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HYPERGROUP EXTENSIONS OF FINITE ABELIAN GROUPS BY HYPERGROUPS OF ORDER TWO

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Abstract. The purpose of the present paper is to establish necessary conditions and sufficient conditions that finite commutative hypergroups are extensions of finite Abelian groups by hypergroups of order two. Applying our results to some concrete cases one can determine all such extensions.

1. Introduction

Let $\mathcal{H}$ and $\mathcal{L}$ be finite commutative hypergroups. A finite commutative hypergroup $\mathcal{K}$ is called an extension of $\mathcal{L}$ by $\mathcal{H}$ if the sequence

$$1 \to \mathcal{H} \to \mathcal{K} \to \mathcal{L} \to 1$$

is exact, i.e., if there exists a hypergroup homomorphism φ from $\mathcal{K}$ onto $\mathcal{L}$ such that $\operatorname{Ker}\varphi = \mathcal{H}$ and $\mathcal{H}$ is embedded in $\mathcal{K}$. The extension problem is to determine all extensions $\mathcal{K}$ of $\mathcal{L}$ by $\mathcal{H}$ when $\mathcal{H}$ and $\mathcal{L}$ are given.

It is known that there exist some methods to construct a new hypergroup from given ones. The structure of hypergroups is not yet known very well even in the case of finite hypergroups of low orders. The structure of hypergroups of order three is determined in 2002 by N.J.Wildberger [10]. In order to understand the full structure of hypergroups, it will play an essential role to determine all extensions $\mathcal{K}$ of $\mathcal{L}$ by $\mathcal{H}$ for given commutative hypergroups $\mathcal{H}$ and $\mathcal{L}$.

Splitting extensions of hypergroups are introduced in [4] and [3]. Extension hypergroups associated with group actions have been constructed in [2] and [3]. Extension problems of the Golden hypergroup were studied in [5] and [6].

By stimulating with Voit's work [8] in 2008 on hypergroup structures on two tori, we have started to consider the extension problem in the category of commutative hypergroups and we succeeded to determine all commutative hypergroup extensions of hypergroups of order two by finite Abelian groups including non-splitting extensions in [7].

In the present paper we investigate the dual version of the above extension problem, namely we analyze the structure of extensions $\mathcal{K}$ of finite Abelian groups $\mathcal{L}$ by hypergroups $\mathcal{H}$ of order two. We give the necessary conditions of such extensions

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in Theorem 3.7 and Corollary 3.8. We give the sufficient conditions of such extensions in the flat case in Theorem 3.10 and Corollary 3.11 and in the crossing case in Theorem 3.12 and Corollary 3.13. Applying these results one can determine all extensions of the cyclic groups $\mathcal{L}$ of low orders by hypergroups $\mathcal{H}$ of order two.

2. Preliminaries

We recall some notions and facts on finite commutative hypergroups which are described in Wildberger's paper [9] and Bloom-Heyer's book [1].

Axiom of a finite commutative hypergroup. A pair $\mathcal{K} := (\mathcal{K}, A)$ is called a *finite commutative hypergroup* if the following conditions (i)–(v) are satisfied.

(i) $A(\mathcal{K})$ is a $*$-algebra over $\mathbb{C}$ with unit c_0.

(ii) $\mathcal{K} = \{c_0, c_1, \ldots, c_n\}$ is a linear basis of A and $\mathcal{K}^* = \mathcal{K}$.

(iii) The structure constants $n_{ij}^k \in \mathbb{C}$ defined by $c_i c_j = \sum_{k=0}^n n_{ij}^k c_k$ satisfy that $n_{ij}^k \geq 0$. Moreover, $c_i^* = c_j$ if and only if $n_{ij}^0 > 0$.

(iv) $\sum_{k=0}^n n_{ij}^k = 1$ for all i, j.

(v) $c_i c_j = c_j c_i$ for all i, j.

The *weight* of an element c_i of $\mathcal{K}$ is defined by $w(c_i) = (n_{ij}^0)^{-1}$ where $c_j = c_i^*$ and the *total weight* of $\mathcal{K}$ is defined by $w(\mathcal{K}) = \sum_{i=0}^n w(c_i)$. Let $\omega_{\mathcal{K}}$ denote the normalized Haar measure of $\mathcal{K}$ which is given by

$$\omega_{\mathcal{K}} = \sum_{k=0}^n \frac{w(c_k)}{w(\mathcal{K})} c_k.$$

Let $M^1(\mathcal{K})$ denote the set of all non-negative probability measures on $\mathcal{K}$, i.e.,

$$M^1(\mathcal{K}) := \left\{ \sum_{k=0}^n a_k c_k;\ a_k \geq 0\ (k = 0, 1, \ldots, n),\ \sum_{k=0}^n a_k = 1 \right\}.$$

