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THE ASYMPTOTIC BEHAVIOR OF GEODESIC CIRCLES IN A 2-TORUS OF REVOLUTION AND A SUB-ERGODIC PROPERTY

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Abstract. Let M be a complete Riemannian manifold with finite volume and G_t the geodesic flow on the unit tangent bundle SM. In the light of the Poincaré recurrence property we study the following properties. (P1) For any point $p \in M$ and any open set $U \subset M$ there exists an $R > 0$ such that $\pi(G_t(S_pM)) \cap U \neq \emptyset$ for all $t > R$. (P2) For any unit tangent vector $x \in SM$ and any point $q \in M$ there exist a sequence of unit tangent vectors $x_n \in SM$ and a sequence $t_n \to \infty$ such that $x_n \to x$ and $\pi(G_{t_n}(x_n)) \to q$.

1. Introduction

Let M be a complete Riemannian manifold with finite volume and SM its unit tangent bundle with the natural projection $\pi : SM \to M$. Let $G_t : SM \to SM$ be the geodesic flow. This means that $G_t(x) = \dot\gamma_x(t)$ for all $x \in SM$ and all $t \in (-\infty, \infty)$ where $\gamma_x(t) = \pi(G_t(x))$, $t \in (-\infty, \infty)$, is the geodesic with $\gamma_x(0) = \pi(x)$ and $\dot\gamma_x(0) = x$. The starting point of our discussion is the Poincaré recurrence property which states that for every vector $x \in SM$ there are a sequence of vectors $x_n \to x$ and a sequence $t_n \to \infty$ such that $G_{t_n}(x_n) \to x$ as $n \to \infty$ (cf. [19]). From the viewpoint of the geometry of geodesics in M this theorem can be restated as follows:

Proposition 1.1. *Let M be a complete Riemannian manifold with finite volume. Let $p, q \in M$. Then there exist a sequence of vectors $x_n \in SM$ and a sequence $t_n \to \infty$ such that $\pi(x_n) \to p$ and $\gamma_{x_n}(t_n) \to q$.*

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This is because there exists a minimizing geodesic connecting p and q in M and Poincaré recurrence property for the unit tangent vector of this geodesic at p concludes Proposition 1.1. This proposition motivates us to study the following properties: Let p be a point in M.

(P1) For any open subset U in M there exists an $R > 0$ such that $\pi(G_t(S_pM)) \cap U \neq \emptyset$ for all $t > R$.

(P2) For any unit vector $x \in S_pM$ and any point $q \in M$ there exist a sequence of unit vectors $x_n \in SM$ and a sequence $t_n \to \infty$ such that $x_n \to x$ and $\gamma_{x_n}(t_n) \to q$.

We first take up the property (P1). In a surface of revolution homeomorphic to the sphere, (P1) fails for the north and south poles if U is a sufficiently small open set. W. Sierpinski and M. N. Huxley ([9]) estimate the asymptotic difference between the area πr^2 of the circle $C(r)$ with radius r and the number $N(r)$ of lattice points contained in $C(r)$ in the Euclidean plane, proving that $|\pi r^2 - N(r)| \leq O(r^{2/3+\varepsilon})$ where $\varepsilon > 0$. Their estimate directly shows that a flat 2-torus $\mathbb{T} = \mathbb{E}^2/\mathbb{Z}^2$ satisfies the property (P1).

We prove that the property (P1) holds in any 2-torus of revolution. Here is the definition of a 2-torus of revolution. Let $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$ and $f(y)$ a periodic positive function with period $b > 0$ and minimum at 0. Give a Riemannian metric on $\mathbb{R}^2$ as

$$ds^2 = f(y)^2 dx^2 + dy^2$$

at (x, y). Let a be a positive constant. The maps $\varphi(x, y) = (x + a, y)$ and $\psi(x, y) = (x, y + b)$ are isometries in $M = (\mathbb{R}^2, ds^2)$. Let Φ be the group generated by $\{\varphi.\psi\}$. The quotient surface $\mathbb{T} = M/\Phi$ is called a *2-torus of revolution.* Let $B(q, \varepsilon) = \{x \mid d(q, x) < \varepsilon\}$.

Theorem 1.2. *Let $\mathbb{T}$ be a 2-torus of revolution and $p, q \in \mathbb{T}$. Given $\epsilon > 0$ there exists an $R > 0$ such that $\pi(G_t(S_p\mathbb{T})) \cap B(q, \varepsilon) \neq \emptyset$ for all $t > R$.*

This theorem follows from the following lemma.

