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CHARACTERIZATION AND AUTOMATIC CONTINUITY OF SEPARATING MAPS BETWEEN BANACH MODULES

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ABSTRACT. A linear map $T : \mathcal{A} \longrightarrow \mathcal{B}$ between algebras (or spaces of functions) $\mathcal{A}$ and $\mathcal{B}$ is called separating if $x \cdot y = 0$ implies $Tx \cdot Ty = 0$ for all $x, y \in \mathcal{A}$. It is well known that a separating map between certain commutative semisimple Banach algebras is very close to being a weighted composition operator on the maximal ideal spaces. In this paper, after introducing the notion of the cozero set for the elements of a Banach module, we first extend the notion of the separating maps to Banach module case. Our approach depends on the notion of point multipliers on a Banach module $\mathcal{X}$ and the relation between hyper maximal submodules of $\mathcal{X}$ and point multipliers on it. Then we generalize some well known results about separating maps between certain subspaces of continuous functions to Banach module case. In particular, we show that, imposing some additional assumptions on Banach modules, such map can be represented as a variation of a weighted composition operator. We also obtain a result concerning the automatic continuity of a bijective separating map whose inverse is also separating.

1. INTRODUCTION

Given two arbitrary algebras (or spaces of functions) $\mathcal{A}$ and $\mathcal{B}$, a linear map $T : \mathcal{A} \longrightarrow \mathcal{B}$ is said to be *separating* if $Tx \cdot Ty = 0$ for all $x, y \in \mathcal{A}$ with $x \cdot y = 0$. If $\mathcal{A}$ and $\mathcal{B}$ are spaces of complex-valued functions on some topological spaces, then a linear map $T : \mathcal{A} \longrightarrow \mathcal{B}$ is separating whenever for each $f, g \in \mathcal{A}$ with disjoint cozero sets, their images under T have disjoint cozero sets as well.

Separating maps between L_p-spaces were studied by Banach in [5]. Later on, J. Lamperti [16] and W. Arendt [4] continued studying such maps called Lamperti operators.

Evidently every algebra homomorphism is a separating map. Weighted composition operators are typical examples of separating maps between algebras of functions.

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In fact, in certain important cases, separating maps are very close to being a homomorphism or a weighted composition operator. For instance, for the supremum norm Banach algebras $C(X)$ and $C(Y)$ of all continuous complex-valued functions on compact Hausdorff spaces X and Y, respectively, a description of a separating map $T : C(X) \longrightarrow C(Y)$ was given by K. Jarosz in [11] and it was shown that if T is bijective, then T is automatically continuous and is a weighted composition operator which induces a homeomorphism between X and Y. These results have been extended by J.J. Font in [9] for the case where $C(X)$ and $C(Y)$ are replaced by regular commutative semisimple Banach algebras. Fredholm separating maps between the Banach algebras $C_0(X)$ and $C_0(Y)$ of all continuous complex-valued functions on locally compact Hausdorff spaces X and Y vanishing at infinity, were studied by J.S. Jeang and N.C. Wong in [13] and it was proved that when such operator exists, X and Y are homeomorphic after removing finite subsets.

In recent years there has been a considerable attention to study separating maps between other algebras. We refer to [18] for some results on separating maps between certain operator algebras, to [15] for the study of such maps between algebras of differentiable functions and to [2] and [14] for separating maps between certain spaces of Lipschitz functions.

We should note that the notion of separating maps between vector lattices E and F is also well-known. For such structures a linear map $T : E \longrightarrow F$ is separating if $|f| \wedge |g| = 0$ implies $|Tf| \wedge |Tg| = 0$, see [1] and [4] for some related results in the lattice case.

In this paper we shall develop the notion of separating maps to Banach module case and investigate the well known results for this more general case. Our approach depends on the notion of point multipliers and hyper maximal submodules of a Banach module and their properties given in [7]. Using this idea we define the notion of cozero set for the elements of a Banach module and then we extend the results of [9] to give a representation for a separating map between certain Banach modules. We show that under certain conditions a linear separating map $T : \mathcal{X} \longrightarrow \mathcal{Y}$ between (left) Banach modules $\mathcal{X}$ and $\mathcal{Y}$ over commutative semisimple Banach algebras can be represented as a variation of weighted composition operators. We also give a result concerning the automatic continuity of a bijective separating map whose inverse is also separating.

2. Preliminaries

Let $\mathcal{A}$ be a Banach algebra with or without unit. We denote the set of all characters (non-trivial complex homomorphisms) on $\mathcal{A}$ by $\sigma(\mathcal{A})$ and in the commutative case we may call it the maximal ideal space of $\mathcal{A}$. The standard unitization of A is denoted by $\mathcal{A}_1$. Hence, for a commutative Banach algebra $\mathcal{A}$, the maximal ideal space of A_1 can be identified by $\sigma_0(\mathcal{A}) := \sigma(\mathcal{A}) \cup \{0\}$. The Jacobson radical of $\mathcal{A}$ will be denoted by $\mathrm{Rad}(\mathcal{A})$ and when $\sigma(\mathcal{A}) \neq \emptyset$, $\widehat{a}$ denotes the *Gelfand transformation* of $a \in \mathcal{A}$.

Let $\mathcal{A}$ be a commutative Banach algebra. For a non-empty subset $S \subseteq \sigma(\mathcal{A})$, the kernel of S is defined by $k_{\mathcal{A}}(S) = \cap_{\varphi \in S} \ker(\varphi)$ and for an ideal J in $\mathcal{A}$ the hull of J is defined by $h_{\mathcal{A}}(J) = \{\varphi \in \sigma(\mathcal{A}) : J \subseteq \ker \varphi\}$.

A commutative Banach algebra $\mathcal{A}$ is said to satisfy the *Ditkin's condition* if for every $a \in \mathcal{A}$ and $\varphi \in \sigma(\mathcal{A})$ with $\widehat{a}(\varphi) = 0$, there exists a sequence $\{a_n\}$ of elements of $\mathcal{A}$ such that each $\widehat{a}_n$, $n \in \mathbb{N}$, vanishes on a neighborhood of φ and $\|a \cdot a_n - a\| \longrightarrow 0$. If, furthermore, $\mathcal{A}$ is not unital, then for every $a \in \mathcal{A}$ there must exist a sequence $\{a_n\}$ in $\mathcal{A}$ such that each $\widehat{a_n}$ has a compact support and $\|a \cdot a_n - a\| \longrightarrow 0$.

We should note that in the above definition of Ditkin's condition (which comes from [10]), for a point $\varphi \in \sigma(\mathcal{A})$ and $a \in \mathcal{A}$ with $\widehat{a}(\varphi) = 0$ it is not assumed that the elements of the corresponding sequence $\{a_n\}$ has compact support. However, some authors consider this additional assumption in the definition, see for example [8].

Let $\mathcal{A}$ be a Banach algebra. We say that a left Banach $\mathcal{A}$-module $\mathcal{X}$ is *essential* if $\overline{\mathcal{A} \cdot \mathcal{X}} = \mathcal{X}$, where $\mathcal{A} \cdot \mathcal{X} = \mathrm{span}\{a \cdot x : a \in \mathcal{A}, x \in \mathcal{X}\}$. It is evident that every left Banach $\mathcal{A}$-module $\mathcal{X}$ can be considered as a left Banach $\mathcal{A}_1$-module with the module action defined by $(a, \lambda) \cdot x = a \cdot x + \lambda x$, for $a \in \mathcal{A}$, $\lambda \in \mathbb{C}$ and $x \in \mathcal{X}$. Clearly, under this module action $\mathcal{X}$ is *unital*, i.e. $1_{\mathcal{A}_1} \cdot x = x$ for every $x \in \mathcal{X}$. Let $\mathcal{X}$ be a left Banach module over a Banach algebra $\mathcal{A}$. Following [7], for a point $\varphi \in \sigma_0(\mathcal{A})$, a linear functional $\xi \in \mathcal{X}^*$ is said to be a *point multiplier at* φ if $\langle \xi, a \cdot x \rangle = \varphi(a)\langle \xi, x \rangle$ for all $a \in \mathcal{A}$ and $x \in \mathcal{X}$. The submodules of $\mathcal{X}$ with codimension one will be referred to as *hyper maximal (left) submodules* of $\mathcal{X}$. Obviously the kernel of each non-trivial point multiplier is a closed hyper maximal left submodule of $\mathcal{X}$. It is easy to see that for any closed hyper maximal submodule P of $\mathcal{X}$ there exists a non-trivial point multiplier $\xi \in \mathcal{X}^*$ at some point $\varphi \in \sigma_0(\mathcal{A})$, such that $P = \ker(\xi)$ (see for example [6] for the unital case). If, moreover, $\mathcal{X}$ is essential, then there is no non-trivial point multiplier at 0 and so in this case any closed hyper maximal submodule P of $\mathcal{X}$ is the kernel of a point multiplier at some point of $\sigma(\mathcal{A})$. We denote the set of all closed hyper maximal left submodules of $\mathcal{X}$ by $\Delta_{\mathcal{A}}(\mathcal{X})$ and we set $\Delta^1_{\mathcal{A}}(\mathcal{X}) = \Delta_{\mathcal{A}}(\mathcal{X}) \cup \{\mathcal{X}\}$.

Let $\mathcal{A}$ be a commutative Banach algebra and $\mathcal{X}$ be an essential left Banach $\mathcal{A}$-module. We define the *natural map* $\nu_X : \Delta_{\mathcal{A}}(\mathcal{X}) \longrightarrow \sigma(\mathcal{A})$ in such a way that for each

$P \in \Delta_{\mathcal{A}}(\mathcal{X})$, $\nu_{\mathcal{X}}(P)$ is the unique point in $\sigma(\mathcal{A})$ corresponding to $P \in \Delta_{\mathcal{A}}(\mathcal{X})$. We also consider the extended map $\widetilde{\nu}_{\mathcal{X}} : \Delta^1_{\mathcal{A}}(\mathcal{X}) \longrightarrow \sigma_0(\mathcal{A})$ of $\nu_{\mathcal{X}}$ by setting $\widetilde{\nu}_{\mathcal{X}}(\mathcal{X}) = 0$.