A complex function χ on $\mathcal{K}$ is called a character of $\mathcal{K}$ if

$$\chi(c_0) = 1, \quad \chi(c_i)\chi(c_j) = \sum_{k=0}^n n_{ij}^k \chi(c_k) \quad \text{and} \quad \chi(c_i^*) = \overline{\chi(c_i)}.$$

Let $\mathcal{K}$ and $\mathcal{L}$ be finite commutative hypergroups. A hypergroup homomorphism φ from $\mathcal{K}$ into $\mathcal{L}$ means that there exists the unique $*$-homomorphism $\tilde{\varphi}$ from $A(\mathcal{K})$ into $A(\mathcal{L})$ such that $\tilde{\varphi}(c_i) = \varphi(c_i)$ for all $c_i \in \mathcal{K}$. Sometimes a hypergroup homomorphism is called simply a homomorphism.

Let $\mathcal{H} = \{h_0, h_1\}$ be a hypergroup of order two with unit h_0. Then the structure of $\mathcal{H}$ is determined by a parameter $0 < q \leq 1$ such that

$$h_1^2 = qh_0 + (1-q)h_1.$$

Hence we denote hypergroup $\mathcal{H}$ by $\mathbb{Z}_q(2)$. The weights and the normalized Haar measure on $\mathbb{Z}_q(2)$ are given by

$$\begin{aligned} &w(h_0) = 1,\ w(h_1) = \frac{1}{q},\ w(\mathbb{Z}_q(2)) = \frac{1+q}{q}, \\ &\omega_{\mathbb{Z}_q(2)} = \frac{w(h_0)}{w(\mathbb{Z}_q(2))}h_0 + \frac{w(h_1)}{w(\mathbb{Z}_q(2))}h_1 = \frac{q}{1+q}h_0 + \frac{1}{1+q}h_1. \end{aligned}$$

The characters χ_0 and χ_1 of $\mathbb{Z}_q(2)$ are given by

$$\begin{aligned} \chi_0(h_0) = \chi_0(h_1) = 1, \\ \chi_1(h_0) = 1,\ \chi_1(h_1) = -q. \end{aligned}$$

3. Extension

Let $\mathcal{H}$ and $\mathcal{L}$ be finite commutative hypergroups. A finite commutative hypergroup $\mathcal{K}$ is called an extension of $\mathcal{L}$ by $\mathcal{H}$ if the sequence

$$1 \to \mathcal{H} \to \mathcal{K} \to \mathcal{L} \to 1$$

is exact, i.e., if $\mathcal{H}$ is embedded in $\mathcal{K}$ and there exists a homomorphism φ from $\mathcal{K}$ onto $\mathcal{L}$ such that Ker $\varphi = \mathcal{H}$. When finite commutative hypergroups $\mathcal{H}$ and $\mathcal{L}$ are given, the extension problem is to determine all finite commutative hypergroup extensions of $\mathcal{L}$ by $\mathcal{H}$ up to equivalence as extensions.

In the present paper we discuss the above extension problem in the case that $\mathcal{H} = \mathbb{Z}_2(q) = \{h_0, h_1\}$ is a hypergroup of order two and $\mathcal{L} = \{\ell_0, \ell_1, \ldots, \ell_n\}$ is a finite Abelian group with unit ℓ_0.

Let $S_i = \varphi^{-1}(\ell_i)$ for $\ell_i \in \mathcal{L}$. Then $\mathcal{K}$ is decomposed to the disjoint union of the sets S_i $(i = 0, 1, 2, \ldots, n)$ as $\mathcal{K} = S_0 \cup S_1 \cup \cdots \cup S_n$, where $S_0 = \mathcal{H}$. We have the following lemma about the cardinal number $|S_i|$ of S_i.

