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THE SUBDIVISION OF THE WINDOW DERIVED FROM FINITE SUBSEQUENCES OF FIBONACCI SEQUENCES

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ABSTRACT. The Fibonacci sequences can be identified with 1-dimensional quasiperiodic tilings by the canonical projection method. We divide the window of the canonical projection method into smaller intervals by using local configurations. Then, we show that the intervals which appears in the window are divided into the ratio at $1 : 1/\tau : 1$ ad infinitum.

1. Introduction

Fibonacci sequences are a well-known example of 1-dimensional quasiperiodic tilings. The Fibonacci sequences can be obtained by several methods. The Fibonacci substitution rule σ is defined on bi-infinite two-symbol sequences with an alphabet $\{A, B\}$ by replacing A by AB and B by A. We say that a sequence S' is a predecessor of a sequence S with respect to the substitution rule σ if $\sigma S' = S$. Fibonacci sequences are defined as bi-infinite two-symbol sequences that have an infinite number of predecessors with respect to σ ([1], [2], [3]).

Another method for constructing the Fibonacci sequences is the canonical projection method ([1], [2]). In this method, we consider the standard lattice $\mathbb{Z}^2$ in $\mathbb{R}^2$ and 1-dimensional subspace E of $\mathbb{R}^2$ with the slope $1/\tau$ ($\tau = (1 + \sqrt{5})/2$) and its orthogonal complement $E^\perp$. We take a interval W in $E^\perp$. We call the interval W the window. Let $\pi : \mathbb{R}^2 \to E$ be the orthogonal projection onto E, and $\pi^\perp : \mathbb{R}^2 \to E^\perp$ the orthogonal projection onto $E^\perp$. Then, 1-dimensional space E is divided into countable many line segments by the point set $\pi((s + \mathbb{Z}^2) \cap (W \times E))$, where $s + \mathbb{Z}^2$ denotes the translation of $\mathbb{Z}^2$ by a vector s in $E^\perp$. These line segments have 2 kinds of the lengths. We identify longer one with the symbol A and shorter one with the symbol B. Here we put $W = (-\cos\theta, \sin\theta]$, where $\tan\theta = 1/\tau$ (see

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Fig. 1). Then we obtain Fibonacci sequences for all s in $E^{\perp}$ (see Fig. 2). We call a point $v \in \pi((s+\mathbb{Z}^2) \cap (W \times E))$ a vertex of a Fibonacci sequence.

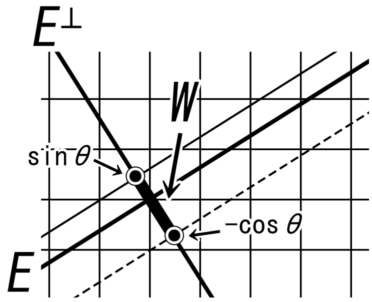

FIG. 1. the canonical projection method

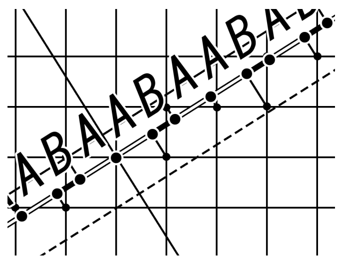

FIG. 2. a Fibonacci sequence by the canonical projection method

We consider a finite subsequence $X_{-n}\dots X_{-2}X_{-1}\cdot X_1X_2\dots X_n$ in a Fibonacci sequence, where X_k denotes the symbol A or B and "$\cdot$" denotes the position of a vertex v. We put $v_0 = v$. For $k > 0$, we take vertices v_{k-1}, v_k such that a line segment $\overline{v_{k-1}v_k}$ is identified with the symbol X_k. For $k < 0$, we take vertices v_k, v_{k+1} such that a line segment $\overline{v_kv_{k+1}}$ is identified with the symbol X_k. By identifying $s+\mathbb{Z}^2$ with the square tiling, $e(1)$ and $e(2)$ denote a horizontal edge and a vertical edge of the square tiling respectively. A line segment corresponding to A is obtained as the image $\pi(e(1))$ of a horizontal edge $e(1)$ of square lattice $s+\mathbb{Z}^2$. A line segment corresponding to B is obtained as the image $\pi(e(2))$ of a vertical edge $e(2)$. Since $\pi|(s+\mathbb{Z}^2)$ is injective, there exists unique point u_k in $(s+\mathbb{Z}^2)\cap(W\times E)$ such that $\pi(u_k) = v_k$ for each $-n \le k \le n$. We define the interval $U(X_{-n}\dots X_{-1}\cdot X_1X_2\dots X_n)$ for $X_{-n}\dots X_{-1}\cdot X_1X_2\dots X_n$ by $U(X_{-n}\dots X_{-1}\cdot X_1X_2\dots X_n) = \{\pi^{\perp}(u)+m \in W \mid m+V \subseteq W, m \in E^{\perp}\}$, where $V = \bigcup_{i=-n}^{n}\{\pi^{\perp}(u_i)\}$ and $m+V = \{m+x \mid x \in V\}$.