In the sequel we assume that $\mathcal{A}$ is a commutative Banach algebra with non-empty maximal ideal space and $\mathcal{X}$ is an essential left Banach $\mathcal{A}$-module with $\Delta_{\mathcal{A}}(\mathcal{X}) \neq \emptyset$.

Based on [7], we call a multiplier T on $\mathcal{X}$ a *simple multiplier* if $T(P) \subseteq P$ for each $P \in \Delta_{\mathcal{A}}(\mathcal{X})$. The *annihilator* of a subset M of $\mathcal{X}$, is the set $\mathrm{ann}_{\mathcal{A}}(M) = \{a \in \mathcal{A} : a \cdot M = \{0\}\}$ and the *Gelfand radical* of $\mathcal{X}$ is defined by $\mathrm{rad}_{\mathcal{A}}(\mathcal{X}) = \bigcap_{P \in \Delta_{\mathcal{A}}(\mathcal{X})} P$ (see [7]). We say that $\mathcal{X}$ is *hyper semisimple* if $\mathrm{rad}_{\mathcal{A}}(\mathcal{X}) = \{0\}$. For a non-empty subset $S \subseteq \Delta^1_{\mathcal{A}}(\mathcal{X})$, the kernel of S denoted by $k_{\mathcal{X}}(S)$ is defined by $k_{\mathcal{X}}(S) = \cap_{P \in S} P$, we also set $k_{\mathcal{X}}(\emptyset) = \mathcal{X}$. For an element $Q \in \Delta^1_{\mathcal{A}}(\mathcal{X})$ we may use the notation $k_{\mathcal{X}}(Q)$ for $k_{\mathcal{X}}(\{Q\})$ which is clearly equal to Q. For a submodule M of $\mathcal{X}$ we set $(M :_{\mathcal{A}} \mathcal{X}) = \{a \in \mathcal{A} : a \cdot \mathcal{X} \subseteq M\}$ and the hull of M is defined by $h_{\mathcal{X}}(M) = \{P \in \Delta_{\mathcal{A}}(\mathcal{X}) : (M :_{\mathcal{A}} \mathcal{X}) \subseteq (P :_{\mathcal{A}} \mathcal{X})\}$. Clearly for any subset S of $\Delta_{\mathcal{A}}(\mathcal{X})$, $k_{\mathcal{X}}(S)$ is a closed submodule of $\mathcal{X}$ containing $\mathrm{rad}_{\mathcal{A}}(\mathcal{X})$. The corresponding topology to the closure operation $S \mapsto h_{\mathcal{X}} k_{\mathcal{X}}(S), S \subseteq \Delta_{\mathcal{A}}(\mathcal{X})$, is called the *hull-kernel topology* on $\Delta_{\mathcal{A}}(\mathcal{X})$. For some preliminaries about the hulls and kernels $h_{\mathcal{X}}(M)$ and $k_{\mathcal{X}}(S)$, we refer the reader to [7].

We note that a linear functional $\xi \in \mathcal{X}^*$, is a non-trivial point multiplier on the left Banach A-module $\mathcal{X}$ at a point $\varphi \in \sigma(\mathcal{A})$ if and only if ξ is a non-trivial point multiplier on $\mathcal{X}$ as a left $\mathcal{A}_1$-module at the extension of φ on $\mathcal{A}_1$. Thus, since $\mathcal{X}$ is assumed to be essential we get easily that $\Delta_{\mathcal{A}}(\mathcal{X}) = \Delta_{\mathcal{A}_1}(\mathcal{X})$ and consequently $\mathcal{X}$ is a hyper semisimple left Banach $\mathcal{A}$-module if and only if it is hyper semisimple as a left Banach $\mathcal{A}_1$-module.

The next proposition has been proved in [7] for the case that $\mathcal{A}$ is unital and $\mathcal{X}$ is a unital left $\mathcal{A}$-module.

Proposition 2.1. *The following statements hold:*

(a) $k_{\mathcal{A}}(\nu_{\mathcal{X}}(S)) = (k_{\mathcal{X}}(S) :_{\mathcal{A}} \mathcal{X})$*, for each* $S \subseteq \Delta_{\mathcal{A}}(\mathcal{X})$.

(b) *If* $\sigma(\mathcal{A})$ *and* $\Delta_{\mathcal{A}}(\mathcal{X})$ *are endowed with their corresponding hull-kernel topologies, then the natural map* $\nu_X : \Delta_{\mathcal{A}}(\mathcal{X}) \longrightarrow \sigma(\mathcal{A})$ *is continuous and sends every closed (open) subset of* $\Delta_{\mathcal{A}}(\mathcal{X})$ *to a closed (open) subset of* $\nu_{\mathcal{X}}(\Delta_{\mathcal{A}}(\mathcal{X}))$.

(c) *If* $\mathcal{X}$ *is hyper semisimple and* $\nu_{\mathcal{X}}$ *is surjective, then* $\mathrm{Rad}(\mathcal{A}) = \mathrm{ann}_{\mathcal{A}}(\mathcal{X}) = \mathrm{ann}_{\mathcal{A}_1}(\mathcal{X})$.

Proof. The proofs of (a) and (b) are minor modifications of Propositions 3.4 and 3.5 in [7].

(c) Using part (a), since $\nu_{\mathcal{X}}$ is surjective and $\mathcal{X}$ is hyper semisimple, we have

$$\mathrm{Rad}(\mathcal{A}) = k_{\mathcal{A}}(\sigma(\mathcal{A})) = k_{\mathcal{A}}(\nu_{\mathcal{X}}(\Delta_{\mathcal{A}}(\mathcal{X}))) = (k_{\mathcal{X}}(\Delta_{\mathcal{A}}(\mathcal{X})) :_{\mathcal{A}} \mathcal{X}) = (0 :_{\mathcal{A}} \mathcal{X}) = \mathrm{ann}_{\mathcal{A}}(\mathcal{X}).$$

Consider now $\mathcal{X}$ as an $\mathcal{A}_1$-module. Then, as we noted before, $\Delta_{\mathcal{A}_1}(\mathcal{X}) = \Delta_{\mathcal{A}}(\mathcal{X})$ and so $\nu_{\mathcal{X}}$ can be considered as the corresponding natural map for the $\mathcal{A}_1$-module $\mathcal{X}$. Hence using part (a) for $\mathcal{A}_1$ instead of $\mathcal{A}$, since $\mathcal{X}$ is hyper semisimple as an $\mathcal{A}_1$-module, we get

$$\begin{aligned}\operatorname{Rad}(\mathcal{A}_1) &= k_{\mathcal{A}_1}(\sigma_0(\mathcal{A})) = k_{\mathcal{A}_1}(\sigma(\mathcal{A})) = k_{\mathcal{A}_1}(\nu_{\mathcal{X}}(\Delta_{\mathcal{A}_1}(\mathcal{X})))\\ &= (k_{\mathcal{X}}(\Delta_{\mathcal{A}_1}(\mathcal{X})) :_{\mathcal{A}_1} \mathcal{X}) = (0 :_{\mathcal{A}_1} \mathcal{X}) = \operatorname{ann}_{\mathcal{A}_1}(\mathcal{X}).\end{aligned}$$

Therefore, $\operatorname{ann}_{\mathcal{A}_1}(\mathcal{X}) = \operatorname{Rad}(\mathcal{A}_1) = \operatorname{Rad}(\mathcal{A}) = \operatorname{ann}_{\mathcal{A}}(\mathcal{X})$.

□

Remark. We note that for a commutative unital Banach algebra A, considering A as a module over itself, it follows easily from Example 2.2 in [7] that $\operatorname{Rad}(A) = \operatorname{rad}_A(A)$. It was shown also in Proposition 3.6 in [7] that if $\mathcal{A}$ is unital and $\mathcal{X}$ is a hyper semisimple unital left Banach $\mathcal{A}$-module, then the natural map $\nu_{\mathcal{X}} : \Delta_{\mathcal{A}}(\mathcal{X}) \longrightarrow \sigma(\mathcal{A})$ is surjective if and only if $\operatorname{ann}_{\mathcal{A}}(\mathcal{X}) = \operatorname{rad}_{\mathcal{A}}(\mathcal{A})$. However, the following example shows that the equality $\operatorname{ann}_{\mathcal{A}}(\mathcal{X}) = \operatorname{rad}_{\mathcal{A}}(\mathcal{A})$ is not sufficient for surjectivity of $\nu_{\mathcal{X}}$. The proof of Proposition 3.6 in [7] shows, indeed, that if $\operatorname{ann}_{\mathcal{A}}(\mathcal{X}) = \operatorname{rad}_{\mathcal{A}}(\mathcal{A})$, then $\nu_{\mathcal{X}}(\Delta_{\mathcal{A}}(\mathcal{X}))$ is hull-kernel dense in $\sigma(\mathcal{A})$.

Example 2.1. Let $\mathcal{A}$ be a commutative unital semisimple Banach algebra satisfying the Ditkin's condition. Identifying $\mathcal{A}$ with its image under the Gelfand transformation, we can assume that each element in $\mathcal{A}$ is a continuous function on $\sigma(\mathcal{A})$. Let $\sigma(\mathcal{A})$ have infinitely many points. Then there exists a non-singular point φ_0 in $\sigma(\mathcal{A})$. Consider now the (maximal) ideal $\mathcal{X} = \{f \in \mathcal{A} : f(\varphi_0) = 0\}$ in $\mathcal{A}$, which is clearly a unital Banach module over $\mathcal{A}$. Clearly for each $\varphi \in \sigma(\mathcal{A})\backslash\{\varphi_0\}$, the restriction $\varphi|_{\mathcal{X}}$ is a non-trivial point multiplier on $\mathcal{X}$. This implies easily that $\mathcal{X}$ is a hyper semisimple Banach left $\mathcal{A}$-module. On the other hand, if $\xi \in \mathcal{X}^*$ is a non-trivial point multiplier at some point $\varphi \in \sigma(\mathcal{A})$, then since $\mathcal{A}$ satisfies the Ditkin's condition, it follows easily that $\varphi \neq \varphi_0$. Choosing $g_0 \in \mathcal{X}$ with $\langle \xi, g_0\rangle = 1$, we conclude that for each $g \in \mathcal{X}$

$$\varphi(g_0)\langle \xi, g\rangle = \langle \xi, g_0 g\rangle = \langle \xi, g g_0\rangle = \varphi(g)\langle \xi, g_0\rangle = \varphi(g),$$

that is $\xi = \lambda\varphi|_{\mathcal{X}}$ for some non-zero scalar λ. Thus $\Delta_{\mathcal{A}}(\mathcal{X}) = \{\ker(\varphi) \cap \mathcal{X} : \varphi \in \sigma(\mathcal{A})\backslash\{\varphi_0\}\}$ and therefore $\nu_{\mathcal{X}} : \Delta_{\mathcal{A}}(\mathcal{X}) \longrightarrow \sigma(\mathcal{A})$ is not surjective, while since φ_0 is a non-singular point of $\sigma(\mathcal{A})$, it can be easily verified that $\operatorname{ann}_{\mathcal{A}}(\mathcal{X}) = \operatorname{rad}_{\mathcal{A}}(\mathcal{A}) = \{0\}$.