Lemma 1.3. *Let $\mathbb{T} = M/\Phi$ be a 2-torus of revolution. Let $p, q \in M$. Given $\varepsilon > 0$ there exists an $R > 0$ such that $S(p, t) \cap \Phi(B(q, \varepsilon)) \neq \emptyset$ for all $t > R$, where $S(p, t) = \{x \in M \mid d(p, x) = t\}$.*

We next take up the condition (P2) in a relation to the disconjugate property of Jacobi vector fields along geodesics. For a unit vector $x \in SM$, we say that γ_x is a *geodesic ray* on $[0, \infty)$ if all points $\gamma_x(t)$, $t > 0$, are not conjugate to $\pi(x) = \gamma_x(0)$ along γ_x, namely all non-trivial Jacobi vector fields along γ_x with $\gamma_x(0) = 0$ do not vanish at any $t > 0$. We say that a point $p \in M$ is a *pole* if γ_x is a geodesic ray for every vector $x \in S_pM$. The property of being a pole does not depend only on the point or its neighborhood, but rather on the whole manifold M.

We think the property (P2) influences the metric structure of M under these properties. A complete Riemannian manifold all of whose points are poles is said to be *without conjugate points*. E. Hopf ([8]) proved that a 2-torus is without conjugate points if and only if it is flat. In a 2-torus of revolution all points lying on the minimum parallel circles are poles (cf. [10]), and, hence, there exists a non-flat 2-torus such that the set of all poles has positive measure ([10]). We say that a complete Riemannian manifold is *without focal points* if all geodesics in M have no focal points as one dimensional submanifolds in M. A manifold without focal points is without conjugate points. N. Innami ([12]) proved that if there exists a point p in a compact Riemannian manifold M such that the point p cannot be a focal point to any geodesic in M, then M is without focal points. Moreover, if there exists a point p in a compact Riemannian manifold M such that all the sectional curvatures $K_{\gamma_x(t)}$ of tangent planes containing $\dot{\gamma}_x(t)$, $x \in S_pM$, are non-positive, then the sectional curvature of M is non-positive.

Thus, “without focal points” and “with non-positive curvature” are controlled by a global property of a single point, but ”without conjugate points” is not. Under the property (P2) we prove the following.

Proposition 1.4. *Let M be a complete Riemannian manifold such that the set of all poles has interior which is not empty. Assume the geodesic flow G_t satisfies* (P2). Then all points in M are poles, namely M is without conjugate points.

Since any Riemannian metric without conjugate points on a Riemannian n-torus is flat ([8], [5]), we have the following corollary.

Corollary 1.5. *Let $\mathbb{T}^n$ be a Riemannian torus such that the set of all poles contains an open set. Then the metric of $\mathbb{T}^n$ is flat if and only if its geodesic flow satisfies* (P2).

Here is another application of Proposition 1.4. Let ω be the volume form induced from the Riemannian metric of M. Let $L_1(M, \omega)$ be the set of all integrable functions defined on M. Then, we consider the following *sub-ergodic* property: If $f \in L_1(M, \omega)$, then

$$(f \circ \pi)^*(v) = \frac{1}{vol(M)} \int_M f \, d\omega, \quad \text{a. e. } v \in SM, \tag{SE}$$

where

$$(f \circ \pi)^*(v) = \lim_{t \to \infty} \frac{1}{t} \int_0^t (f \circ \pi)(G_t(v)) \, dt.$$

Obviously, (SE) implies (P2). The property (SE) has been introduced in [13] to characterize flat metrics on 2-tori. The ergodic property of geodesic flows implies (SE). However, the geodesic flow of a flat n-torus satisfies (SE) but it is not ergodic ([1]). On the other hand, V. J. Donnay ([7]) proved that there exists a Riemannian metric on a 2-torus whose geodesic flow is ergodic. Let P be the set of all unit vectors $x \in S\mathbb{T}^n$ such that the geodesic γ_x is without conjugate points on $[0, \infty)$, namely a geodesic ray. The following corollary shows that the measure of P is zero if the geodesic flow of $\mathbb{T}$ is ergodic.

Corollary 1.6. *Let M be a complete Riemannian manifold with finite volume. Let P be the set of all vectors $x \in SM$ such that γ_x is a geodesic ray on $[0, \infty)$. Assume that P has positive measure. Then, M is without conjugate points if the geodesic flow G_t satisfies* (SE). In particular, a Riemannian torus $\mathbb{T}^n$ such that P has positive measure is flat if and only if its geodesic flow satisfies (SE).

