Nihonkai Math. J.
Vol.23(2012), 11–19

ON TOTALLY GEODESIC COMPLEX SUBMANIFOLDS OF PSEUDO-BOCHNER-FLAT LOCALLY CONFORMAL KÄHLER MANIFOLDS IN THE HERMITIAN SENSE

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Dedicated to Professor Kentaro Mikami on the occasion of his 65-th birthday.

ABSTRACT. We prove the l.c.K. version of the characterization theorem, due to B.-Y. Chen, L. Vanhecke and L. Verstraelen, for totally geodesic complex submanifolds of Bochner-flat Kähler manifolds.

1. Introduction

In [1], B.-Y. Chen, L. Vanhecke and L. Verstraelen proved a characterization theorem for totally geodesic complex submanifolds of Bochner-flat Kähler manifolds:

Theorem A ([1]) *Let (M, J, g) be a complex m-dimensional Kähler submanifold of a complex n-dimensional Bochner-flat Kähler manifold $(\widetilde{M}, \widetilde{J}, \widetilde{g})$. Then M is totally geodesic if and only if the Ricci tensors R_1 and $\widetilde{R}_1$ of M and $\widetilde{M}$ satisfy the following relation*

$$\widetilde{R}_1 = \frac{n+2}{m+2}R_1 + \left\{ \frac{\widetilde{r}}{4(n+1)} - \frac{(n+2)r}{4(m+1)(m+2)} \right\} g,$$

where r and $\widetilde{r}$ denote the scalar curvatures of M and $\widetilde{M}$ respectively.

In [5] the author introduced the notion of the pseudo-Bochner curvature tensor on a Hermitian manifold which is constructed out of the curvature tensor of the Hermitian (or Chern) connection and is conformally invariant. In the Kähler case, this tensor coincides with the original Bochner curvature tensor.

In this paper, the ambient manifolds are assumed to be locally conformal Kähler (briefly, l.c.K.) manifolds. Then their complex submanifolds inherit l.c.K. structures. By an *l.c.K. submanifold*, we mean a complex submanifold with the induced l.c.K. structure.

2010 *Mathematics Subject Classification.* Primary 53C55, 53C40.

Key words and phrases. Hermitian pseudo-curvature tensor, pseudo-Bochner curvature tensor, l.c.K. submanifolds, PG-equation, totally geodesic in the Hermitian sense.

Our purpose of this paper is, with respect to l.c.K. submanifolds, to prove the following theorem corresponding to Theorem A mentioned above.

Theorem 1.1. *Let (M, J, g) be a complex m-dimensional l.c.K. submanifold of a complex n-dimensional pseudo-Bochner-flat l.c.K. manifold $\widetilde{M}$. Then M is totally geodesic in the Hermitian sense if and only if the pseudo-Ricci tensors P_1 and $\widetilde{P}_1$ satisfy the following relation*

$$\widetilde{P}_1 = \frac{n+2}{m+2}P_1 + \left\{\frac{\widetilde{p}}{4(n+1)} - \frac{(n+2)p}{4(m+1)(m+2)}\right\} g, \qquad (*)$$

where p and $\widetilde{p}$ denote the pseudo-scalar curvatures of M and $\widetilde{M}$ respectively.

Throughout this paper, we work in C^∞-category and deal with connected complex manifolds of complex dimension ≥ 2 without boundary only.

2. Pseudo-Bochner curvature tensor

Let M be a Hermitian manifold with complex structure J and compatible Riemannian metric g. The Kähler form Ω on M is defined by $\Omega(X, Y) = g(X, JY)$ for all vector fields X, Y on M. The Hermitian (or Chern) connection of M is a unique affine connection D on M such that $DJ = 0, Dg = 0$, and the torsion tensor T satisfies $T(JX, Y) = JT(X, Y)$ (cf. [4]). The Hermitian connection D and the Levi-Civita connection ∇ are related by

$$g(D_X Y, Z) = g(\nabla_X Y, Z) + \tfrac{3}{2}\, d\Omega(JX, Y, Z) \qquad (2.1)$$

for all vector fields X, Y, Z on M. Let H be the Hermitian curvature tensor (the curvature tensor of D) on M, i.e.,