For an example of a Banach algebra satisfying the hypotheses of the above example, we can refer to the supremum norm Banach algebra $C(K)$ where K is an infinite compact Hausdorff space. For $0 < \alpha \leq 1$ and an infinite compact metric space (K, d), the Banach algebra $\operatorname{lip}(K, \alpha)$ consisting of all complex-valued functions

f on K satisfying the Lipschitz condition of order α such that $\lim \frac{|f(x)-f(y)|}{d(x,y)^\alpha} = 0$ as $d(x,y) \to 0$, has also the desired properties.

3. LINEAR SEPARATING MAPS BETWEEN BANACH MODULES

In this section we introduce the notion of cozero sets for elements of an essential left Banach module over a commutative Banach algebra. Then we extend the concept of separating maps for Banach module case and give some results related to the representation and the automatic continuity of separating maps. We need to recall that in our earlier work [19] we introduced the concept of cozero set for the elements of a Banach module, based on point multipliers, and gave some results concerning Banach module valued separating maps defined on a commutative Banach algebra. Here we apply an alternative idea and obtain more general results.

As before, in this section we assume that $\mathcal{A}$ is a commutative Banach algebra with $\sigma(\mathcal{A}) \neq \emptyset$ and $\mathcal{X}$ is an essential left Banach $\mathcal{A}$-module with $\Delta_{\mathcal{A}}(\mathcal{X}) \neq \emptyset$ unless otherwise is specified.

Definition 3.1. For each $x \in \mathcal{X}$ we define the *hyper cozero set* and the *cozero set* of x by $\mathrm{coz}_h(x) := \{P \in \Delta_{\mathcal{A}}(\mathcal{X}) : x \notin P\}$ and $\mathrm{coz}(x) := \nu_{\mathcal{X}}(\mathrm{coz}_h(x))$, respectively.

If $\mathcal{A}$ is a commutative Banach algebra such that $\overline{\mathcal{A}^2} = \mathcal{A}$, then considering $\mathcal{A}$ as a left $\mathcal{A}$-module with usual action, we see that $\Delta_{\mathcal{A}}(\mathcal{A}) = \{\ker(\varphi) : \varphi \in \sigma(\mathcal{A})\}$ and for any $a \in \mathcal{A}$, $\mathrm{coz}(a)$ is, indeed, the cozero set of the continuous function $\widehat{a}$ on $\sigma(\mathcal{A})$.

We note that since $\Delta_{\mathcal{A}_1}(\mathcal{X}) = \Delta_{\mathcal{A}}(\mathcal{X})$, it follows that the cozero set of each $x \in \mathcal{X}$ as an element of the $\mathcal{A}$-module $\mathcal{X}$ is the same as its cozero set when we consider $\mathcal{X}$ as an $\mathcal{A}_1$-module.

In the next proposition we state some elementary properties of the defined cozero sets.

Proposition 3.1. *The following statements hold:*

(a) *For each* $a \in \mathcal{A}$ *and* $x \in \mathcal{X}$*,* $\mathrm{coz}(a \cdot x) \subseteq \mathrm{coz}(\widehat{a}) \cap \mathrm{coz}(x)$.

(b) *If* $\mathcal{X}$ *is hyper semisimple and* $\mathrm{coz}(x) = \emptyset$*, then* $x = 0$.

(c) $\mathrm{coz}(x_1 + x_2) \subseteq \mathrm{coz}(x_1) \cup \mathrm{coz}(x_2)$ *for each* x_1, x_2 *in* $\mathcal{X}$.

Proof. It is straightforward. □

It should be noted that, the notion of the support for an element of a left Banach $\mathcal{A}$-module $\mathcal{X}$ is well-known, see for example [7] and [17]. In fact, the support $\mathrm{supp}_{\mathcal{A}}(x)$ of an element $x \in \mathcal{X}$ is defined as $h_A(\mathrm{ann}_A(\mathcal{X}))$, i.e. the hull of the ideal

$\mathrm{ann}_{\mathcal{A}}(x)$ in $\mathcal{A}$. The next proposition shows that if $\mathcal{X}$ is hyper semisimple, then the support of $x \in \mathcal{X}$ is the hull-kernel closure of the above defined $\mathrm{coz}(x)$.

Proposition 3.2. *If $\mathcal{X}$ is hyper semisimple, then for each $x \in \mathcal{X}$, $\mathrm{ann}_{\mathcal{A}}(x) = k_{\mathcal{A}}(\mathrm{coz}(x))$. In particular, $\mathrm{supp}_{\mathcal{A}}(x) = h_{\mathcal{A}} k_{\mathcal{A}}(\mathrm{coz}(x))$.*

Proof. Let $a \in \mathrm{ann}_{\mathcal{A}}(x)$ and $P \in \mathrm{coz}_h(x)$. Let $\xi \in \mathcal{X}^*$ be a non-trivial point multiplier such that $P = \ker(\xi)$. Then $\nu_{\mathcal{X}}(P)(a)\langle \xi, x\rangle = \langle \xi, a \cdot x\rangle = 0$ thus $\nu_{\mathcal{X}}(P)(a) = 0$, since $x \notin P$. Consequently $a \in \ker(\nu_{\mathcal{X}}(P))$ for any $P \in \mathrm{coz}_h(x)$, that is $a \in k_{\mathcal{A}}(\mathrm{coz}(x))$. Conversely, assume that $a \in k_{\mathcal{A}}(\mathrm{coz}(x))$, then for each $P \in \mathrm{coz}_h(x)$, $a \in \ker(\nu_{\mathcal{X}}(P))$ and consequently $\langle \xi, a \cdot x\rangle = \nu_{\mathcal{X}}(P)(a)\langle \xi, x\rangle = 0$. Hence $a \cdot x \in \ker(\xi) = P$ and this implies easily that $a \cdot x \in \mathrm{rad}_{\mathcal{A}}(\mathcal{X}) = \{0\}$, i.e. $a \cdot x = 0$ as desired. □

Definition 3.2. Let A and B be commutative Banach algebras and X and Y be essential left Banach modules over A and B, respectively. We say that a linear map $T : X \longrightarrow Y$ is *separating* if $\mathrm{coz}(Tx_1) \cap \mathrm{coz}(Tx_2) = \emptyset$ whenever $\mathrm{coz}(x_1) \cap \mathrm{coz}(x_2) = \emptyset$, for each $x_1, x_2 \in X$. A linear bijective map $T : X \longrightarrow Y$ is *biseparating* if T and T^{-1} are both separating.

If A and B are commutative semisimple Banach algebras which are unital, or more generally each one is essential as a Banach module over itself, then, since the cozero sets of the algebra elements are the cozero sets of their Gelfand representations, every linear separating map $T : \mathcal{A} \longrightarrow \mathcal{B}$ in the usual sense is separating for the above defined module case and vice versa. The other examples of separating mapa are as follows.

Example 3.1. (a) If X is an essential left Banach module over a commutative Banach algebra A and $T : X \longrightarrow X$ is a linear map satisfying $T(P) \subseteq P$ for each $P \in \Delta_A(X)$, then it is easy to see that T is a separating map. In particular, every simple multiplier on X is a separating map.

(b) Let A be a commutative Banach algebra and X, Y be two essential left Banach A-modules such that ν_Y is injective, for instance for $A = C([0,1])$ and $Y = C([0,1])^*$, the natural map ν_Y is injective by Example 4.4 in [7]. Let $\Phi : \Delta_A(Y) \longrightarrow \Delta_A(X)$ be an arbitrary map and $T : X \longrightarrow X$ be a linear map such that $T(\Phi(P)) \subseteq P$ for each $P \in \Delta_A(Y)$. Then it can be easily verified that T is separating.

(c) Let A be a commutative semisimple Banach algebra such that A^2 is dense in A. Considering A as a Banach A_1-module, we have $\Delta_{A_1}(A) = \{\ker(\varphi) : \varphi \in \sigma(A)\}$. Moreover, every multiplier on A is a simple multiplier by Example 4.3 in [7], and so is a separating map.

Remark. It should be noted that for a commutative semisimple Banach algebra A, since closed codimensional one ideals in A are not necessarily modular, it follows that $\Delta_{A_1}(A)$ is not necessarily the set of all maximal modular ideals in A without assuming that A^2 is dense in A. Consider, for example, the Banach space $A = C([0,1])$ of all continuous complex-valued functions on $[0,1]$ equipped with the new defined product $f \circ g(t) = tf(t)g(t)$, for $f, g \in A$ and $t \in [0,1]$. Then one can see that $\sigma(A) = \{t\delta_t : t \in (0,1]\}$, where for each $t \in [0,1]$, δ_t is the evaluation functional at t. Hence A is semisimple and $\ker(\delta_0)$ is a one codimensional closed ideal in A which is not modular. However, it is easy to see that, this is true if we assume, furthermore, that A^2 is dense in A.