In Section 2 we review the properties of geodesics in the universal covering space of a 2-torus of revolution and prove some lemmas which will be needed in the proof of Lemma 1.3. In Section 3 we prove Lemma 1.3. Proposition 1.4 will be proved in Section 4. In Section 5, apart from the geodesic flows, we take up the properties (P2) and (SE) on a flow having a first integral. In Section 6 we also take up the properties (P2) and (SE) on the convex billiards in relation to the set of poles having positive measure.

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2. Preliminaries

The theory provided in this section has been developed by H. Busemann ([6]) for straight G-spaces and it has been modified by V. Bangert ([2], [3]) and N. Innami ([10], [11], [18]) to be applicable for not straight spaces and convex billiards.

Throughout this section let $\mathbb{T}^2 = M/\Phi$ be a 2-torus of revolution whose definition was given in Section 1. Let $\{\varphi, \psi\}$ generate Φ. If $\tau = \varphi^m \circ \psi^n \in \Phi$, then $\tau(x, y) = (x + ma, y + nb)$ for all $(x, y) \in \mathbb{R}^2$. Let $d_\tau : M \to \mathbb{R}$ be the *displacement function* of $\tau \in \Phi$ which is given by $d_\tau(x) = d(x, \tau(x))$ for all $x \in M$. Since $\mathbb{T}$ is compact and Φ is abelian, every displacement function of $\tau \in \Phi$ attains its minimum in M. Recall that $d_\tau(p) = \min d_\tau =: c$ if and only if there exists a straight line $\gamma : \mathbb{R} \to M$ through p such that $\tau(\gamma(t)) = \gamma(t + c)$ for all $t \in \mathbb{R}$ and $\gamma(0) = p$. Here a *straight line* $\gamma : \mathbb{R} \to M$ is by definition a geodesic such that $d(\gamma(t), \gamma(t')) = |t - t'|$ for all $t, t' \in \mathbb{R}$. Such a straight line is called an *axis* of τ. All points in an axis of τ are also minimum points of d_τ.

2.1. Geodesic

We recall the facts about geodesics in the universal covering M.

(1) The set of poles is $\{p \in M \mid f(p) = \min f\}$ and hence $\{(x, mb) \mid x \in \mathbb{R}, m \in \mathbb{Z}\}$ is a subset of poles.
(2) At any pole all the displacement functions d_τ of isometries $\tau \in \Phi$ attain their minimums.
(3) Every geodesic passing through a pole is straight line in M.
(4) If $f(p) = \min f$ and $p = (x(p), y(p))$, then $y = y(p)$ is a straight line. In particular, $y = mb$, $m \in \mathbb{Z}$, are straight lines.
(5) $x = c$ is a straight line for any $c \in \mathbb{R}$.
(6) All geodesics crossing $y = mb$, $m \in \mathbb{Z}$, are straight lines.

2.2. Slope

Let $\gamma : [0, \infty) \to M$ be a geodesic with unit speed and $\gamma(s) = (u(s), v(s))$ for all $s \in \mathbb{R}$. We set

$$A(\gamma) := \liminf_{s\to\infty} \frac{v(s)}{u(s)}.$$

We call $A(\gamma)$ the *slope* of γ. We say that a geodesic $\gamma : [0, \infty) \to M$ is a *ray* if $d(\gamma(s), \gamma(0)) = s$ for all $s \in [0, \infty)$. If γ is a ray, then

$$A(\gamma) = \lim_{s\to\infty} \frac{v(s)}{u(s)}$$

exists. If $\gamma : \mathbb{R} \to M$ is a straight line, then $A(\gamma) = A(\gamma_-)$ where γ_- is given by $\gamma_-(s) = \gamma(-s)$ for all $s \in \mathbb{R}$. In general, there exist at least two rays from each point p with slope h for each $h \in \mathbb{R}$. Let $\gamma(s) = (u(s), v(s))$, $s \in [0, \infty)$, be a ray. Then $A(\gamma) \neq \pm\infty$ if and only if $u(s) \to \pm\infty$ as $s \to \infty$ if and only if $u(s)$ is not constant in $s > 0$.