The finite subsequence with the length $2n$ on both sides of a vertex v is $X_{-n}\dots X_{-1}\cdot X_1X_2\dots X_n$ if and only if $\pi^{\perp}(u) \in U(X_{-n}\dots X_{-1}\cdot X_1\dots X_n)$, where u in $(s+\mathbb{Z}^2)\cap(W\times E)$ such that $\pi(u)=v$.

For each n, the window W is divided into the intervals $U(X_{-n}\dots X_{-1}\cdot X_1\dots X_n)$. Since the Fibonacci sequences form a local isomorphism class, we have the same division up to scale of the window for any Fibonacci sequence.

The point set D_n is defined to be the set of the end-points of intervals $U(X_{-n}\dots X_{-1}\cdot X_1\dots X_n)$ for all finite subsequences $X_{-n}\dots X_{-1}\cdot X_1\dots X_n$ with the length $2n$. Note that $D_k \subset D_{k+1}$. For example, all finite subsequences with the length 2 are $A\cdot B$, $A\cdot A$ and $B\cdot A$. Then we can see that $U(A\cdot B) = (-\cos\theta, -\sin\theta/\tau]$, $U(A\cdot A) = (-\sin\theta/\tau, 0]$ and $U(B\cdot A) = (0, \sin\theta]$, and that $D_1 = \{-\sin\theta/\tau, 0, \sin\theta\}$.

Let S be the Fibonacci sequence obtained by the canonical projection method with respect to the window $W = (-\cos\theta, \sin\theta]$ and $s + \mathbb{Z}^2$, and S' be the bi-infinite two-symbol sequence with an alphabet $\{A, B\}$ with respect to the window $W' = (-1/\tau)W = [-\sin\theta/\tau, \sin\theta)$ and $s + \mathbb{Z}^2$. It is well-known that S' is the predecessor of the Fibonacci sequence S and that S' is a Fibonacci sequence (cf. [3]).

By a similar way, we can define intervals $U(X_{-n} \dots X_{-1} \cdot X_1 \dots X_n)$ and the point set D_n with respect to the window $W' = (-1/\tau)W = [-\sin\theta/\tau, \sin\theta)$ and $s + \mathbb{Z}^2$.

We define the point set $D_{1:1/\tau:1}$ by the following process: We divide the window into 3 intervals in ratio $1 : 1/\tau : 1$. Then, we divide each one of the intervals which appears in the window into 3 intervals in ratio $1 : 1/\tau : 1$. We continue this process ad infinitum, and denote the set of the dividing points by $D_{1:1/\tau:1}$.

We obtain the following theorem:

Theorem 1.1. *The subdivision of the window has the following properties* (1)–(4)*:*

(1) $\bigcup_{n=1}^{\infty} D_n = D_{1:1/\tau:1}$ *(see Fig. 3).*
(2) *The subdivision of the window* $(-1/\tau)W$ *is the restriction to* $(-1/\tau)W$ *of the subdivision of the window* W*.*
(3) *The lengths of the intervals in the division by* D_n *take two or three values for any* n*.*
(4) *The points of* $D_{n+1} - D_n$ *divide intervals with the longest length which appear in the division by* D_n *for any* n*.*