In the sequel we assume that $\mathcal{A}$ is a commutative semisimple regular Banach algebra, $\mathcal{B}$ is a commutative Banach algebra and $\mathcal{X}$ and $\mathcal{Y}$ are essential left Banach modules over $\mathcal{A}$ and $\mathcal{B}$, respectively, such that $\Delta_{\mathcal{A}}(\mathcal{X})$ and $\Delta_{\mathcal{B}}(Y)$ are non-empty and the natural map $\widetilde{\nu}_{\mathcal{X}} : \Delta^1_{\mathcal{A}}(\mathcal{X}) \longrightarrow \sigma_0(\mathcal{A})$ is surjective. We also assume that $T : \mathcal{X} \longrightarrow \mathcal{Y}$ is a linear separating map.

For every $P \in \Delta_{\mathcal{B}}(\mathcal{Y})$, let $\delta_P : \mathcal{Y} \longrightarrow \mathcal{Y}/P$ be the quotient map $\delta_P(y) = y + P$, $y \in \mathcal{Y}$. We now divide $\Delta_{\mathcal{B}}(\mathcal{Y})$ into three disjoint parts as follows

$$\Delta_0(\mathcal{Y}) = \{P \in \Delta_{\mathcal{B}}(\mathcal{Y}) : \delta_P \circ T = 0\},$$
$$\Delta_c(\mathcal{Y}) = \{P \in \Delta_{\mathcal{B}}(\mathcal{Y}) \backslash \Delta_0(\mathcal{Y}) : \delta_P \circ T \text{ is continuous}\},$$
$$\Delta_d(\mathcal{Y}) = \{P \in \Delta_{\mathcal{B}}(\mathcal{Y}) \backslash \Delta_0(\mathcal{Y}) : \delta_P \circ T \text{ is discontinuous}\}.$$

Definition 3.3. For any $P \in \Delta_{\mathcal{B}}(\mathcal{Y}) \backslash \Delta_0(\mathcal{Y})$, we define $V(\delta_P \circ T)$ as the set of all $Q \in \Delta^1_{\mathcal{A}}(\mathcal{X})$ such that for any open neighborhood U of $\widetilde{\nu}_{\mathcal{X}}(Q)$ in $\sigma_0(\mathcal{A})$, there exists a point $x \in \mathcal{X}$ with $\text{coz}(x) \subseteq U$ and $Tx \notin P$. We call the set $V(\delta_P \circ T)$ the *support* of T at P.

Proposition 3.3. *For any $P \in \Delta_{\mathcal{B}}(\mathcal{Y}) \backslash \Delta_0(\mathcal{Y})$, the set $V(\delta_P \circ T)$ is non-empty.*

Proof. Assume on the contrary that $P \in \Delta_{\mathcal{B}}(\mathcal{Y}) \backslash \Delta_0(\mathcal{Y})$ and $V(\delta_P \circ T) = \emptyset$. Then by the definition of $V(\delta_P \circ T)$, for each $Q \in \Delta^1_{\mathcal{A}}(\mathcal{X})$, there exists an open neighborhood U_Q of $\widetilde{\nu}_{\mathcal{X}}(Q)$ in $\sigma_0(\mathcal{A})$ such that for any $x \in \mathcal{X}$ with $\text{coz}(x) \subseteq U_Q$, we have $Tx \in P$. Since $\widetilde{\nu}_{\mathcal{X}}$ is assumed to be surjective $\sigma_0(\mathcal{A}) = \bigcup_{Q \in \Delta^1_{\mathcal{A}}(\mathcal{X})} U_Q$ and so the compactness of $\sigma_0(\mathcal{A})$ implies that there exist $Q_1, ..., Q_n \in \Delta^1_{\mathcal{A}}(\mathcal{X})$ such that $\sigma_0(\mathcal{A}) = \bigcup_{i=1}^n U_{Q_i}$. By the regularity of $\mathcal{A}_1$, we can find elements $a_i \in \mathcal{A}_1$, $i = 1, 2, ..., n$, such that $\text{coz}(\widehat{a}_i) \subseteq U_{Q_i}$ for each $1 \leq i \leq n$ and $\sum_{i=1}^n \widehat{a}_i = 1$ on $\sigma_0(\mathcal{A})$. Hence $\sum_{i=1}^n a_i = 1_{\mathcal{A}_1}$ since $\mathcal{A}_1$ is semisimple. Considering $\mathcal{X}$ as an $\mathcal{A}_1$-module we have $x = 1_{\mathcal{A}_1} \cdot x = (\sum_{i=1}^n a_i) \cdot x = \sum_{i=1}^n a_i \cdot x$ for each $x \in \mathcal{X}$. Since for $i = 1, ..., n$ the cozero set $\text{coz}(a_i \cdot x)$ is contained in $\text{coz}(\widehat{a}_i)$ it follows that $\text{coz}(a_i \cdot x) \subseteq U_{Q_i}$. Therefore, $T(a_i \cdot x) \in P$ for each $i = 1, ..., n$, and consequently $\delta_P \circ T(x) = \sum_{i=1}^n \delta_P(T(a_i \cdot x)) = 0$, that is, $P \in \Delta_0(\mathcal{Y})$, which is a contradiction. $\square$

Proposition 3.4. *Let $P \in \Delta_{\mathcal{B}}(\mathcal{Y})\backslash\Delta_0(\mathcal{Y})$.*

(a) *If $Q \in V(\delta_P \circ T)$ and $x \in \mathcal{X}$ such that $\widetilde{\nu}_{\mathcal{X}}(Q) \notin \mathrm{cl}_{\sigma_0(\mathcal{A})}(\mathrm{coz}(x))$, where $\mathrm{cl}_{\sigma_0(\mathcal{A})}(\cdot)$ denotes the closure in $\sigma_0(\mathcal{A})$, then $Tx \in P$.*

(b) *If $Q_1, Q_2 \in V(\delta_P \circ T)$, then $\widetilde{\nu}_{\mathcal{X}}(Q_1) = \widetilde{\nu}_{\mathcal{X}}(Q_2)$.*

Proof. (a) Set $U = \sigma_0(\mathcal{A})\backslash\mathrm{cl}_{\sigma_0(\mathcal{A})}(\mathrm{coz}(x))$. Then U is a neighborhood of $\widetilde{\nu}_{\mathcal{X}}(Q)$ in $\sigma_0(\mathcal{A})$ and so, there exists $x_0 \in \mathcal{X}$ such that $\mathrm{coz}(x_0) \subseteq U$ and $Tx_0 \notin P$, i.e. $\nu_{\mathcal{Y}}(P) \in \mathrm{coz}(Tx_0)$. Since $\mathrm{coz}(x_0)\cap\mathrm{coz}(x) = \emptyset$, it follows that $\mathrm{coz}(Tx_0)\cap\mathrm{coz}(Tx) = \emptyset$ and so $\nu_{\mathcal{Y}}(P) \notin \mathrm{coz}(Tx)$, that is $Tx \in P$.

(b) Let $Q_1, Q_2 \in V(\delta_p \circ T)$. Assume that $\widetilde{\nu}_{\mathcal{X}}(Q_1) \neq \widetilde{\nu}_{\mathcal{X}}(Q_2)$ and choose disjoint neighborhoods U_1 and U_2 of $\widetilde{\nu}_{\mathcal{X}}(Q_1)$ and $\widetilde{\nu}_{\mathcal{X}}(Q_2)$, respectively. Since $Q_1, Q_2 \in V(\delta_P \circ T)$, we can find elements $x_1, x_2 \in \mathcal{X}$ such that $\mathrm{coz}(x_1) \subseteq U_1$, $\mathrm{coz}(x_2) \subseteq U_2$, $Tx_1 \notin P$ and $Tx_2 \notin P$. Then clearly $\mathrm{coz}(x_1) \cap \mathrm{coz}(x_2) = \emptyset$, and therefore $\mathrm{coz}(Tx_1) \cap \mathrm{coz}(Tx_2) = \emptyset$ while $\nu_{\mathcal{Y}}(P) \in \mathrm{coz}(Tx_1) \cap \mathrm{coz}(Tx_2)$, which is a contradiction. □

In the following we impose an additional assumption on $\mathcal{X}$ which will be called hyper Ditkin's condition to ensure that for each $P \in \Delta_{\mathcal{B}}(\mathcal{Y})\backslash\Delta_0(\mathcal{Y})$, the support set $V(\delta_P \circ T)$ is a singleton.

Definition 3.4. Let A be a commutative Banach algebra and X be a left essential Banach A-module. We say that X satisfies the *hyper Ditkin's* condition if for each $x \in X$ and $Q \in \Delta_A^1(X)$ with $x \in Q$, there exists a sequence $\{a_n\}$ in A such that each $\widehat{a_n}$ vanishes on a neighborhood of $\widetilde{\nu}_X(Q)$ in $\sigma_0(A)$ and $\|a_n.x - x\| \longrightarrow 0$ in X.

Remark. If A is a commutative Banach algebra satisfying the Ditkin's condition, then since $\overline{A^2} = A$, we have $\Delta_A(A) = \{\ker(\varphi) : \varphi \in \sigma(A)\}$ and it follows easily that A satisfies the hyper Ditkin's condition as a module over itself. It is also easy to see that for each $\varphi_0 \in \sigma(A)$ the maximal modular ideal $M = \ker(\varphi_0)$ in A is an essential A-module with $\Delta_A(M) = \{\ker(\varphi|_M) : \varphi \in \sigma(A)\backslash\{\varphi_0\}\}$, and consequently M satisfies the hyper Ditkin's condition as well.

Example 3.2. Let A be a non-unital commutative Banach algebra and let X be a dense ideal in A. Assume, furthermore, that $\|.\|_X$ is a norm on X making it a Banach algebra satisfying the Ditkin's condition such that $\|a \cdot x\|_X \leq \|a\|\,\|x\|_X$ holds for all $a \in A$ and $x \in X$. Then the restriction map $\varphi \mapsto \varphi|_X$ defines a continuous injective map from $\sigma(A)$ onto $\sigma(X)$ and it is easy to see that X is an essential A-module with $\Delta_A(X) = \{\ker(\varphi|_X) : \varphi \in \sigma(A)\}$, and consequently X satisfies the hyper Ditkin's condition. In particular, every Segal algebra on a locally compact abelian group G satisfies the hyper Ditkin's condition as an $L_1(G)$-module.