$$H(X, Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z.$$

We also use the Hermitian curvature tensor H of type $(0, 4)$ given by

$$H(X, Y, Z, W) = g(H(Z, W)Y, X).$$

Since $DJ = 0$ and $Dg = 0$, the curvature tensor H have the following properties: For any vector fields X, Y, Z, W on M,

$$H(X, Y, Z, W) = -H(Y, X, Z, W) = -H(X, Y, W, Z), \qquad (2.2)$$

$$H(JX, JY, Z, W) = H(X, Y, JZ, JW) = H(X, Y, Z, W). \qquad (2.3)$$

Moreover, a general affine connection D satisfies the Bianchi's first identity

$$\mathfrak{S}_{X,Y,Z} \left[\, H(X, Y)Z - T(T(X, Y), Z) - (D_X T)(Y, Z) \,\right] = 0 \quad \text{(cf. [3])},$$

or equivalently,

$$\mathfrak{S}_{Y,Z,W}\,\big[\,H(X,Y,Z,W)-g(X,T(T(Y,Z),W))-g(X,(D_YT)(Z,W))\,\big]=0.$$

Since the torsion tensor T of the Hermitian connection D satisfies $T(JX,Y) = JT(X,Y)$, we then also have

$$\begin{aligned}&\mathfrak{S}_{Y,Z,W}\,\big[\,H(JX,JY,JZ,JW)\\ &\qquad +g(X,T(T(Y,Z),W))+g(JX,(D_{JY}T)(Z,W))\,\big]=0.\end{aligned}$$

Since, by (2.3), $H(JX,JY,JZ,JW) = H(X,Y,Z,W)$, the *Bianchi's first identity* for the Hermitian connection D reduces to

$$\begin{aligned}&\mathfrak{S}_{Y,Z,W}\,H(X,Y,Z,W)\\ &\quad =\tfrac{1}{2}\,\mathfrak{S}_{Y,Z,W}\,\big[\,g(X,(D_YT)(Z,W))-g(JX,(D_{JY}T)(Z,W))\,\big]\,.\end{aligned}\tag{2.4}$$

For each unit vector X in T_xM, the Hermitian holomorphic sectional curvature $\mathcal{H}(X)$ for the holomorphic plane spanned by X and JX is given by

$$\mathcal{H}(X)=H(X,JX,X,JX).$$

We have the unique tensor P , called the *Hermitian pseudo-curvature tensor* ([5]), on M defined by

$$\begin{aligned}P(X,Y,Z,W)=\tfrac{1}{8}\,\big[\,&H(X,Z,Y,W)-H(X,W,Y,Z)\\ &+H(Y,W,X,Z)-H(Y,Z,X,W)\\ &+H(X,JZ,Y,JW)-H(X,JW,Y,JZ)\\ &+H(Y,JW,X,JZ)-H(Y,JZ,X,JW)\\ &+2H(X,Y,Z,W)+2H(Z,W,X,Y)\,\big]\end{aligned}\tag{2.5}$$

for all vector fields X,Y,Z,W on M. This tensor P has the same symmetries as the Riemannian curvature tensor on a Kähler manifold.

Proposition 2.1. ([5]) *For any vector fields X,Y,Z,W on M,*

$$P(X,Y,Z,W)=-P(Y,X,Z,W)=-P(X,Y,W,Z),$$

$$P(JX,JY,Z,W)=P(X,Y,JZ,JW)=P(X,Y,Z,W),$$

$$P(X,Y,Z,W)=P(Z,W,X,Y),$$

$$\mathfrak{S}_{X,Y,Z}\,P(X,Y)Z=0\qquad\big(\,\mathfrak{S}_{Y,Z,W}\,P(X,Y,Z,W)=0\,\big)$$

where $g(P(X,Y)Z,W)=P(W,Z,X,Y)$.

In particular, it holds that $P(X,JX,X,JX) = H(X,JX,X,JX)$ for all vector field X on M. From this, we have the following theorem.