Lemma 2.1. *Let $\gamma : [0, \infty) \to M$ be a ray. If $\tau \in \Phi$, then $A(\tau \circ \gamma) = A(\gamma)$.*

Proof. Let $\gamma(s) = (u(s), v(s))$ and $\tau = \varphi^m \circ \psi^n$. Then $\tau \circ \gamma(s) = (u(s)+ma, v(s)+nb)$ for all $s \in [0, \infty)$ where a and b are the minimum positive periods of φ and ψ, respectively. Therefore, we have

$$A(\tau \circ \gamma) = \lim_{s\to\infty} \frac{v(s) + nb}{u(s) + ma} = \lim_{s\to\infty} \frac{v(s)}{u(s)} = A(\gamma).$$

□

If a geodesic γ is an axis of $\varphi^m \circ \psi^n$ through (u, v), then it is a straight line through $(u + kma, v + knb)$ for all $k \in \mathbb{Z}$ and $A(\gamma) = nb/ma$. If the axis γ of $\varphi^m \circ \psi^n$ through

$(0, v_0)$ passes through (ia, v_i) for each $i \in \mathbb{Z}$, then $v_{i+km} = v_i + knb$ for all $k \in \mathbb{Z}$ and $i = 0, 1, \dots, m-1$.

2.3. Co-ray and parallel

Let $\gamma : [0, \infty) \to M$ be a ray with $A(\gamma) = h$. Let p_k be a sequence of points in M converging to p and s_k a sequence going to ∞. Then a sequence of minimizing geodesic segments from p_k to $\gamma(s_k)$ contains a subsequence which converges to a ray $\beta : [0, \infty) \to M$ with $\beta(0) = p$. We call such a ray β a *co-ray* from p to γ.

Lemma 2.2. *If β is a co-ray to a ray γ, then $A(\beta) = A(\gamma)$.*

Proof. Let $\gamma(s) = (u(s), v(s))$, $s > 0$, and $\beta(s) = (w(s), z(s))$, $s > 0$. Suppose $A(\gamma) \neq \pm\infty$. Then, $u(s), w(s) \to \pm\infty$ as $s \to \infty$, say ∞. There exists an $x_0 \in \mathbb{R}$ such that both γ and β cross $x = x_0$. Hence, there exist an isometry $\tau \in \Phi$ and an s_0 such that $\beta([s_0, \infty))$ is contained in the half strip bounded by γ, $x = x_0$ and $\tau \circ \gamma$. This implies that $A(\beta) = A(\gamma)$ because of Lemma 2.1.

Suppose $A(\gamma) = \pm\infty$. Then, $\beta(s) = (c, z(s))$, $s > 0$ for some $c \in \mathbb{R}$. This means that $A(\beta) = \pm\infty$. □

The co-ray relation is not symmetric, in general. Let $\gamma : (-\infty, \infty) \to M$ be a straight line. We say that a straight line $\beta : (-\infty, \infty) \to M$ is a *parallel* to γ if, for any s_0, $\beta|_{[s_0,\infty)}$ and $\beta|_{(-\infty,s_0]}$ are co-rays to γ and γ_-, respectively. The parallel relation is not symmetric, in general. However, it follows from the following lemma that the co-ray and parallel relations are equivalence relations in the set of all geodesics γ with $A(\gamma) \neq 0$ in M.

Lemma 2.3. *Let $h \in \mathbb{R} \setminus \{0\}$. Let γ be a straight line with $A(\gamma) = h$. Then, M is foliated uniquely by parallels to γ. Moreover, if $h = nb/ma$ for some $m, n \in \mathbb{Z}$, then those parallels are the axes of $\varphi^m \circ \psi^n$.*

Let $\pi : M \to \mathbb{T}$ be the covering map.

Lemma 2.4. *Let γ be a straight line in M. Assume that $A(\gamma) \neq 0$. If $A(\gamma)$ is rational or $\pm\infty$, then $\pi \circ \gamma$ is a closed geodesic in $\mathbb{T}$. If $A(\gamma)$ is irrational, then $\pi \circ \gamma$ is dense in $\mathbb{T}$.*

2.4. Busemann function

Let $\gamma : [0, \infty) \to M$ be a ray with $A(\gamma) = h$. Define a function f_γ on M by

$$f_\gamma(q) = \lim_{t\to\infty} (d(q, \gamma(t)) - t)$$

for all $q \in M$. The function f_γ is called the *Busemann function* for γ. Its level set $\{q \in M \mid f_\gamma(q) = c\}$ is denoted by $[f_\gamma = c]$. If γ is a ray with $A(\gamma) \neq 0$, then the Busemann function f_γ is differentiable on M and its gradient vector at p is the reverse of the initial tangent vector of the co-ray from p to γ. Therefore, if γ_v, $v \in S_pM$, is a geodesic with $A(\gamma_v) = A(\gamma)$, then $-v$ is the gradient vector of f_γ at p.