Lemma 3.1. *Let $\mathcal{X}$ satisfy the hyper Ditkin's condition. Then for each $P \in \Delta_{\mathcal{B}}(\mathcal{Y})\backslash\Delta_0(\mathcal{Y})$, the support $V(\delta_P \circ T)$ of T is a singleton.*

Proof. Let $P \in \Delta_B(\mathcal{Y})\backslash\Delta_0(\mathcal{Y})$. For each $Q \in \Delta_A^1(\mathcal{X})$ we set

$$J_Q = \{x \in \mathcal{X} : \widetilde{\nu}_{\mathcal{X}}(Q) \notin \mathrm{cl}_{\sigma_0(\mathcal{A})}(\mathrm{coz}(x))\}.$$

Then for each $x \in J_Q$, since $\widetilde{\nu}_{\mathcal{X}}(Q) \notin \mathrm{coz}(x)$, it follows that $x \in Q$. This shows that J_Q (and hence its closure) is contained in Q. We shall show that $\overline{J_Q} = Q$. For suppose that $x \in Q$, then by the hypothesis, there exists a sequence $\{a_n\}$ in $\mathcal{A}$ such that each $\widehat{a_n}$ vanishes on a neighborhood V_n of $\widetilde{\nu}_{\mathcal{X}}(Q)$ and $\|a_n \cdot x - x\| \longrightarrow 0$. Since $\mathrm{coz}(a_n \cdot x) \subseteq \mathrm{coz}(\widehat{a_n}) \subseteq \sigma_0(\mathcal{A})\backslash V_n$, it follows that $V_n \cap \mathrm{coz}(a_n \cdot x) = \emptyset$ for all $n \in \mathbb{N}$. Therefore, $\widetilde{\nu}_{\mathcal{X}}(Q) \notin \mathrm{cl}_{\sigma_0(\mathcal{A})}(\mathrm{coz}(a_n \cdot x))$, that is, $a_n \cdot x \in J_Q$. Hence $x \in \overline{J_Q}$ and consequently $\overline{J_Q} = Q$.

Now let Q_1 and Q_2 be two distinct points in $V(\delta_P \circ T)$. By Proposition 3.4(b), $\widetilde{\nu}_{\mathcal{X}}(Q_1) = \widetilde{\nu}_{\mathcal{X}}(Q_2)$, and therefore $J_{Q_1} = J_{Q_2}$. Hence $Q_1 = \overline{J_{Q_1}} = \overline{J_{Q_2}} = Q_2$, as desired. □

In the case where $\mathcal{X}$ satisfies the hyper Ditkin's condition, the above lemma allows us to define a map $\Phi : \Delta_{\mathcal{B}}(\mathcal{Y})\backslash\Delta_0(\mathcal{Y}) \longrightarrow \Delta_{\mathcal{A}}^1(\mathcal{X})$ by $\{\Phi(P)\} = V(\delta_P \circ T)$, $P \in \Delta_{\mathcal{B}}(\mathcal{Y})\backslash\Delta_0(\mathcal{Y})$.

Lemma 3.2. *Let $\mathcal{X}$ satisfy the hyper Ditkin's condition and $P \in \Delta_{\mathcal{B}}(\mathcal{Y})\backslash\Delta_0(\mathcal{Y})$. Then $P \in \Delta_c(\mathcal{Y})$ if and only if $\Phi(P) = T^{-1}(P)$.*

Proof. Assume first that $P \in \Delta_c(\mathcal{Y})$. For each $Q \in \Delta_{\mathcal{A}}^1(\mathcal{X})$, let J_Q be defined as in the previous lemma. If $x \in J_{\Phi(P)}$, then $\widetilde{\nu}_{\mathcal{X}}(\Phi(P)) \notin \mathrm{cl}_{\sigma_0(\mathcal{A})}(\mathrm{coz}(x))$ and so by Proposition 3.4(a), $T(x) \in P$, i.e. $x \in \ker(\delta_P \circ T)$. This shows that $J_{\Phi(P)} \subseteq \ker(\delta_P \circ T)$ and since $\delta_P \circ T$ is continuous $\overline{J_{\Phi(P)}} \subseteq \ker(\delta_P \circ T)$. As it was shown in the proof of the preceding lemma, $\Phi(P) = \overline{J_{\Phi(P)}}$, hence $\Phi(P) \subseteq \ker(\delta_P \circ T)$. On the other hand, $\ker(\delta_P \circ T) = T^{-1}(P) \neq \mathcal{X}$, since $\delta_P \circ T \neq 0$, thus $\Phi(P) \neq \mathcal{X}$ and consequently $\Phi(P) = T^{-1}(P)$, since $\Phi(P)$ is a hyper maximal submodule.

Assume now that $P \in \Delta_{\mathcal{B}}(\mathcal{Y})\backslash\Delta_0(\mathcal{Y})$ such that $\Phi(P) = T^{-1}(P)$. Then clearly $\Phi(P) \neq \mathcal{X}$. Let $\omega(P) : \mathcal{X}/\Phi(P) \longrightarrow \mathcal{Y}/P$ be defined by $\omega(P)(x + \Phi(P)) = Tx + P = \delta_P \circ T(x)$, $x \in \mathcal{X}$. Since $\Phi(P)$ has codimension one in $\mathcal{X}$, it follows that the linear map $\omega(P)$ is continuous, which implies that $\delta_P \circ T$ is continuous as well. □

The proof of the above lemma shows, in particular, that $\Phi(\Delta_c(\mathcal{Y})) \subseteq \Delta_{\mathcal{A}}(\mathcal{X})$.

Now we give a description of the separating map T as follows. Consider first the following subset of $\Pi_{P\in\Delta_{\mathcal{A}}(\mathcal{X})}\mathcal{X}/P$:

$$\underline{\mathcal{X}} = \{\underline{x} = (x_P + P)_{P\in\Delta_{\mathcal{A}}(\mathcal{X})} : \sup_{P\in\Delta_{\mathcal{A}}(\mathcal{X})} \|x_P + P\| < \infty\}.$$

Then $\underline{\mathcal{X}}$ is a Banach space under the norm defined by $\|\underline{x}\| = \sup_{P\in\Delta_{\mathcal{A}}(\mathcal{X})} \|x_p + P\|$, $\underline{x} = (x_P + P)_{P\in\Delta_{\mathcal{A}}(\mathcal{X})} \in \underline{\mathcal{X}}$ which is actually a left Banach $\mathcal{A}$-module in a natural way (see [7] for the case that $\mathcal{A}$ is unital). Furthermore, for each $x \in \mathcal{X}$ the map $G_{\mathcal{X}} : \mathcal{X} \longrightarrow \underline{\mathcal{X}}$ defined by $G_{\mathcal{X}}(x) = \widehat{x}$, where for each $x \in \mathcal{X}$, $\widehat{x} = (x+P)_{P\in\Delta_{\mathcal{A}}(\mathcal{X})}$ is a norm decreasing map which is injective if $\mathcal{X}$ is hyper semisimple. Similar notations will be applied for the left Banach module $\mathcal{Y}$.

Theorem 3.1. *Under the hypotheses of Lemma 3.2, there exists a complex-valued function $\omega : \Delta_c(Y) \longrightarrow \mathbb{C}$ such that $\widehat{Tx}(P) = \omega(P) \cdot \widehat{x}(\Phi(P))$ for all $P \in \Delta_c(Y)$.*

Proof. Let $P \in \Delta_c(Y)$ and let $\omega(P) : X/\Phi(P) \longrightarrow Y/P$ be defined by $\omega(P)(x + \Phi(P)) = Tx + P = \delta_P \circ T(x)$, $x \in X$. Then clearly $\omega(P)$ is well-defined since $\Phi(P) = T^{-1}(P)$. We note that since $\Phi(P)$ and P have codimension one in X and Y, respectively, we can regard the linear map $\omega(P)$ as a scalar in the complex field. Considering the function $\omega : \Delta_c(Y) \longrightarrow \mathbb{C}$ defined in this way we see that for each $P \in \Delta_c(Y)$, $\widehat{Tx}(P) = Tx + P = \omega(P) \cdot (x + \Phi(P)) = \omega(P) \cdot \widehat{x}(\Phi(P))$. □

Another way to give a description of a separating map in module case is as follows:

For each $Q \in \Delta_{\mathcal{A}}(\mathcal{X})$, set $(Q) = \{\xi \in \mathcal{X}^* : \ker(\xi) = Q\}$ and then put $\widetilde{\Delta_c(\mathcal{Y})} = \bigcup_{P\in\Delta_c(\mathcal{Y})} (P) \times (\Phi(P))$. We consider $\widetilde{\Delta_c(\mathcal{Y})}$ with the relative product topology inherited from $\mathcal{Y}^* \times \mathcal{X}^*$, where $\mathcal{X}^*$ and $\mathcal{Y}^*$ are equipped with their corresponding weak-star topologies.