Theorem 2.1. ([5]) *A Hermitian manifold M is of pointwise constant Hermitian holomorphic sectional curvature k if and only if $P = \frac{k}{8}\, g \triangle g$.*

Here, for any two tensors a, b of type (0,2), $a \triangle b$ is defined by

$$a \triangle b = a \owedge b + \overline{a} \owedge \overline{b} + 2\overline{a} \otimes \overline{b} + 2\overline{b} \otimes \overline{a},$$

where $a \owedge b$ denotes a tensor of type (0,4) given by

$$\begin{aligned}(a \owedge b)(X, Y, Z, W) = a(X, Z)b(Y, W) - a(X, W)b(Y, Z) \\ +a(Y, W)b(X, Z) - a(Y, Z)b(X, W),\end{aligned}$$

and $\overline{a}(X, Y) = a(X, JY)$.

By means of the Hermitian pseudo-curvature tensor P, we also define the (Hermitian) *pseudo-Ricci tensor* P_1 and the (Hermitian) *pseudo-scalar curvature* p as follows:

$$P_1(X, Y) = \mathrm{tr}\left[\ Z \to P(Z, Y)X\ \right], \quad p = \mathrm{tr}\, P_1.$$

Pseudo-Ricci tensor P_1 is symmetric and compatible with J:

$$P_1(X, Y) = P_1(Y, X), \quad P_1(JX, JY) = P_1(X, Y).$$

Theorem 2.2. ([5]) *Let M be a complex m-dimensional Hermitian manifold of pointwise constant Hermitian holomorphic sectional curvature k. Then*

$$P_1 = \frac{(m+1)k}{2} g, \quad p = m(m+1)k.$$

Moreover we define a tensor B_H, called the *pseudo-Bochner curvature tensor* ([5]), on M as follows:

$$B_H = P - \frac{1}{2(m+2)}\, g \triangle P_1 + \frac{p}{8(m+1)(m+2)}\, g \triangle g. \tag{2.6}$$

Theorem 2.3. ([5]) *The pseudo-Bochner curvature tensor on a Hermitian manifold is conformally invariant.*

Remark 2.1. (cf. [5]) Note that, on Kähler manifold, the pseudo-quantities P, P_1, p and B_H defined above coincide with the curvature tensor, the Ricci tensor, the scalar curvature and the original Bochner curvature tensor of the Levi-Civita connection respectively.

3. Locally conformal Kähler manifolds

A Hermitian manifold M is said to be *locally conformal Kähler* (briefly, *l.c.K.*) if there is a closed 1-form ω on M, called the *Lee form*, such that $d\Omega = \omega \wedge \Omega$ (cf. [7],[2]). In particular, if ω is exact, M is said to be *globally conformal Kähler* (briefly, *g.c.K.*). If $\dim_{\mathbb{C}} M = m \geq 3$, the closedness of ω follows from the condition $d\Omega = \omega \wedge \Omega$. From (2.1), we have immediately the following lemma.

Lemma 3.1. *The condition $d\Omega = \omega \wedge \Omega$ is equivalent to the condition*

$$2T(X,Y) = \omega(X)Y - \omega(Y)X - \omega(JX)JY + \omega(JY)JX,$$

where T is the torsion tensor of the Hermitian connection D of M.

We shall give a typical example of pseudo-Bochner-flat l.c.K. manifolds.

Example 3.1. Let α be any non-zero complex number with $|\alpha| \neq 1$, and let G_α be the cyclic group generated by the transformation $(z^1, \dots, z^m) \to (\alpha z^1, \dots, \alpha z^m)$ of $\mathbb{C}^m - \{0\}$. Then G_α acts freely on $\mathbb{C}^m - \{0\}$ as a properly discontinuous group of complex analytic transformations. Thus the quotient space $H_\alpha^m = (\mathbb{C}^m - \{0\})/G_\alpha$ has the structure of a complex manifold. This manifold H_α^m is called the *Hopf manifold*. As is well-known (cf. [4]), H_α^m is diffeomorphic with the product $S^1 \times S^{2m-1}$ of two odd-dimensional spheres. In particular H_α^m is compact, and does not admit any Kähler metric. On $\mathbb{C}^m - \{0\}$, we consider a Hermitian metric

$$ds^2 = \frac{2}{\|z\|^2} \sum_{i=1}^{m} dz^i d\bar{z}^i,$$

where $\|z\|^2 = \sum_{i=1}^m z^i \bar{z}^i$. Since this metric is invariant under the action of G_α, it induces a Hermitian metric, called the *Boothby metric*, on H_α^m (cf. [2]). The Hopf manifold H_α^m with the Boothby metric is an l.c.K. manifold whose local Kähler metrics are flat. On such a manifold, the pseudo-Bochner curvature tensor vanishes everywhere.