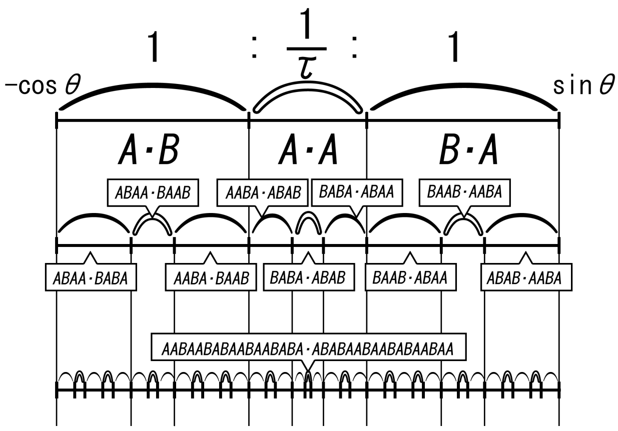

Fig. 3. the subdivision of the window

2. Proof of Theorem

By scaling up and translating, we can set that $W = (0, 1+\tau]$ and that $\pi^{\perp}(e(1))$ has the length 1 and $\pi^{\perp}(e(2))$ has the length τ. For simplicity, we use this setting in the argument that follows, and for convenience, we use the previous symbols and notations.

We consider a finite subsequence $\cdot X_1 X_2 \ldots X_n$ in a Fibonacci sequence, where X_k denotes the symbol A or B and "$\cdot$" denotes the position of the vertex v. We put $v_0 = v$. For $k = 1, 2, \ldots, n$, we take vertices v_{k-1}, v_k such that a line segment $\overline{v_{k-1}v_k}$ is identified with the symbol X_k.

Since $\pi|(s+\mathbb{Z}^2)$ is injective, there exists unique point u_k in $(s+\mathbb{Z}^2)\cap(W\times E)$ such that $\pi(u_k) = v_k$ for each $k = 0, 1, \ldots, n$. We define the interval $U_R(\cdot X_1 X_2 \ldots X_n)$ for $\cdot X_1 X_2 \ldots X_n$ by $U_R(\cdot X_1 X_2 \ldots X_n) = \{\pi^{\perp}(u) + m \in W \mid m + V \subseteq W, m \in E^{\perp}\}$, where $V = \bigcup_{i=0}^{n} \pi^{\perp}(u_i)$ and $m + V = \{m + x \mid x \in V\}$. For each n, the window W is divided into the intervals $U(\cdot X_1 \ldots X_n)$.

For a subsequence $\cdot A$, a line segment corresponding to A is obtained as the image of a horizontal edge $e(1)$ of $s + \mathbb{Z}^2$. Then, $\pi^{\perp}(e(1)) = [\pi^{\perp}(u_0) - 1, \pi^{\perp}(u_0)]$, where $u_0 \in (s + \mathbb{Z}^2) \cap (W \times E))$ such that $v = \pi(u_0)$. For a subsequence $\cdot B$, a line segment corresponding to B is obtained as the image of a vertical edge e_2 of $s+\mathbb{Z}^2$. Then, $\pi^{\perp}(e(2)) = [\pi^{\perp}(u_0), \pi^{\perp}(u_0) + \tau]$, where $u_0 \in (s + \mathbb{Z}^2) \cap (W \times E))$ such that $v = \pi(u_0)$. Note that $\pi^{\perp}(u_0)-1 = \pi^{\perp}(u_0)+\tau$ (modulo $1+\tau$). The point 1 divides the window into $U(\cdot A) = (1, 1+\tau]$ and $U(\cdot B) = (0, 1]$. Since any Fibonacci sequence has $n + 1$ kinds of finite subsequences with the length n, we see that $\{(s_R)^k(1 + \tau)\}_{k=0}^{n}$ divides the window into intervals $U(\cdot X_1 X_2 \ldots X_n)$ for finite subsequences $\cdot X_1 X_2 \ldots X_n$, where $s_R : (0, 1+\tau] \to (0, 1+\tau]$ is defined by $s_R(x) = x + 1$ (modulo $1+\tau$)(see Fig. 4).

By a similar way, when symbols are added to the left of the vertex v, we can see that $\{(s_L)^k(1 + \tau)\}_{k=0}^{n}$ divides the window into intervals $U_L(X_{-n} \ldots X_{-2}X_{-1}\cdot)$ for finite subsequences $X_{-n} \ldots X_{-2}X_{-1}\cdot$, where $s_L : (0, 1+\tau] \to (0, 1+\tau]$ is defined by $s_L(x) = x - 1$ (modulo $1+\tau$).