Lemma 3.1. *$|S_i| = 1$ or 2 for each $i = 1, 2, \ldots, n$.*

Proof. Suppose that $|S_i| \geq 3$. Take $s_0, s_1, s_2 \in S_i$. Let φ be a homomorphism from $\mathcal{K}$ onto $\mathcal{L}$. Since

$$\varphi(s_0 s_0^*) = \varphi(s_0)\varphi(s_0^*) = \varphi(s_0)\varphi(s_0)^* = \ell_i \ell_i^* = \ell_0,$$

the product $s_0 s_0^*$ must be in $M^1(\mathcal{H})$. In a similar way, $s_0 s_1^*$, $s_0 s_2^*$ and $s_1 s_2^*$ must be in $M^1(\mathcal{H})$. Then

$$s_0^* s_0 = \tau h_0 + (1-\tau)h_1 \quad \text{and} \quad s_0^* s_1 = s_0 s_2^* = s_1 s_2^* = h_1,$$

where $\tau = w(s_0)^{-1} > 0$. Hence we have on one hand

$$s_0^* s_1 s_0 s_2^* = (s_0^* s_1)(s_0 s_2^*) = h_1^2 = q h_0 + (1-q) h_1$$

and on the other hand

$$\begin{aligned} s_0^* s_1 s_0 s_2^* &= (s_0^* s_0)(s_1 s_2^*) \\ &= (\tau h_0 + (1-\tau) h_1) h_1 \\ &= q(1-\tau) h_0 + (\tau + (1-q)(1-\tau))\, h_1. \end{aligned}$$

Compare the coefficients of h_0. Then we have $q = q(1-\tau)$. Since $q \neq 0$, we obtain $\tau = 0$. This contradicts the fact that $\tau > 0$. Therefore $|S_i|$ must be one or two. □

When $|S_i| = 2$, we put $S_i = \{s_0(\ell_i), s_1(\ell_i)\}$ and $\gamma_i = w(s_1(\ell_i))/w(s_0(\ell_i))$, where $s_0(\ell_0) = h_0$, $s_1(\ell_0) = h_1$, and $\gamma_0 = 1/q$. This positive real numbers γ_i will play an important role to determine the structure of extensions of $\mathcal{L}$ by $\mathcal{H}$. When $|S_i| = 1$, we put $S_i = \{s(\ell_i)\}$.

Lemma 3.2. *The products of each element in S_i are given as follows:*

(i) *In the case that $S_i = \{s_0(\ell_i), s_1(\ell_i)\}$, one has*

$$\begin{aligned} s_0(\ell_i) s_0(\ell_i)^* &= \frac{q(1+\gamma_i)}{1+q} h_0 + \frac{1 - q\gamma_i}{1+q} h_1, \\ s_1(\ell_i) s_1(\ell_i)^* &= \frac{q(1+\gamma_i^{-1})}{1+q} h_0 + \frac{1 - q\gamma_i^{-1}}{1+q} h_1, \\ s_0(\ell_i) s_1(\ell_i)^* &= s_0(\ell_i)^* s_1(\ell_i) = h_1, \quad \textit{where} \quad q \leq \gamma_i \leq 1/q. \end{aligned}$$

(ii) *In the case that $S_i = \{s(\ell_i)\}$, one has $s(\ell_i) s(\ell_i)^* = \omega_{\mathcal{H}}$.*

Proof. (i) In the case that $|S_i| = 2$, we have

$$\begin{aligned} s_0(\ell_i) s_0(\ell_i)^* &= \tau_i h_0 + (1-\tau_i) h_1, \\ s_1(\ell_i) s_1(\ell_i)^* &= \rho_i h_0 + (1-\rho_i) h_1, \\ s_0(\ell_i) s_1(\ell_i)^* &= h_1, \end{aligned}$$

where $\tau_i = w(s_0(\ell_i))^{-1} > 0$ and $\rho_i = w(s_1(\ell_i))^{-1} > 0$. Note that $\tau_i = \gamma_i \rho_i$. We have

$$\begin{aligned} & s_0(\ell_i) s_0(\ell_i)^* s_1(\ell_i) s_1(\ell_i)^* = (s_0(\ell_i) s_0(\ell_i)^*)(s_1(\ell_i) s_1(\ell_i)^*) \\ &= (\tau_i h_0 + (1-\tau_i) h_1)(\rho_i h_0 + (1-\rho_i) h_1) \\ &= \tau_i \rho_i h_0 + (1-\tau_i)\rho_i h_1 + \tau_i (1-\rho_i) h_1 + (1-\tau_i)(1-\rho_i) h_1^2 \\ &= \tau_i \rho_i h_0 + (1-\tau_i)\rho_i h_1 + \tau_i (1-\rho_i) h_1 + (1-\tau_i)(1-\rho_i)(q h_0 + (1-q) h_1) \\ &= (\tau_i \rho_i + q(1-\tau_i)(1-\rho_i)) h_0 \\ &\quad + ((1-\tau_i)\rho_i + \tau_i(1-\rho_i) + (1-q)(1-\tau_i)(1-\rho_i)) h_1. \end{aligned}$$