4. Locally conformal Kähler submanifolds

Let $\psi : M \to \widetilde{M}$ be a holomorphic immersion of a complex manifold (M, J) into an l.c.K. manifold $(\widetilde{M}, \widetilde{J}, \widetilde{g})$. Then the Riemannian metric $g = \psi^* \widetilde{g}$ induced on M is Hermitian. Let $\widetilde{\Omega}$ and $\widetilde{\omega}$ be the Kähler form and Lee form on $\widetilde{M}$ respectively. Then $d\widetilde{\Omega} = \widetilde{\omega} \wedge \widetilde{\Omega}$. Putting $\Omega = \psi^* \widetilde{\Omega}$ and $\omega = \psi^* \widetilde{\omega}$, it is easy to see that Ω is the Kähler form associated with g and satisfies $d\Omega = \omega \wedge \Omega$. Hence (M, J, g) is an l.c.K. manifold. The normal space $T_x^\perp M$ is the orthogonal complement of $T_x M$ in $T_x \widetilde{M}$

with respect to $\widetilde{g}$. The tangent bundle of $\widetilde{M}$, restricted to M, is the Whitney sum of the tangent bundle TM and the normal bundle $T^{\perp}M$;

$$T\widetilde{M}|M = TM \oplus T^{\perp}M. \tag{4.1}$$

We denote by $\widetilde{D}$ the Hermitian connection of $\widetilde{M}$ with respect to $\widetilde{g}$. Let X and Y be any vector fields on M, and ξ any normal vector field on M. From (4.1) we may then decompose $\widetilde{D}_X Y$ and $\widetilde{D}_X \xi$ respectively as follows ([6]):

$$\widetilde{D}_X Y = D_X Y + \sigma(X, Y), \tag{4.2}$$

$$\widetilde{D}_X \xi = -A_\xi X + D^{\perp}_X \xi, \tag{4.3}$$

where $D_X Y$ (resp. $-A_\xi X$) and $\sigma(X, Y)$ (resp. $D^{\perp}_X \xi$) are the tangential and normal components respectively of $\widetilde{D}_X Y$ (resp. $\widetilde{D}_X \xi$). We call (4.2) (resp. (4.3)) *G-formula* (resp. *W-formula*). D defines the Hermitian connection of M with respect to the induced Hermitian metric $g = \psi^* \widetilde{g}$. σ is a normal vector bundle valued symmetric bilinear form on M. Since $\widetilde{D}\widetilde{J} = 0$, σ satisfies

Lemma 4.1. $\sigma(JX, Y) = \sigma(X, JY) = \widetilde{J}\sigma(X, Y)$.

As is an immediate consequence of Lemma 4.1, we have

Lemma 4.2. $\operatorname{tr} \sigma = 0$.

This is corresponding to the well-known fact that every Kähler submanifold is minimal. We call σ the *Hermitian second fundamental form* of l.c.K. submanifold M.

With respect to general affine connections, totally geodesic submanifolds are defined as follows:

Definition 4.1. A submanifold N of a manifold $\widetilde{N}$ is *totally geodesic* if geodesics of N are carried into geodesics of $\widetilde{N}$ by the immersion.

In our l.c.K. case, we have the following.