Hence, we obtain that $D_n = \{(s_R)^k(1+\tau)\}_{k=0}^{n} \cup \{(s_L)^k(1+\tau)\}_{k=0}^{n}$ (see Fig. 5). We put $D_R = \{(s_R)^k(1+\tau)\}_{k=0}^{\infty}$ and $D_L = \{(s_L)^k(1+\tau)\}_{k=0}^{\infty}$. Then, $\bigcup_{k=1}^{\infty} D_k = D_R \cup D_L$.

(1) We need the following lemma in the proof of (1).

Lemma 2.1. *For $n \in \mathbb{N}$ with $n \geq 1$,*

$$\frac{1}{\tau^n} = (-1)^{n+1}(a_{n-1}(1+\tau) - a_{n+1})$$

where a_n are Fibonacci numbers with the recurrence relation $a_{n+1} = a_n + a_{n-1}$ and $a_0 = a_1 = 1$.

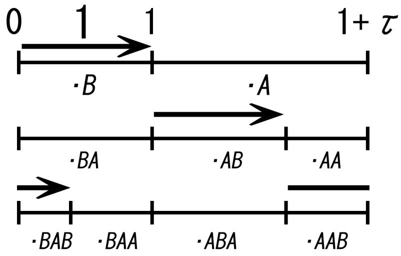

FIG. 4. the division of the window W into the intervals $U_R(\cdot X_1 \dots X_n)$

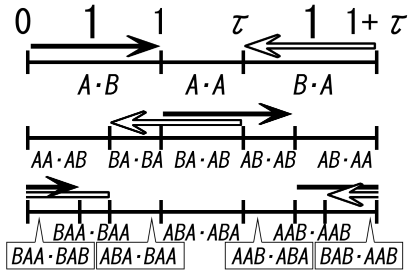

FIG. 5. the division of the window W by $D_R \cup D_L$

Proof of Lemma 2.1. For $n = 1$, we have that $1/\tau = -1 + \tau = (-1)^2(a_0(1+\tau) - a_2)$ by $\tau^2 - \tau - 1 = 0$. For $n = 2$, we have that $1/\tau^2 = 2 - \tau = (-1)^3(a_1(1+\tau) - a_3)$ by $1/\tau = -1 + \tau$. We assume that the equation of Lemma 2.1 is true for $n = k-1, k$. By $1/\tau^{k+1} = 1/\tau^{k-1} - 1/\tau^k$ and the assumption, $1/\tau^{k+1} = (-1)^k(a_{k-2}(1+\tau) - a_k) - (-1)^{k+1}(a_{k-1}(1+\tau) - a_{k+1}) = (-1)^{k+2}((a_{k-2} + a_{k-1})(1+\tau) - (a_k + a_{k+1})) = (-1)^{k+2}(a_k(1+\tau) - a_{k+2})$

Hence, the equation of Lemma 2.1 is true for $n = k+1$. $\square$

We prove that $D_{1:1/\tau:1} = D_R \cup D_L$. First, we show that $D_{1:1/\tau:1} \subseteq D_R \cup D_L$. Two points 1 and τ divide the window W into 3 intervals $(0, 1]$, $(1, \tau]$ and $(\tau, 1+\tau]$ in ratio at $1 : 1/\tau : 1$, and one end-point of each 3 intervals is in D_R and the other is in D_L.

We assume that an interval $(\alpha, \beta]$ appears in the division of the window in ratio at $1 : 1/\tau : 1$, and that $\alpha \in D_L$, $\beta \in D_R$ if $(\alpha, \beta]$ has the length $1/\tau^{2k}$ $(k \geq 0)$, and that $\alpha \in D_R$ and $\beta \in D_L$ if $(\alpha, \beta]$ has the length $1/\tau^{2k+1}$ $(k \geq 0)$. We have only to show that we can take two points γ, δ which divide $(\alpha, \beta]$ into 3 intervals in ratio at $1 : 1/\tau : 1$, and that one end-point of each 3 intervals is in D_R and the other is in D_L. By Lemma 2.1 we have the equation $-1/\tau^{2k+1} = a_{2k+2}$ (modulo $1+\tau$). If $(\alpha, \beta]$ has the length $1/\tau^{2k}$ $(k \geq 0)$, then we can take $\gamma = \beta - (1/\tau^{2k+1}) \in D_R$, $\delta = \alpha + (1/\tau^{2k+1}) \in D_L$. By Lemma 2.1 we have the equation $1/\tau^{2k+2} = a_{2k+3}$ (modulo $1+\tau$). If $(\alpha, \beta]$ has the length $1/\tau^{2k+1}$ $(k \geq 0)$, then we can take $\gamma = \beta - (1/\tau^{2k+2}) \in D_L$, $\delta = \alpha + (1/\tau^{2k+2}) \in D_R$.

