.

Theorem 13. *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $r \in \mathbb{R}$ with $r < 1$ and let U be a widely r-strict pseudo-contraction of C into H and let f be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A1) - (A4)$. Let T_r be the resolvent of f for $r > 0$. Let $0 < k < 1$ and let g be a k-contraction of H into itself. Let G be a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma} > 0$. Let $0 < \gamma < \frac{\overline{\gamma}}{k}$ and suppose $F(U) \cap EP(f) \neq \emptyset$. Let $x_1 = x \in H$ and let $\{x_n\} \subset H$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)\{\alpha_n \gamma g(x_n) + (I - \alpha_n G)\{(1 - t_n)U + t_n I\}T_{r_n} x_n\}$$

for all $n \in \mathbb{N}$*, where* $\{t_n\} \subset (-\infty, 1)$*,* $\{\beta_n\} \subset (0,1)$*,* $\{\alpha_n\} \subset (0,1)$ *and* $\{r_n\} \subset (0,\infty)$ *satisfy*

$$r \le t_n \le b < 1, \quad 0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1,$$

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \textit{and} \quad \liminf_{n\to\infty} r_n > 0.$$

Then, the sequence $\{x_n\}$ *converges strongly to a point* z_0 *of* $F(U) \cap EP(f)$*, where* $z_0 = P_{F(U)\cap EP(f)}(I - G + \gamma g)z_0$*.*

Proof. For the bifunction $f : C \times C \to \mathbb{R}$, we can define A_f in Lemma 12. Putting $A = I - U$, $Bx = 0$ for all $\in H$ and $F = A_f$ in Theorem 7, we obtain from Lemma 12 that $J_{\lambda_n} = I$ for all $\lambda_n > 0$ and $T_{r_n} = (I + r_n A_f)^{-1}$ for all $r_n > 0$. As in the proof of Theorem 9, the sequence $\{t_n\}$ and U are changed in $\{\lambda_n\}$ and A. We have also from Lemma 12 that $EP(f) = (A_f)^{-1}0 = F^{-1}0$. Furthermore, we have that for $z \in C$,

$$\begin{aligned} z \in (A+B)^{-1}0 &\Longleftrightarrow 0 = Az + Bz \\ &\Longleftrightarrow 0 = Az \\ &\Longleftrightarrow z = Uz \\ &\Longleftrightarrow z \in F(U). \end{aligned}$$

So, we obtain the desired result by Theorem 7. □

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