On the other hand

$$\begin{aligned} s_0(\ell_i)s_0(\ell_i)^*s_1(\ell_i)s_1(\ell_i)^* &= (s_0(\ell_i)s_1(\ell_i)^*)(s_1(\ell_i)s_0(\ell_i)^*) \\ &= h_1^2 = qh_0 + (1-q)h_1. \end{aligned}$$

By comparing the coefficients of h_0 we obtain

$$\tau_i\rho_i + q(1-\tau_i)(1-\rho_i) = q.$$

Since $\tau_i\rho_i + q(1-\tau_i)(1-\rho_i) = q$ and $\tau_i = \gamma_i\rho_i$, we have

$$\tau_i = q(1+\gamma_i)/(1+q)$$

and

$$\rho_i = \gamma_i^{-1}\tau_i = q(1+\gamma_i)\gamma_i^{-1}/(1+q) = q(1+\gamma_i^{-1})/(1+q).$$

It is easy to see that $q \le \gamma_i \le 1/q$ from the facts that $1-\tau_i = 1-q\gamma_i \ge 0$ and $1-\rho_i = 1-q\gamma_i^{-1} \ge 0$.

(ii) In the case that $S_i = \{s(\ell_i)\}$, it is obvious that $h_1 s(\ell_i) = s(\ell_i)$, so that $\omega_{\mathcal{H}} s(\ell_i) = s(\ell_i)$. Since $s(\ell_i)s(\ell_i)^* = c \in M^1(\mathcal{H})$ and $\omega_{\mathcal{H}} c = \omega_{\mathcal{H}}$,

$$s(\ell_i)s(\ell_i)^* = (\omega_{\mathcal{H}} s(\ell_i))s(\ell_i)^* = \omega_{\mathcal{H}}(s(\ell_i)s(\ell_i)^*) = \omega_{\mathcal{H}} c = \omega_{\mathcal{H}}.$$

Therefore $s(\ell_i)s(\ell_i)^* = \omega_{\mathcal{H}}$. □

For $\ell_i \in \mathcal{L}$, we will use a notation i^* given by $\ell_{i^*} = \ell_i^*$.

Lemma 3.3. $L_0 = \{\ell_i \in \mathcal{L};\ |S_i| = 2\}$ *is a subgroup of* $\mathcal{L}$.

Proof. First we show that $\ell_k = \ell_i\ell_j \in L_0$ for $\ell_i,\ \ell_j \in L_0$. Suppose that $\ell_k = \ell_i\ell_j \notin L_0$ for $\ell_i, \ell_j \in L_0$. It is easy to see that

$$s_0(\ell_i)s_0(\ell_j) = s_1(\ell_i)s_1(\ell_j) = s(\ell_k)$$

and

$$s_0(\ell_i)s_1(\ell_i)^* = s_0(\ell_j)s_1(\ell_j)^* = h_1.$$

It is shown that $s(\ell_k)s(\ell_k)^* = \omega_{\mathcal{H}}$ by Lemma 3.2. Hence we have

$$\begin{aligned} s_0(\ell_i)s_0(\ell_j)s_1(\ell_i)^*s_1(\ell_j)^* &= (s_0(\ell_i)s_0(\ell_j))(s_1(\ell_i)s_1(\ell_j))^* \\ &= s(\ell_k)s(\ell_k)^* = \omega_{\mathcal{H}}. \end{aligned}$$

On the other hand

$$s_0(\ell_i)s_0(\ell_j)s_1(\ell_i)^*s_1(\ell_j)^* = (s_0(\ell_i)s_1(\ell_i)^*)(s_0(\ell_j)s_1(\ell_j)^*) = h_1^2.$$

Hence we have $\omega_{\mathcal{H}} = h_1^2$. This is a contradiction. Therefore ℓ_k must be in L_0.

Since for $\ell_i \in L_0$ we have $|S_i^*| = 2$, the involution ℓ_i^* of ℓ_i must be in L_0. Consequently we see that L_0 is a subgroup of $\mathcal{L}$. □

For $s_0(\ell_i), s_1(\ell_i) \in S_i$, there are two possibilities that $s_0(\ell_i)^* = s_0(\ell_i^*)$ or $s_0(\ell_i)^* = s_1(\ell_i^*)$ for $\ell_i \in L_0$. Here we give some notations which are used in our discussion. We call $F = \{\ell_i \in L_0;\ s_0(\ell_i)^* = s_0(\ell_i^*)\}$ the *flat part* of L_0 and $C = \{\ell_i \in L_0;\ s_0(\ell_i)^* = s_1(\ell_i^*)\}$ the *crossing part* of L_0. We note that $L_0 = C \cup F$ and $C \cap F = \emptyset$. It is obvious that $F^* = F$ and $C^* = C$. Moreover, we put $C_0 = \{\ell_i \in C;\ \ell_i^* = \ell_i\}$.