Proposition 4.1. *An l.c.K. submanifold M of an l.c.K. manifold $\widetilde{M}$ is totally geodesic in the Hermitian sense if and only if $\sigma = 0$ identically.*

5. Fundamental Equations

Let M be a complex m-dimensional l.c.K. submanifold of a complex n-dimensional l.c.K. manifold $\widetilde{M}$. Let $\widetilde{H}$ be the curvature tensor of the Hermitian connection $\widetilde{D}$ of $\widetilde{M}$. Then, for all vector fields X, Y, Z on M, we have

$$\widetilde{H}(X, Y)Z = \widetilde{D}_X \widetilde{D}_Y Z - \widetilde{D}_Y \widetilde{D}_X Z - \widetilde{D}_{[X,Y]} Z.$$

Thus, by using G-formula (4.2) and W-formula (4.3), we obtain

$$\begin{aligned}\widetilde{H}(X,Y)Z = H(X,Y)Z - A_{\sigma(Y,Z)}X + A_{\sigma(X,Z)}Y \\ +\sigma(X, D_Y Z) - \sigma(Y, D_X Z) - \sigma([X,Y],Z) \\ +D^{\perp}_X\sigma(Y,Z) - D^{\perp}_Y\sigma(X,Z),\end{aligned}$$

where H is the curvature tensor of the Hermitian connection D of M. Therefore, for all vector fields X, Y, Z, W on M, we have

$$\begin{aligned}H(X,Y,Z,W) &= \widetilde{H}(X,Y,Z,W) \\ &+\widetilde{g}(\sigma(X,Z),\sigma(Y,W)) - \widetilde{g}(\sigma(X,W),\sigma(Y,Z)).\end{aligned} \tag{5.1}$$

Equation (5.1) is called *G-equation*. Moreover, between the Hermitian pseudo-curvature tensors P and $\widetilde{P}$, there is a similar relation:

$$\begin{aligned}P(X,Y,Z,W) &= \widetilde{P}(X,Y,Z,W) \\ &+\widetilde{g}(\sigma(X,Z),\sigma(Y,W)) - \widetilde{g}(\sigma(X,W),\sigma(Y,Z)).\end{aligned} \tag{5.2}$$

Equation (5.2) is called *PG-equation*.

Now, suppose the ambient space $\widetilde{M}$ is pseudo-Bochner-flat. Then, by (2.6), we get

$$\widetilde{P} = \frac{1}{2(n+2)}\,\widetilde{g} \triangle \widetilde{P}_1 - \frac{\widetilde{p}}{8(n+1)(n+2)}\,\widetilde{g} \triangle \widetilde{g}\,. \tag{5.3}$$

Moreover, suppose that the pseudo-Ricci-tensors P_1 and $\widetilde{P}_1$ of M and $\widetilde{M}$ satisfy the relation

$$\widetilde{P}_1(X,Y) = \alpha P_1(X,Y) + \beta g(X,Y) \tag{**}$$

for any vector fields X, Y on M and for some functions α, β on M. Let $E_1, ..., E_{2m}$ be an orthonormal basis of the tangent space of M. Then, from (5.3), we find

$$\begin{aligned}&\sum_{i=1}^{2m} \widetilde{P}(E_i, X, E_i, Y) \\ &= \frac{m+2}{n+2}\widetilde{P}_1(X,Y) + \frac{1}{2(n+2)}\left\{ \sum_{i=1}^{2m} \widetilde{P}_1(E_i,E_i) - \frac{(m+1)\widetilde{p}}{n+1} \right\} g(X,Y)\end{aligned} \tag{5.4}$$

Substituting (**) to (5.4), we get

$$\begin{aligned}(n+2)\sum_{i=1}^{2m} \widetilde{P}(E_i, X, E_i, Y) &= (m+2)\alpha P_1(X,Y) \\ &+\left\{\frac{\alpha p}{2} + 2(m+1)\beta - \frac{(m+1)\widetilde{p}}{2(n+1)}\right\} g(X,Y).\end{aligned} \tag{5.5}$$

On the other hand, PG-equation (5.2) implies

$$\sum_{i=1}^{2m} \widetilde{P}(E_i, X, E_i, Y) = P_1(X,Y) + \sum_{i=1}^{2m} \widetilde{g}(\sigma(X,E_i), \sigma(Y,E_i)).$$

Therefore we obtain

$$\lambda P_1(X,Y) - \mu g(X,Y) = -(n+2)\sum_{i=1}^{2m} \widetilde{g}(\sigma(X,E_i), \sigma(Y,E_i)), \tag{5.6}$$

where

$$\lambda = n+2-(m+2)\alpha, \quad \mu = \frac{\alpha p}{2} + 2(m+1)\beta - \frac{(m+1)\widetilde{p}}{2(n+1)}. \tag{5.7}$$

Consequently we have the following.