Next, we show that $D_{1:1/\tau:1} \supseteq D_R \cup D_L$. We give the proof that $D_{1:1/\tau:1} \supseteq D_R$. By using the similar argument, we can prove that $D_{1:1/\tau:1} \supseteq D_L$. For any positive integer n, we can take k_i $(k_{i+1} > k_i \geq 1)$ such that $n = a_{k_m} + a_{k_{m-1}} + \dots + a_{k_1}$. Then we have the equation $n = (-1)^{k_1-1}/\tau^{k_1-1} + \dots + (-1)^{k_m-1}/\tau^{k_m-1}$ (modulo $1+\tau$).

We put $C(j) = (-1)^{k_1-1}/\tau^{k_1-1} + \dots + (-1)^{k_j-1}/\tau^{k_j-1}$ for $1 \leq j \leq m$. We show that $C(j)$ is in $D_{1:1/\tau:1}$ (modulo $1+\tau$) by the induction on j.

Assume that $C(j-1)$, $C(j)$ are in $D_{1:1/\tau:1}$ (modulo $1+\tau$) for $2 \le j$. By the definition of $D_{1:1/\tau:1}$, $C(j-1)$, $C(j)$ satisfy one of the following cases (a)–(d) in the process of dividing the window in ratio $1 : 1/\tau : 1$ repeatedly. :

(a) $C(j), \delta$ $(C(j) < \delta)$ divide $(C(j-1), \beta]$ into 3 intervals in ratio at $1 : 1/\tau : 1$ for some $\beta, \delta \in D_{1:1/\tau:1}$.

(b) $\gamma, C(j)$ $(\gamma < C(j))$ divide $(C(j-1), \beta]$ into 3 intervals in ratio at $1 : 1/\tau : 1$ for some $\beta, \delta \in D_{1:1/\tau:1}$.

(c) $\gamma, C(j)$ $(\gamma < C(j))$ divide $(\alpha, C(j-1)]$ into 3 intervals in ratio at $1 : 1/\tau : 1$ for some $\beta, \delta \in D_{1:1/\tau:1}$.

(d) $C(j), \delta$ $(C(j) < \delta)$ divide $(\alpha, C(j-1)]$ into 3 intervals in ratio at $1 : 1/\tau : 1$ for some $\beta, \delta \in D_{1:1/\tau:1}$.

If an interval $(a, b]$ with length $1/\tau^k$ appears in the process, $a+(1/\tau^j)$ and $b-(1/\tau^j)$ are in $D_{1:1/\tau:1}$ for any $j > k$ by the definition of $D_{1:1/\tau:1}$. Then we can see that $C(j+1) = C(j) + (-1)^{k_j+1}/\tau^{k_j+1} \in D_{1:1/\tau:1}$ (modulo $1+\tau$).

Hence we obtain that $n \in D_{1:1/\tau:1}$ (modulo $1+\tau$). By a similar way, we can show that $-n \in D_{1:1/\tau:1}$ (modulo $1+\tau$).

(2) In the original setting, $W = (-\cos\theta, \sin\theta]$ and $(-1/\tau)W = [-\sin\theta/\tau, \sin\theta)$. In the present setting, $W' = [1, 1+\tau)$ is corresponding to $(-1/\tau)W$.

We define $s'_L : W' \to W'$ by $s'_L(x) = x - (1/\tau)$ (modulo τ). Since $W' = [1, 1+\tau)$ and $s_R^2(1+\tau) = -1/\tau$ (modulo $1+\tau$), we see that $W' \cap \{(s_R)^k(1+\tau)\}_{k=0}^n = \{(s'_L)^k(1+\tau)\}_{k=0}^n$.

We define $s'_R : W' \to W'$ by $s'_R(x) = x + (1/\tau)$ (modulo τ). Since $W' = [1, 1+\tau)$ and $s_L(1+\tau) = 1 + 1/\tau$ (modulo $1+\tau$), we see that $W' \cap \{(s_L)^k(1+\tau)\}_{k=0}^n = \{(s'_R)^k(1+\tau)\}_{k=0}^n$.