Lemma 3.3. *There exists a continuous map $\widetilde{\omega} : \widetilde{\Delta_c(\mathcal{Y})} \longrightarrow \mathbb{C}$ such that for each non-trivial point multiplier ξ on $\mathcal{X}$, $\xi \circ T = \widetilde{\omega}(\xi, \zeta) \cdot \zeta$, for all $\zeta \in X^*$ with $(\xi, \zeta) \in \widetilde{\Delta_c(\mathcal{Y})}$*

Proof. Suppose that $P \in \Delta_c(\mathcal{Y})$ and $(\xi, \zeta) \in (P) \times (\Phi(P))$. Since $\Phi(P) = T^{-1}(P)$, it follows that $\ker(\zeta) = \ker(\xi \circ T)$, which implies that there exists a nonzero scalar $\widetilde{\omega}(\xi, \zeta)$ such that $\xi \circ T = \widetilde{\omega}(\xi, \zeta).\zeta$. Now we shall prove that the function $\widetilde{\omega} : \widetilde{\Delta_c(\mathcal{Y})} \longrightarrow \mathbb{C}$ obtained in this way is continuous. For suppose that $(\xi_0, \zeta_0) \in \widetilde{\Delta_c(\mathcal{Y})}$. Then there exists a non-zero element $x \in \mathcal{X}$ such that $\langle \zeta_0, x\rangle \neq 0$ and there exists a weak-star open neighborhood U of ζ_0 in $\mathcal{X}^*$ such that for each $\eta \in U$, $\langle \eta, x\rangle \neq 0$. Therefore, we can write $\widetilde{\omega}(\xi, \zeta) = \frac{\langle \xi, Tx\rangle}{\langle \zeta, x\rangle}$ for all (ξ, ζ) in the neighborhood $(\mathcal{Y}^* \times U) \cap \widetilde{\Delta_c(\mathcal{Y})}$ of (ξ_0, ζ_0) in $\widetilde{\Delta_c(\mathcal{Y})}$, which concludes that $\widetilde{\omega}$ is continuous as desired. □

In the case where $\mathcal{X}$ satisfies the Ditkin's condition, we define a map

$$\Psi : \Delta_{\mathcal{B}}(\mathcal{Y})\backslash\Delta_0(\mathcal{Y}) \longrightarrow \sigma_0(\mathcal{A})$$

by $\Psi(P) = \widetilde{\nu}_{\mathcal{X}}(\Phi(P))$, $P \in \Delta_{\mathcal{B}}(\mathcal{Y})\backslash\Delta_0(Y)$. In the next theorem, we generalize Propositions 3 and 6 in [9] and give more properties for the given partition of $\Delta_{\mathcal{B}}(\mathcal{Y})$.

Theorem 3.2. *Let $\mathcal{X}$ be hyper semisimple and satisfy the hyper Ditkin's condition. Then the following statements hold;*

(a) $\Psi(\mathrm{coz}_h(Tx)) \subseteq cl_{\sigma_0(A)}(\mathrm{coz}(x))$.

(b) *If* $x \in \mathcal{X}$ *and* U *is an open subset of* $\sigma_0(\mathcal{A})$ *such that* $x \in k_{\mathcal{X}}(\widetilde{\nu}_{\mathcal{X}}^{-1}(U))$, *then* $Tx \in k_{\mathcal{Y}}(\Psi^{-1}(U))$.

(c) $\Psi(\Delta_d(\mathcal{Y}))$ *is a finite subset of* $\sigma_0(\mathcal{A})$ *consisting of non-singular points of* $\sigma_0(\mathcal{A})$.

(d) *If* T *is surjective, then* $\Delta_0(\mathcal{Y}) = \emptyset$.

(e) *If* T *is injective and* $\mathcal{Y}$ *is hyper semisimple, then* $\mathrm{cl}_{\sigma_0(\mathcal{A})}(\Psi(\Delta_c(\mathcal{Y}) \cup \Delta_d(\mathcal{Y}))) = \mathrm{cl}_{\sigma_0(\mathcal{A})}(\Psi(\Delta_c(\mathcal{Y})) = \sigma_0(\mathcal{A})$ *and* $h_{\mathcal{X}} k_{\mathcal{X}}(\Phi(\Delta_c(\mathcal{Y}))) = \Delta_{\mathcal{A}}(\mathcal{X})$.

Proof. (a) It is immediate from Proposition 3.4 (a).

(b) Let $x \in \mathcal{X}$ and U be an open subset of $\sigma_0(\mathcal{A})$ such that $x \in k_{\mathcal{X}}(\widetilde{\nu}_{\mathcal{X}}^{-1}(U))$. Let $P \in \Psi^{-1}(U)$, then U is an open neighborhood of $\Psi(P) = \widetilde{\nu}_{\mathcal{X}}(\Phi(P))$. Since $\widetilde{\nu}_{\mathcal{X}}$ is assumed to be surjective, for every $\varphi \in \sigma_0(\mathcal{A}))\backslash U$ there exists an element $Q_\varphi \in \Delta_{\mathcal{A}}^1(\mathcal{X})$ with $\varphi = \widetilde{\nu}_{\mathcal{X}}(Q_\varphi)$. Clearly $Q_\varphi \neq \Phi(P)$, and hence there exists an open neighborhood U_φ of φ in $\sigma_0(\mathcal{A})$ such that for any $t \in \mathcal{X}$ with $\mathrm{coz}(t) \subseteq U_\varphi$ we have $Tt \in P$. Since $\sigma_0(\mathcal{A})\backslash U \subseteq \bigcup_{\varphi \in \sigma_0(\mathcal{A})\backslash U} U_\varphi$, it follows that there exist $\varphi_1, ..., \varphi_n \in \sigma_0(\mathcal{A})\backslash U$ such that $\sigma_0(\mathcal{A})\backslash U \subseteq \bigcup_{i=1}^n U_{\varphi_i}$. Set $U_i = U_{\varphi_i}$ for $i = 1, ..., n$ and $U_{n+1} = U$. Then $\sigma_0(\mathcal{A}) \subseteq \bigcup_{i=1}^{n+1} U_i$ and so, by the regularity of $\mathcal{A}_1$, there exist $a_i \in \mathcal{A}_1$, $i = 1, ..., n+1$ such that $\mathrm{coz}(\widehat{a_i}) \subseteq U_i$, $i = 1, 2, ..., n+1$, and $\sum_{i=1}^{n+1} \widehat{a_i} = 1$ on $\sigma_0(\mathcal{A})$. We claim that $a_{n+1} \cdot x = 0$. Since $x \in k_{\mathcal{X}}(\widetilde{\nu}_{\mathcal{X}}^{-1}(U))$, it follows that $x \in Q$ for each $Q \in \widetilde{\nu}_{\mathcal{X}}^{-1}(U)$. Hence $\mathrm{coz}_h(x) \subseteq \Delta_{\mathcal{A}}^1(\mathcal{X})\backslash \widetilde{\nu}_{\mathcal{X}}^{-1}(U) = \widetilde{\nu}_{\mathcal{X}}^{-1}(\sigma_0(\mathcal{A})\backslash U)$, and so $\mathrm{coz}(x) \subseteq \sigma_0(\mathcal{A})\backslash U$, since $\widetilde{\nu}_{\mathcal{X}}$ is surjective. Thus $\mathrm{coz}(a_{n+1} \cdot x) \subseteq \mathrm{coz}(x) \cap \mathrm{coz}(\widehat{a_{n+1}}) \subseteq (\sigma_0(\mathcal{A})\backslash U) \cap U = \emptyset$ and so $a_{n+1} \cdot x = 0$ by Proposition 3.1(b). This establishes the claim. Now since for $i = 1, ..., n$, $\mathrm{coz}(a_i \cdot x) \subseteq \mathrm{coz}(\widehat{a_i}) \subseteq U_i$ it follows that $T(a_i \cdot x) \in P$ and consequently $Tx + P = \sum_{i=1}^n T(a_i \cdot x) + P = P$. Hence $Tx \in P$ for all $P \in \Psi^{-1}(U)$, that is $Tx \in k_{\mathcal{Y}}(\Psi^{-1}(U))$.

(c) Assume on the contrary that $\Psi(\Delta_d(\mathcal{Y}))$ has infinitely many points and let $\{P_n\}_{n\in\mathbb{N}}$ be a sequence in $\Delta_d(\mathcal{Y})$ such that $\{\Psi(P_n)\}_{n\in\mathbb{N}}$ is a sequence of distinct points in $\Psi(\Delta_d(\mathcal{Y}))$. Since for each n, $P_n \notin \Delta_c(\mathcal{Y})$, it follows from Lemma 3.2 that $T^{-1}(P_n) \neq \Phi(P_n)$ and consequently $\Phi(P_n)$ is not contained in $T^{-1}(P_n)$, since $\Phi(P_n)$ is either a hyper maximal submodule of $\mathcal{X}$ or is $\mathcal{X}$ itself. Hence for each $n \in \mathbb{N}$, there exists $x_n \in \mathcal{X}$ such that $x_n \in \Phi(P_n)$ and $x_n \notin T^{-1}(P_n)$. Replacing each x_n by a scalar multiple of x_n, we can assume that for each $n \in \mathbb{N}$, $\|Tx_n + P_n\| \geq n$. Since $\mathcal{X}$ satisfies the hyper Ditkin's condition, for each $n \in \mathbb{N}$ there exists an element $a_n \in \mathcal{A}$ such that $\widehat{a_n}$ vanishes on a neighborhood U_n of $\Psi(P_n)$ and $\|a_n \cdot x_n - x_n\| \leq \frac{1}{n^2}$. We may assume that $U_n \cap U_m = \emptyset$ for all $n \neq m$.