Lemma 2.5. *Let $h \in \mathbb{R} \setminus \{0\}$. If γ and γ_1 are straight lines with slope $A(\gamma) = A(\gamma_1) = h$, then $f_\gamma - f_{\gamma_1}$ is constant on M.*

This lemma shows that the foliation of M by the level sets of f_γ is independent of the choice of γ with $A(\gamma) = h$. From Lemma 2.1 we have the following.

Lemma 2.6. *Let $h \in \mathbb{R} \setminus \{0\}$ and γ a ray with $A(\gamma) = h$. If $\tau \in \Phi$, then $\tau([f_\gamma = c])$ is also a level of f_γ, namely the foliation by the level sets of f_γ is invariant under Φ.*

Let $p \in M$ and $h \in \mathbb{R} \setminus \{0\}$. Take the straight line γ with $A(\gamma) = h$ and $\gamma(0) = p$. Set $f_h = f_\gamma$.

Lemma 2.7. *The level sets $[f_h = 0]$ continuously depend on $h \in \mathbb{R} \setminus \{0\}$.*

Let $Y = \{\tau([f_h = 0]) \mid \tau \in \Phi\}$.

Lemma 2.8. *Let $m \neq 0$ and $n \neq 0$ be relative prime integers. Then there exists the unique $h \in \mathbb{R} \setminus \{0\}$ such that $[f_h = 0]$ passes through $(x(p) + ma, y(p) + nb)$. Moreover, Y divides the curves $C(p, \varphi(p)) = \{(x, y(p)) \mid x(p) \le x \le x(p) + a\}$ and $C(p, \psi(p)) = \{(x(p), y) \mid y(p) \le y \le y(p) + b\}$ into n and m subarcs, respectively.*

Proof. Lemmas 2.5 to 2.7 show the first part of this lemma. The numbers of divided subsegments are equal to the numbers of points at which Y intersects $C(p, \varphi(p))$ and $C(p, \psi(p))$, respectively. □

The following property is important to prove Lemma 1.3.

Lemma 2.9. *Let $h \in \mathbb{R} \setminus \{0\}$. Let C and B be compact subarcs of the level $[f_h = 0]$ with $C \subset B$. Then, for any $\varepsilon > 0$ there exists a $K > 0$ such that* $\max\{d(u, S(q,t)) \mid u \in C\} < \varepsilon$ *for any $t > K$ and any q such that $q = \sigma(t)$ where σ is a straight line intersecting B at $\sigma(0)$ and with $A(\sigma) = h$.*

Proof. Since $B(u, t - d(u,v)) \subset B(v,t)$ for any points u, v with $u \in B(v,t)$, $S(\sigma(t), t)$ lies between $[f_h = 0]$ and $S(\sigma(s), s)$ for $0 < s < t$. Hence, $d(u, S(\sigma(s), s)) \ge d(u, S(\sigma(t), t))$ for $0 < s < t$ and $u \in C$. Since $S(\sigma(t), t)$ continuously depends on σ with $A(\sigma) = h$ and t, we conclude this lemma. □

3. Proof of Lemma 1.3

In this section we use the notation as in Lemma 1.3.

We may assume that $q \in M$ lies in the rectangle $R(p)$ with vertices $(x(p), y(p))$, $(x(p), y(p)+b)$, $(x(p)+a, y(p))$ and $(x(p)+a, y(p)+b)$, namely the domain surrounded by $C(p, \varphi(p))$, $C(p, \psi(p))$, $C(\varphi(p), \psi \circ \varphi(p))$ and $C(\psi(p), \varphi \circ \psi(p))$. Let m and n be relatively prime integers such that $a/n < \varepsilon/2$ or $b/m < \varepsilon/2$. Assume that $a/n < \varepsilon/2$.

From Lemma 2.8 there exists the unique slope $h \in \mathbb{R} \setminus \{0\}$ such that $q' = (x(q) + ma, y(q) + nb) \in [f_h = f_h(q)]$. The distance between the adjacent elements in the set Y as in Lemma 2.7 is less than $\varepsilon/2$. In fact, the length of one of subarcs cut by adjacent intersection points of Y and the parallel to the y-axis through q is less than $\varepsilon/2$. Since adjacent elements in Y have equal distance, the claim is proved.