These imply that the subdivision of the window $(-1/\tau)W$ is the restriction to $(-1/\tau)W$ of the subdivision of the window W by the similar argument of the proof of (1) of Theorem1.1.

(3) As mentioned above, when the window W is divided into the intervals $U(X_{-n} \dots X_{-1} \cdot X_1 \dots X_n)$, the window W is divided in $2n+1$ points $D_n = \{(s_R)^k(1+\tau)\}_{k=0}^n \cup \{(s_L)^k(1+\tau)\}_{k=0}^n$. We see that $\{(s_R)^k(1+\tau)\}_{k=0}^n \cup \{(s_L)^k(1+\tau)\}_{k=0}^n = \{(s_R)^k((s_L)^n(1+\tau))\}_{k=0}^{2n}$ which is obtained by s_R starting from $s_L^n(1+\tau)$.

By the three distance theorem (see [4] and also [5]), the lengths of the intervals in the division by D_n take at most three values. By (1) of Theorem 1.1, an interval with the length $1/\tau^n$ is divided into an interval with the length $1/\tau^{n+1}$ and an interval with the length $1/\tau^{n+2}$. So, the lengths of the intervals cannot take one value.

(4) We need the following lemma in the proof of (4).

Lemma 2.2. *For $n \in \mathbb{N}$ with $n \ge 1$,*

$$\frac{a_{n+1}}{\tau^{n-1}} + \frac{a_n}{\tau^n} = 1 + \tau,$$

where $\{a_n\}$ is Fibonacci number.

Proof of Lemma 2.2. For $n = 1$, we have that $a_2/\tau^0 + a_1/\tau = 2/1 + 1/\tau = 1 + (1 + (1/\tau)) = 1 + \tau$. For $n = k$, we assume that $a_{k+1}/\tau^{k-1} + a_k/\tau^k = 1 + \tau$. Then, $a_{k+2}/\tau^k + a_{k+1}/\tau^{k+1} = (a_k + a_{k+1})/\tau^k + a_{k+1}/\tau^{k+1} = a_k/\tau^k + a_{k+1}(1/\tau^k + 1/\tau^{k+1})$ $= a_k/\tau^k + a_{k+1}/\tau^{k-1} = 1 + \tau$ Hence, the equation of Lemma2.2 is true for n = k+1 $\square$

Assume that the points of $D_{n+1} - D_n$ does not necessarily divide intervals with the longest length which appear in the division by D_n. Since the lengths of the intervals take two or three values before and after dividing, the following two cases (i), (ii) might be left.

(i) The lengths of the intervals take two values and there are 1 interval with the length $1/\tau^{k+2}$ and some intervals with the length $1/\tau^{k+1}$. Now, we divide the interval with the length $1/\tau^{k+2}$.

(ii) The lengths of the intervals take three values and there are 1 interval with the length $1/\tau^{k+2}$ and some intervals with the lengths $1/\tau^{k+1}$ or $1/\tau^{k+3}$. Now, we divide the interval with the length $1/\tau^{k+2}$.

If the case (i) occurs, then we have the equation $1/\tau^{k+2} + m/\tau^{k+1} = 1 + \tau$ $(m \in \mathbb{N})$. By Lemma 2.2 we have $1/\tau + 2/1 = 1 + \tau$. However, we actually check the case correspondind to this equation and we can see that the interval with the length $1/\tau^{k+1} = 1$ is divided. This implies the contradiction.

If the case (ii) occurs, then we have the equation $1/\tau^{k+2} + m_1/\tau^{k+1} + m_2/\tau^{k+3} = 1 + \tau$ for some $m_1, m_2 \in \mathbb{N}$. By $1/\tau^{k+1} = 1/\tau^{k+2} + 1/\tau^{k+3}$, we have $(m_1+1)/\tau^{k+2} + (m_1 + m_2)/\tau^{k+3} = 1 + \tau$. By Lemma 2.2 we have the equations $a_{k+4} = m_1 + 1$ and $a_{k+3} = m_1 + m_2$. So, we get that $a_{k+2} \leq 0$. This implies the contradiction. Hence the points of $D_{n+1} - D_n$ divide intervals with the longest length which appear in the division by D_n.

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