We now claim that for each $n \in \mathbb{N}$, there exists an element $z_n \in \mathcal{X}$ such that $\|Tz_n + P_n\| \geq n$, $\mathrm{coz}(z_n) \subseteq U_n$ and $\|z_n\| \leq \frac{1}{n^2}$. Set $t_n = x_n - a_n \cdot x_n$, $n \in \mathbb{N}$, then clearly for each $n \in \mathbb{N}$, $t_n \in \Phi(P_n)$. We note that, similar to Proposition 2.1(a), it is easy to see that for each $n \in \mathbb{N}$, $k_{\mathcal{A}_1}(U_n) = (k_{\mathcal{X}}(\widetilde{\nu}_{\mathcal{X}}^{-1}(U_n)) :_{\mathcal{A}_1} \mathcal{X})$ and consequently $a_n \cdot \mathcal{X} \subseteq k_{\mathcal{X}}(\widetilde{\nu}_{\mathcal{X}}^{-1}(U_n))$. Hence for each $n \in \mathbb{N}$, $x_n - t_n = a_n \cdot x_n \in k_{\mathcal{X}}(\widetilde{\nu}_{\mathcal{X}}^{-1}(U_n))$ and it follows from part (b) that $Tx_n - Tt_n \in k_{\mathcal{Y}}(\Psi^{-1}(U_n)) \subseteq P_n$. Thus $\|Tt_n + P_n\| = \|Tx_n + P_n\| \geq n$. For each $n \in \mathbb{N}$, choose an open neighborhood W_n of $\Psi(P_n)$ such that $\mathrm{cl}_{\sigma_0(\mathcal{A})}(W_n) \subseteq U_n$. Then there exists a sequence $\{b_n\}$ in $\mathcal{A}_1$ such that $\widehat{b_n} = 1$ on W_n and $\widehat{b_n} = 0$ on $\sigma_0(\mathcal{A})\backslash U_n$. Put $s_n = b_n \cdot t_n$, then clearly $s_n \in \Phi(P_n)$ and since $b_n - 1_{\mathcal{A}_1} = 0$ on W_n, i.e. $b_n - 1_{\mathcal{A}} \in k_{\mathcal{A}_1}(W_n) = (k_{\mathcal{X}}(\widetilde{\nu}_{\mathcal{X}}^{-1}(W_n)) :_{\mathcal{A}_1} \mathcal{X})$, it follows that $s_n - t_n = (b_n - 1_{\mathcal{A}_1})t_n \in k_{\mathcal{X}}(\widetilde{\nu}_{\mathcal{X}}^{-1}(W_n))$. Therefore, $Ts_n - Tt_n \in k_{\mathcal{Y}}(\Psi^{-1}(W_n)) \subseteq P_n$ by part (b), which implies that $\|Ts_n + P_n\| = \|Tt_n + P_n\| \geq n$. We note also that $\mathrm{coz}(s_n) = \mathrm{coz}(b_n \cdot t_n) \subseteq \mathrm{coz}(\widehat{b_n}) \subseteq U_n$, $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ choose $c_n \in \mathcal{A}_1$ such that $\widehat{c_n}$ vanishes on a neighborhood of $\Psi(P_n)$ and $\|s_n - c_n \cdot s_n\| \leq \frac{1}{n^2}$. Setting $z_n = s_n - c_n \cdot s_n$, we see that $z_n \in \Phi(P_n)$, $\|z_n\| \leq \frac{1}{n^2}$ and $\mathrm{coz}(z_n) \subseteq \mathrm{coz}(s_n) \subseteq U_n$. Moreover, since for each $n \in \mathbb{N}$, $z_n - s_n = c_n \cdot s_n$ and c_n vanishes on a neighborhood of $\psi(P_n)$ an argument as above shows that $Tz_n - Ts_n \in P_n$. Hence $\|Tz_n + P_n\| = \|Ts_n + P_n\| \geq n$ as we claimed.

Obviously for the above sequence $\{z_n\}$ we have $\mathrm{coz}(z_n) \cap \mathrm{coz}(z_m) = \emptyset$, for all $n \neq m$, which implies that $\mathrm{coz}(Tz_n) \cap \mathrm{coz}(Tz_m) = \emptyset$. Since for each $n \in \mathbb{N}$, $Tz_n \notin P_n$ it follows that $P_n \in \mathrm{coz}_h(Tz_n)$. Therefore, for each $n \neq m$, $P_n \notin \mathrm{coz}_h(Tz_m)$ that is $Tz_m \in P_n$. Put now $z = \sum_{n=1}^{\infty} z_n$. Then since for each $n \neq m$, $\mathrm{coz}(z_m) \cap U_n = \emptyset$, we can easily deduce that $z - z_n \in k_{\mathcal{X}}(\widetilde{\nu}_{\mathcal{X}}^{-1}(U_n))$. This implies that $Tz - Tz_n \in k_{\mathcal{Y}}(\Psi^{-1}(U_n)) \subseteq P_n$ by part (b). Hence $\|Tz\| \geq \|Tz + P_n\| = \|Tz_n + P_n\| \geq n$, for each $n \in \mathbb{N}$, which is impossible. This contradiction shows that $\Psi(\Delta_d(\mathcal{Y}))$ is finite.

We shall show that each point in $\Psi(\Delta_d(\mathcal{Y}))$ is a non-singular point of $\sigma_0(\mathcal{A})$. Let $P_0 \in \Delta_d(\mathcal{Y})$ such that $\Psi(P_0)$ is a singular point of $\sigma_0(\mathcal{A})$ and set $U = \{\Psi(P_0)\}$. Then U is an open subset of $\sigma_0(\mathcal{A})$. Let $x \in \Phi(P_0)$, then since $\mathcal{X}$ satisfies the hyper Ditkin's condition there exists a sequence $\{a_n\}$ in $\mathcal{A}$ such that $\widehat{a_n}(\Psi(P_0)) = 0$ for each $n \in \mathbb{N}$, and $\|a_n \cdot x - x\| \longrightarrow 0$. Furthermore, for any $Q \in \widetilde{\nu}_{\mathcal{X}}^{-1}(U)$, since $\ker(\psi(P_0)) = \ker(\widetilde{\nu}_{\mathcal{X}}(Q)) = k_{\mathcal{A}_1}(\widetilde{\nu}_{\mathcal{X}}(Q)) = (k_{\mathcal{X}}(Q) :_{\mathcal{A}_1} X)$ and $a_n \in \ker(\psi(P_0))$ it follows that $a_n \cdot x \in k_{\mathcal{X}}(Q) = Q$, which implies that $x \in Q$. This shows that $\Phi(P_0) \subseteq k_{\mathcal{X}}(\widetilde{\nu}_{\mathcal{X}}^{-1}(U)) \subseteq \Phi(P_0)$, that is $\Phi(P_0) = k_{\mathcal{X}}(\widetilde{\nu}_{\mathcal{X}}^{-1}(U))$. Thus using part (b) we conclude that $T(\Phi(P_0)) \subseteq k_{\mathcal{Y}}(\Psi^{-1}(U)) \subseteq P_0$, that is $\Phi(P_0) \subseteq T^{-1}(P_0)$, and consequently $\Phi(P_0) = T^{-1}(P_0) \neq \mathcal{X}$. Hence $P_0 \in \Delta_c(\mathcal{Y})$, by Lemma 3.2, which is a contradiction. Therefore, each point in $\Psi(\Delta_d(\mathcal{Y}))$ is non-singular.

(d) Assume on the contrary that T is surjective and $\Delta_0(\mathcal{Y}) \neq \emptyset$. Let $P \in \Delta_0(\mathcal{Y})$, then $T\mathcal{X} \subseteq P \subseteq \mathcal{Y}$ which implies, by the surjectivity of T, that $\mathcal{Y} = P$, a contradiction.

(e) Let T be injective and $\mathcal{Y}$ be hyper semisimple. Let $\varphi \in \sigma_0(\mathcal{A})$ and assume that there exists an open neighborhood V of φ with $V \cap \Psi(\Delta_c(\mathcal{Y}) \cup \Delta_d(\mathcal{Y})) = \emptyset$. Choosing a neighborhood U of φ with $\mathrm{cl}_{\sigma_0(\mathcal{A})}(U) \subseteq V$, we can find a non-zero element $a \in \mathcal{A}_1$

such that $\mathrm{coz}(\widehat{a}) \subseteq U$. Then $\mathrm{cl}_{\sigma_0(\mathcal{A})}(\mathrm{coz}(a \cdot x)) \subseteq \mathrm{cl}_{\sigma_0(\mathcal{A})}(U) \subseteq V$ for all $x \in \mathcal{X}$ and consequently for any $P \in \Delta_c(\mathcal{Y}) \cup \Delta_d(\mathcal{Y})$, $\Psi(P) \notin \mathrm{cl}_{\sigma_0(\mathcal{A})}(\mathrm{coz}(a \cdot x))$. Therefore, $T(a \cdot x) \in P$, by Proposition 3.4(a). This clearly implies that $T(a \cdot x) \in \mathrm{rad}_{\mathcal{B}}(\mathcal{Y}) = \{0\}$ for all $x \in \mathcal{X}$. The injectivity of T concludes that $a \cdot x = 0$ for all $x \in \mathcal{X}$, i.e. $a \in \mathrm{ann}_{\mathcal{A}_1}(\mathcal{X}) = \mathrm{Rad}(\mathcal{A}) = \{0\}$ by Proposition 2.1(c), which is a contradiction. Thus $\Psi(\Delta_c(\mathcal{Y}) \cup \Delta_d(\mathcal{Y}))$ is dense in $\sigma_0(\mathcal{A})$.

For the other equality we need only to show that $\Psi(\Delta_d(\mathcal{Y})) \subseteq \mathrm{cl}_{\sigma_0(\mathcal{A})}(\Psi(\Delta_c(\mathcal{Y})))$. For suppose that $\varphi \in \Psi(\Delta_d(\mathcal{Y}))$ and let U be an open neighborhood of φ such that $U \cap \Psi(\Delta_c(\mathcal{Y})) = \emptyset$. By (c) we can assume that $(U \backslash \{\varphi\}) \cap \Psi(\Delta_d(\mathcal{Y})) = \emptyset$. Then obviously $(U \backslash \{\varphi\}) \cap \Psi(\Delta_c(\mathcal{Y}) \cup \Delta_d(\mathcal{Y})) = \emptyset$, while since φ is a non-singular point of $\sigma_0(\mathcal{A})$, $U \backslash \{\varphi\}$ is a non-empty open subset of $\sigma_0(\mathcal{A})$ and so it has a non-empty intersection with $\Psi(\Delta_c(\mathcal{Y}) \cup \Delta_d(\mathcal{Y}))$. This contradiction shows that $\mathrm{cl}_{\sigma_0(\mathcal{A})}(\Psi(\Delta_c(\mathcal{Y})) = \sigma_0(\mathcal{A})$

We now prove that $\Phi(\Delta_c(\mathcal{Y}))$ is dense in $\Delta_{\mathcal{A}}(\mathcal{X})$ with respect to the hull-kernel topology. Since $\Psi(\Delta_c(\mathcal{Y}))$ is dense in $\sigma_0(\mathcal{A})$ and $\mathcal{A}$ is regular, we have

$$h_{\mathcal{A}_1} k_{\mathcal{A}_1}(\Psi(\Delta_c(\mathcal{X}))) = \sigma_0(\mathcal{A})$$

which concludes that $k_{\mathcal{A}_1}(\Psi(\Delta_c(\mathcal{Y})) = \{0\}$. Similar to Proposition 2.1(a) we have

$$k_{\mathcal{A}_1}(\Psi(\Delta_c(\mathcal{Y})) = (k_{\mathcal{X}}(\Phi(\Delta_c(\mathcal{Y}))) :_{\mathcal{A}_1} \mathcal{X})$$

and consequently $(k_{\mathcal{X}}(\Phi(\Delta_c(\mathcal{Y}))) :_{\mathcal{A}} \mathcal{X}) \subseteq (k_{\mathcal{X}}(\Phi(\Delta_c(\mathcal{Y}))) :_{\mathcal{A}_1} \mathcal{X}) = \{0\}$. Thus, by the definition of the hull, $h_{\mathcal{X}} k_{\mathcal{X}}(\Phi(\Delta_c(\mathcal{Y}))) = \Delta_{\mathcal{A}}(\mathcal{X})$, as desired. □

Definition 3.5. Let X be a left Banach module over a commutative Banach algebra A. We say that X is a *Banach multiplication module* if for any closed left submodule N of X there exists an ideal I in A such that $N = \overline{I \cdot X}$, where $I \cdot X = \mathrm{span}\{a \cdot x : a \in I, x \in X\}$.