Let $\pi : M \to \mathbb{T}$ be the covering map. Let C be the subarc of $[f_h = f_h(q)]$ from q to q'. Then $\pi(C)$ is a closed curve in $\mathbb{T}$. Hence, if the length of C is $L(C)$, then all subarcs B of $[f_h = f_h(q)]$ with length greater than $L(C)$ contain $\tau(q)$ for some $\tau \in \Phi$.

Let γ be the straight line with $\gamma(0) = p$ and $A(\gamma) = h$. Let $\Phi(\gamma)$ be the set of those isometries $\tau \in \Phi$ such that γ intersects $\tau(R(p))$. If the diameter of $R(p)$ is d, then for any isometry $\tau \in \Phi(\gamma)$ the distance from $\tau(p)$ to the closest point $\tau_1(p)$, $\tau_1 \in \Phi(\gamma)$, is less than $2d$.

We consider a compact set X such that the length of $X \cap [f_h = s]$ is greater than $L(C)$ $(0 \leq s \leq 2d)$, $\max\{f_h(u) \,|\, u \in X\} = 2d$, $\min\{f_h(u) \,|\, u \in X\} = 0$ and all parallels to γ with distance to γ less than $2d$ pass through X.

Apply Lemma 2.9 to $X \cap [f_h = 2d]$ and $\varepsilon/2$, and we have a $K > 0$ satisfying the property in Lemma 2.9. Therefore, if $\tau(p)$, $\tau \in \Phi(\gamma)$, satisfies that $f_h(\tau(p)) > K+2d$, then for any t with $f_h(\tau(p)) < t < f_h(\tau(p))+2d$, there exists an isometry $\tau_1 \in \Phi$ such that $\tau_1(q) \in X$ and $d(\tau_1(q), S(\tau(p), t)) < \varepsilon$, in other words, $d(\tau^{-1} \circ \tau_1(q), S(p, t)) < \varepsilon$. This completes the proof.

4. Proof of Proposition 1.4

In this section we use the notation as in Proposition 1.4.

Let A be an open set consisting of poles. Let p be any point in M. Let $v \in S_pM$ be any vector and $B(-v, 1/k)$ the open ball in SM centered at $-v$ with radius $1/k$ for $k = 1, 2, \ldots$. It follows from (P2) that there exist a point $-v_k \in B(-v, 1/k)$ and a parameter t_k such that $G_{t_k}(-v_k) \in \pi^{-1}(A)$. Since $\pi(G_{t_k}(-v_k)) = \gamma_{-v_k}(t_k)$ is a pole and $v_k = \dot{\gamma}_{w_k}(t_k)$ where $w_k = -\dot{\gamma}_{-v_k}(t_k)$, any point is not conjugate to $\gamma_{v_k}(0)$ along the geodesic γ_{v_k} for each k. Since the sequence of unit vectors v_k converges to v, there is no point conjugate to $\pi(v)$ along the geodesic γ_v. Thus the point p is a pole, and, hence, all the points in M are poles. This completes the proof of Proposition 1.4.

5. Flows with first integrals

In this section we take up the condition (SE) on a class of flows that have first integrals.

Let M be a smooth manifold and $\varphi_t : M \to M$ be a flow which has a smooth first integral $H : M \to \mathbb{R}$, namely $H(\varphi_t(p)) = H(p)$ for all $p \in M$ and $t \in \mathbb{R}$. Let μ be a measure on M with $\mu(M) = 1$ which is invariant under φ_t. Let $\psi_t : M \to M$ be the flow generated by the gradient vector field of H. We define an equivalence relation $\sim$ as follows; $p \sim q$ if and only if $q = \psi_t(p)$ for some $t \in \mathbb{R}$. Set $N = M/\sim$. Let $\pi : M \to N$ be the natural projection. Define a measure ω on N by $\omega(A) = \mu(\widetilde{A})$ for $A \subset N$ where $\widetilde{A} = \pi^{-1}(A)$.

In this situation the property (P2) is stated as follows:

(P2) Let p be any point in N and U any open subset in N. For every point $x \in \pi^{-1}(p)$ there exist a sequence of points $x_n \in M$ and a sequence $t_n \to \infty$ such that $x_n \to x$ and $\pi(\varphi_{t_n}(x)) \in U$.

We prove the following.

Proposition 5.1. *Assume that the gradient vector of H does not vanish. If φ_t satisfies* (P2), then all $H^{-1}(c)$, $c \in H(M)$, are diffeomorphic to one another. In addition, we assume that there exists a measure ω on N such that $\mu = \omega \wedge \eta$ for some measure η on $H(M)$. Then, φ_t satisfies (SE) if and only if $\varphi_t|H^{-1}(c)$ is ergodic for a. e. $c \in H(M)$.