Theorem 5.1. *Let M be a complex m-dimensional l.c.K. submanifold of a complex n-dimensional pseudo-Bochner-flat l.c.K. manifold $\widetilde{M}$ satisfying $(**)$. Then $\lambda P_1(X,X) - \mu g(X,X) \le 0$. The equality sign holds if and only if M is totally geodesic in the Hermitian sense.*

Corollary 5.1. *Let M be a complex m-dimensional l.c.K. submanifold of a complex n-dimensional pseudo-Bochner-flat l.c.K. manifold $\widetilde{M}$ satisfying $(**)$. Then*

$$(n+1)\left[\,\{n+2-2(m+1)\alpha\}p - 4m(m+1)\beta\,\right] \le -m(m+1)\widetilde{p}.$$

The equality sign holds if and only if M is totally geodesic in the Hermitian sense.

6. Proof of Theorem 1.1

If P_1 and $\widetilde{P}_1$ satisfy $(*)$, then they satisfy $(**)$ with

$$\alpha = \frac{n+2}{m+2}, \quad \beta = \frac{\widetilde{p}}{4(n+1)} - \frac{(n+2)p}{4(m+1)(m+2)}$$

from which we find $\lambda = \mu = 0$, where λ and μ are given by (5.7). Thus, by Theorem 5.1, we see that M is totally geodesic in the Hermitian sense.

Conversely, if M is a totally geodesic submanifold of $\widetilde{M}$ in the Hermitian sense, then PG-equation (5.2) and (5.4) imply

$$P_1 = \frac{1}{2(n+2)}\sum_{i=1}^{2m} \widetilde{P}_1(E_i,E_i)g + \frac{m+2}{n+2}\widetilde{P}_1 - \frac{(m+1)\widetilde{p}}{2(n+1)(n+2)}g \tag{6.1}$$

on M. From this, we find

$$p = \frac{2(m+1)}{n+2}\sum_{i=1}^{2m} \widetilde{P}_1(E_i,E_i) - \frac{m(m+1)\widetilde{p}}{(n+1)(n+2)} \tag{6.2}$$

on M. Substituting (6.2) into (6.1), we obtain $(*)$.

Remark 6.1. On an l.c.K. submanifold M of $\widetilde{M}$, by (2.1) we have

$$\widetilde{D}_X Y = \widetilde{\nabla}_X Y - \frac{1}{2}\omega(X)Y - \frac{1}{2}\omega(JX)JY + \frac{1}{2}g(X,Y)\widetilde{B},$$
$$D_X Y = \nabla_X Y - \frac{1}{2}\omega(X)Y - \frac{1}{2}\omega(JX)JY + \frac{1}{2}g(X,Y)B$$

for all vector fields X, Y on M, where $B = \omega^{\#}$ and $\widetilde{B} = \widetilde{\omega}^{\#}$ are the Lee vector fields of M and $\widetilde{M}$, respectively. From these equations, we get

$$\sigma(X,Y) = h(X,Y) + \frac{1}{2}g(X,Y)\widetilde{B}^{\perp},$$

where h denotes the (Riemannian) second fundamental form and $\widetilde{B}^{\perp}$ the normal component of $\widetilde{B}$, i.e., $\widetilde{B}^{\perp} = \widetilde{B} - B$. Therefore $\sigma = 0$ means that for any normal vector field ξ on M, we have

$$g(A^{\nabla}_{\xi}(X), Y) = \widetilde{g}(h(X,Y),\xi) = -\frac{1}{2}\widetilde{\omega}(\xi)g(X,Y),$$

where A^{∇}_{ξ} denotes the Weingarten operator corresponding to ξ. Namely, an l.c.K. submanifold M with $\sigma = 0$ is a totally umbilical submanifold of $\widetilde{M}$ in the usual Riemannian sense.

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Received January 10, 2012