Clearly every commutative Banach algebra with approximate identity is a Banach multiplication module over itself. Commutative semisimple Banach algebras satisfying the Ditkin's condition are other examples of Banach multiplication modules over itself.

Let $S(G)$ be a Segal algebra on a locally compact abelian group G. Then since $S(G)$ satisfies the Ditkin's condition, for any closed $L_1(G)$-submodule N of $S(G)$ we have $\overline{S(G) \cdot N} = N$. Hence every Segal algebra on G is a Banach multiplication $L_1(G)$-module.

Remark. We note that if X is an essential left Banach multiplication module over a commutative Banach algebra A, then it can be easily verified that for closed submodules M and P of X, $(M :_A X) \subseteq (P :_A X)$ if and only if $M \subseteq P$. We observe that in this case for each $x \in X$, $h_{\mathcal{X}} k_{\mathcal{X}}(\Delta_A(X) \backslash \mathrm{coz}_h(x)) = \Delta_A(X) \backslash \mathrm{coz}_h(x)$, which implies that $\operatorname{coz}_h(x)$ is an open subset of $\Delta_A(X)$ in the hull-kernel topology. Hence if the natural map $\widetilde{\nu}_X$ is surjective, it follows from Proposition 2.1(b) that $\operatorname{coz}(x)$ is an open subset of $\sigma_0(A)$ in the hull-kernel topology.

Lemma 3.4. *Let $\mathcal{X}$ satisfy the hyper Ditkin's condition. If $\mathcal{Y}$ is, in addition, a Banach multiplication module, then the restriction of Φ to $\Delta_c(\mathcal{Y})$ is continuous.*

Proof. We first show that Ψ is continuous on $\Delta_{\mathcal{B}}(\mathcal{Y})\backslash\Delta_0(\mathcal{Y})$. Assume on the contrary that there exists a net $\{P_\alpha\}_{\alpha\in I}$ in $\Delta_{\mathcal{B}}(\mathcal{Y})\backslash\Delta_0(\mathcal{Y})$ converging to a point $P_0 \in \Delta_{\mathcal{B}}(\mathcal{Y})\backslash\Delta_0(\mathcal{Y})$ and $\{\Psi(P_\alpha)\}_{\alpha\in I}$ does not converge to $\Psi(P_0)$. By compactness of $\sigma_0(\mathcal{A})$, passing through a subnet, we can assume that $\{\Psi(P_\alpha)\}$ converges to a point φ distinct from $\Psi(P_0)$. Let U_0 and U be disjoint neighborhoods of $\Psi(P_0)$ and φ in $\sigma_0(\mathcal{A})$, respectively. By the definition of $\Psi(P_0)$, there exists an element $x_0 \in \mathcal{X}$ such that $\operatorname{coz}(x_0) \subseteq U_0$ and $Tx_0 \notin P_0$. Since $\mathcal{Y}$ is a Banach multiplication module $\operatorname{coz}_h(Tx_0)$ is an open subset of $\Delta_{\mathcal{B}}(\mathcal{Y})$ and so for sufficiently large α, we have $\Psi(P_\alpha) \in U$ and $P_\alpha \in \operatorname{coz}_h(Tx_0)$. Thus for such α we can find $x_\alpha \in \mathcal{X}$ such that $\operatorname{coz}(x_\alpha) \subseteq U$ and $Tx_\alpha \notin P_\alpha$. Therefore, $\operatorname{coz}(x_0) \cap \operatorname{coz}(x_\alpha) = \emptyset$ while $\nu_{\mathcal{Y}}(P_\alpha) \in \operatorname{coz}(Tx_0) \cap \operatorname{coz}(Tx_\alpha)$ which is a contradiction. Hence $\Psi : \Delta_{\mathcal{B}}(\mathcal{Y})\backslash\Delta_0(\mathcal{Y}) \longrightarrow \sigma_0(\mathcal{A})$ is continuous.

To prove that Φ is continuous on $\Delta_c(\mathcal{Y})$, we shall show that for each subset S of $\Delta_c(\mathcal{Y})$, $\Phi(h_{\mathcal{Y}}k_{\mathcal{Y}}(S) \cap \Delta_c(\mathcal{Y})) \subseteq h_{\mathcal{X}}k_{\mathcal{X}}(\Phi(S))$. We note that the restriction of Ψ to $\Delta_c(\mathcal{Y})$ maps this set into $\sigma(\mathcal{A})$, hence the continuity of Ψ implies that $\Psi(h_{\mathcal{Y}}k_{\mathcal{Y}}(S)\cap \Delta_c(\mathcal{Y})) \subseteq h_{\mathcal{A}}k_{\mathcal{A}}(\Psi(S))$. Now let $P_0 \in h_{\mathcal{Y}}k_{\mathcal{Y}}(S) \cap \Delta_c(\mathcal{Y})$, then $\Psi(P_0) \in h_{\mathcal{A}}k_{\mathcal{A}}(\Psi(S))$, i.e. $k_{\mathcal{A}}(\Psi(S)) \subseteq \ker \Psi(P_0)$, which implies that $(k_{\mathcal{X}}(\Phi(S)) :_{\mathcal{A}} \mathcal{X}) \subseteq (\Phi(P_0) :_{\mathcal{A}} \mathcal{X})$, by Proposition 2.1 (a). Hence $\Phi(P_0) \in h_{\mathcal{X}}k_{\mathcal{X}}(\Phi(S))$, that is $\Phi(h_{\mathcal{Y}}k_{\mathcal{Y}}(S) \cap \Delta_c(\mathcal{Y})) \subseteq h_{\mathcal{X}}k_{\mathcal{X}}(\Phi(S))$ as desired.

□

Theorem 3.3. *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{X}, \mathcal{Y}$ be as in Theorem 3.2. If, in addition, $\mathcal{X}$ is a Banach multiplication module, $\mathcal{Y}$ is hyper semisimple and $T : \mathcal{X} \longrightarrow \mathcal{Y}$ is a bijective separating map, then*

(a) *T is continuous.*
(b) *If T is biseparating and $\mathcal{B}$, $\mathcal{Y}$ satisfy the same conditions as $\mathcal{A}$ and $\mathcal{X}$, respectively, then Φ is a homeomorphism.*

Proof. (a) By Theorem 3.2(d), $\Delta_0(\mathcal{Y}) = \emptyset$ and as it was shown in the proof of part (e) of this theorem, $(k_{\mathcal{X}}(\Phi(\Delta_c(\mathcal{Y}))) :_{\mathcal{A}} \mathcal{X}) = \{0\} \subseteq (0 :_{\mathcal{A}} \mathcal{X})$. Since $\mathcal{X}$ is a Banach multiplication module this inclusion implies $k_{\mathcal{X}}(\Phi(\Delta_c(\mathcal{Y}))) = \{0\}$. Let now $\{x_n\}$ be a sequence in $\mathcal{X}$ converging to 0 and $Tx_n \longrightarrow Tz$ for some $z \in \mathcal{X}$. Then given any $P \in \Delta_c(\mathcal{Y})$, we have $\delta_P \circ T(x_n) \longrightarrow \delta_P \circ T(z)$ and $\delta_P \circ T(x_n) \longrightarrow \delta_P \circ T(0) = P$. Thus $Tz + P = P$, for any $P \in \Delta_c(\mathcal{Y})$ which implies that $Tz \in k_{\mathcal{Y}}(\Delta_c(\mathcal{Y}))$. Hence

$z \in T^{-1}(P) = \Phi(P)$ for all $P \in \Delta_c(\mathcal{Y})$, that is, $z \in k_{\mathcal{X}}(\Phi(\Delta_c(\mathcal{Y}))) = \{0\}$. This shows that T is continuous.

(b) The above discussion shows, in particular, that $\Delta_{\mathcal{B}}(\mathcal{Y}) = \Delta_c(\mathcal{Y})$ and so by Lemma 3.4, Φ is a continuous map on $\Delta_{\mathcal{B}}(\mathcal{Y})$ since $\mathcal{Y}$ is assumed to be a Banach multiplication module. By the hypotheses T^{-1} is separating and hence it is continuous by the first part. Therefore, the corresponding support map $\Gamma : \Delta_{\mathcal{A}}(\mathcal{X}) \longrightarrow \Delta_{\mathcal{B}}(\mathcal{Y})$ of T^{-1} is a continuous map with a dense range. Moreover, by Lemma 3.2, we have $\Gamma(\Phi(P)) = T(\Phi(P))$, for each $P \in \Delta_{\mathcal{B}}(\mathcal{Y})$. On the other hand, for each $P \in \Delta_{\mathcal{B}}(\mathcal{Y})$ we have $\Phi(P) = T^{-1}(P)$, which implies that $\Gamma(\Phi(P)) = P$. In a similar manner, we can prove that $\Phi(\Gamma(Q)) = Q$, for every $Q \in \Delta_{\mathcal{A}}(\mathcal{X})$. Hence Φ is a homeomorphism from $\Delta_{\mathcal{B}}(\mathcal{Y})$ onto $\Delta_{\mathcal{A}}(\mathcal{X})$. □

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