Proof. Let $a, b \in H(M)$. We prove that $H^{-1}(a)$ is diffeomorphic to $H^{-1}(b)$. To do this we have only to show that for any point $p \in H^{-1}(a)$ there exists a point $q \in H^{-1}(b)$ such that $\pi(p) = \pi(q)$. Take a point x with $H(x) = b$. Let $B(\pi(p), 1/n)$ be the $1/n$-ball around $\pi(p)$. It follows from (P2) that there exist a sequence of points $x_n \to x$ and a sequence of parameters $t_n \to \infty$ such that $\varphi_{t_n}(x_n) \in \pi^{-1}(B(p, 1/n))$. Since $H^{-1}(b)$ is invariant under φ_t, we have $\varphi_{t_n}(x_n)$ converges to a point q in $\pi^{-1}(p) \cap H^{-1}(b)$.

Let $X \subset H(M)$ be the set of all values a such that $\varphi_t|H^{-1}(a)$ is not ergodic. Suppose that X has positive measure. There exist subsets $Y(a) \subset H^{-1}(a)$, $a \in X$, such that $0 < \omega(Y(a)) < \omega(H^{-1}(a))$ and $Y(a)$ is invariant under $\varphi_t|H^{-1}(a)$. Let $p(a) \in M$ such that $H(p(a)) = a \in X$ and $\pi(p(a))$ are constant for all $a \in X$. One of $Y(a)$ and $H^{-1}(a) - Y(a)$ contains $p(a)$ for each $a \in X$, say $Y'(a)$. Set $Y = \cup_{a \in X} Y'(a)$ and $Z = N - \pi(Y)$. If χ_Z is the characteristic function of Z, we see $(\chi_Z \circ \pi)^*$ is not constant a. e. $p \in M$. In fact, $(\chi_Z \circ \pi)^*(p) = 0$ if $p \in Y$ and the set of those points $p \in M$ such that $(\chi_Z \circ \pi)^*(p) > 0$ has positive measure.

Suppose that $\varphi_t|H^{-1}(a)$ is ergodic for a. e. $a \in H(M)$. Then we have

$$(f \circ \pi)^*(p) = \frac{1}{\operatorname{vol} M} \int_M f \circ \pi \, d\mu = \frac{1}{\operatorname{vol} N} \int_N f \, d\omega$$

for any function $f \in L_1(N, \omega)$ and a. e. $p \in M$. Thus (SE) holds. This completes the proof. □

6. Convex billiards

Let M^{n+1} be a complete Riemannian manifold with boundary $\partial M =: B \neq \emptyset$ which is a union of smooth hypersurfaces. Let $q \in B$ be an arbitrary point at which B is smooth and Q_q the symmetry with respect to T_qB, i.e.,

$$Q_q(W) = w - 2\langle w, N(q)\rangle N(q)$$

for any $w \in T_qM$, where $\langle \cdot, \cdot \rangle$ is the Riemannian metric in M and N is the inward unit normal vector field to B. We say that $\gamma : [a, b] \to M$ is a *reflecting geodesic* or briefly a geodesic if there exists the partition $a = a_0 < a_1 < \cdots < a_m = b$ such that

(1) $\gamma(a_i) \in B$, B is smooth at $\gamma(a_i)$ and $\dot\gamma(a_i - 0) \notin T_{\gamma(a_i)}B$ for $i-1, 2, \ldots, m-1$.
(2) $\gamma_i = \gamma|[a_{i-1}, a_i]$ is a geodesic in M in the usual sense for $i = 1, 2, \ldots, m$.
(3) $Q(\dot\gamma(a_i - 0)) = \dot\gamma(s_i + 0)$ for $i = 1, 2, \ldots, m-1$.

A variation of a geodesic γ through geodesics yields a Jacobi vector field Y along γ which satisfies the following properties at the boundary:

(1) $Q(Y(a_i - 0)) = Y(a_i + 0)$
(2) $Q(\nabla_{\dot\gamma(a_i-0)}Y) - \nabla_{\dot\gamma(a_i+0)}Y = A(\dot\gamma(A_i + 0))(Y^\perp(a_i + 0))$

where $A(\dot\gamma(a_i + 0))$ is a symmetric endomorphism of n-dimensional subspace $\dot\gamma(a_i + 0)^\perp$ of $T_{\gamma(a_i)}M$ which is perpendicular to $\dot\gamma(a_i+0)$. We say that $\gamma(t_1)$, $t_1 \neq t_0 \in [a, b]$, is *conjugate* to $\gamma(t_0)$, $t_0 \in [a, b]$, along γ if there exists a nontrivial Jacobi vector field Y along γ with $Y(t_0) = Y(t_1) = 0$.

Let SM be the unit tangent bundle of M. For a vector $v \in SM$ let γ_v be the geodesic with $\dot\gamma_v(0) = v$. If $\pi(v) \in B$, then $\dot\gamma_v(0)$ is considered either $\dot\gamma(+0)$ of $\dot\gamma(-0)$. The geodesic γ_v are defined on the whole real line $(-\infty, \infty)$ for almost all $v \in SM$. Let the set of all such vectors be denoted by SM, also.

Let $\varphi_t \,|\, SM \to SM$ be a flow given by $\varphi_t(v) = \dot\gamma_v(t)$ for any $v \in SM$. The symbol ∂SM denotes the set of all vectors $v \in SM$ with $\pi(v) = q \in B$ and $\langle v, N(q)\rangle > 0$.

Let T be the *ceiling function* on ∂SM, i.e., $T(v)$ is the first parameter such that $\gamma_v(T(v)) \in B$, $T(v) > 0$. We assume that $T(v) < \infty$ for all $v \in \partial SM$.

Let $\varphi : \partial SM \to \partial SM$ be a map given by $\varphi(v) = \dot\gamma_v(T(v) + 0)$ for all $v \in \partial SM$. We call φ the *billiard ball map* of B. We say that conjugate points along a geodesic γ_v are separated by the boundary B if there exists only one point $\gamma_v(t) \in B$ between any pair of adjacent conjugate points. We say that a point $p \in B$ is a *pole* if the geodesic $\gamma_v(t)$ satisfies this property for every vector $v \in \partial SM$ with $\pi(v) = p$. This naming comes from the geometry of geodesics for convex billiards.

Let S be the second fundamental form of B at differentiable points with respect to N and λ_S the maximal eigenvalue function of S. The following theorem is proved in [17].

Theorem 6.1 ([17])**.** *Let M^{n+1} be a compact Riemannian manifold with nonpositive sectional curvature and $T(v) < \infty$ for all $v \in \partial SM$. If all points in B are poles, then we have the inequality*

$$\int_B \lambda_S \, dB \geq \frac{\mathrm{vol}\,(B)^2}{(n+1)\mathrm{vol}\,(M)}$$

and the equality holds if and only if λ_S is constant and M is a spherical domain with flat metric of radius ${\lambda_S}^{-1}$.

If M is a simply connected domain in the 2-dimensional Euclidean space $\mathbb{E}^2$, then the right hand side is greater than or equal to 2π because of the isoperimetric inequality. Therefore we have the following corollary which is a slight generalization of the theorem in [4], [20].

Corollary 6.2 ([17])**.** *Suppose $M \subset \mathbb{E}^2$ is a simply connected domain bounded by a closed curve B and the sum of outer angles of B is nonnegative. Then all the points of B are poles if and only if M is a circular domain in $\mathbb{E}^2$.*

We consider the same properties as (P2) and (SE) on the billiard ball map. We can prove the following.

Theorem 6.3. *Let B be a simple closed convex curve in the Euclidean plane such that the set of poles has positive measure. Then, B is a circle if and only if its billiard ball map φ satisfies* (SE)*.*

Proof. We assume that B is the unit circle. If M denotes the disk bounded by B, then $\partial SM = B \times (-1, 1)$ and the measure on it is $ds \wedge dt$ where ds is the line element of B and dt is the natural measure. Note that $\varphi = \cup_{0<\theta<\pi} R(2\theta) \times I$ where $R(2\theta)$ is the rotation around the center of $B \times \{\cos\theta\}$ by 2θ and I is the identity map on $(-1, 1)$. Since $R(2\theta)$ is ergodic on $B \times \{\cos\theta\}$ if $2\pi/\theta$ is irrational, Proposition 5.1 shows that φ satisfies the condition (SE).

Assume that the billiard ball map φ satisfies (SE). Let Z be the set of all poles in B. Let $v \in \partial SM$ be an arbitrary point. Since Z has positive measure, the orbit $\varphi^n(v)$ of v meets Z. Then the conjugate pairs of the geodesic $\gamma_v(t)$, $t \in \mathbb{R}$, are separated by the points in the boundary B. This implies that all points in B are poles. Corollary 6.2 completes the proof. $\square$

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