Lemma 3.4. *For $\ell_i \in L_0$, the relations of between γ_i and γ_{i^*} are as follows:*

$$\gamma_{i^*} = \begin{cases} \gamma_i & \text{if } \ell_i \in F, \\ \gamma_i^{-1} & \text{if } \ell_i \in C \cap C_0^c, \\ \gamma_i = 1 & \text{if } \ell_i \in C_0. \end{cases}$$

Proof. For $\ell_i \in L_0$, we have

$$s_0(\ell_{i^*})s_0(\ell_{i^*})^* = q(1+\gamma_{i^*})(1+q)^{-1}h_0 + (1-q\gamma_{i^*})(1+q)^{-1}h_1 \tag{3.1}$$

by Lemma 3.2. In the case that $\ell_i \in F$, we have

$$\begin{aligned} s_0(\ell_{i^*})s_0(\ell_{i^*})^* &= s_0(\ell_i)^* s_0(\ell_i) \\ &= q(1+\gamma_i)(1+q)^{-1}h_0 + (1-q\gamma_i)(1+q)^{-1}h_1. \end{aligned}$$

By comparing the coefficients of h_0 in the above equality and (3.1) we obtain $\gamma_{i^*} = \gamma_i$. In the case that $\ell_i \in C$, we have

$$\begin{aligned} s_0(\ell_{i^*})s_0(\ell_{i^*})^* &= s_1(\ell_i)^* s_1(\ell_i) \\ &= q(1+\gamma_i^{-1})(1+q)^{-1}h_0 + (1-q\gamma_i^{-1})(1+q)^{-1}h_1. \end{aligned}$$

Compare the coefficients of h_0 in the above equality and (3.1) we obtain $\gamma_{i^*} = \gamma_i^{-1}$. When $\ell_i \in C_0$, $\gamma_i^{-1} = \gamma_i$ by $\ell_{i^*} = \ell_i$. Then we obtain $\gamma_i = 1$ by $\gamma_i > 0$. □

Lemma 3.5. *If $\ell_i, \ell_j \in C_0$, then $\ell_i\ell_j \in F$.*

Proof. Let χ be a character of $\mathcal{K}$ with $\chi(h_1) = -q$. Note that for $\ell \in C_0$, $s_0(\ell)^2 = s_0(\ell)s_0(\ell^*) = s_0(\ell)s_1(\ell)^* = h_1$ by Lemma 3.2. Since $\chi(s_0(\ell))^2 = \chi(s_0(\ell)^2) = \chi(h_1) = -q$,

$$\chi(s_0(\ell)) = \pm\sqrt{-q}.$$

For $\ell_i, \ell_j \in C_0$, there exists $a \in [0,1]$ such that

$$s_0(\ell_i)s_0(\ell_j) = a s_0(\ell_k) + (1-a)s_1(\ell_k)$$

where $\ell_k = \ell_i\ell_j$. Suppose $\ell_k \in C$. Then since $\ell_k^2 = \ell_0$ and $s_0(\ell_k)^* = s_1(\ell_k^*)$, we know that ℓ_k must be in C_0. Hence we have on one hand

$$\chi(s_0(\ell_i)s_0(\ell_j)) = \chi(s_0(\ell_i))\chi(s_0(\ell_j)) = \pm q,$$

and on the other hand

$$\begin{aligned}\chi(s_0(\ell_i)s_0(\ell_j)) &= a\chi(s_0(\ell_k)) + (1-a)\chi(s_1(\ell_k)) \\ &= a\chi(s_0(\ell_k)) + (1-a)\overline{\chi(s_0(\ell_k))} \\ &= \pm\sqrt{-q}a + \mp\sqrt{-q}(1-a).\end{aligned}$$

It is obvious that $\pm\sqrt{-q}a + \mp\sqrt{-q}(1-a)$ is a purely imaginary or zero. This contradicts the fact $q > 0$. Therefore ℓ_k must be in F. $\square$

Proposition 3.6. *The products by $h_1 \in \mathcal{H}$ and $s_0(\ell_i), s_1(\ell_i) \in S_i$ for $\ell_i \in L_0$ are the following:*

(i) $h_1 s_0(\ell_i) = \dfrac{1-q\gamma_i}{1+\gamma_i} s_0(\ell_i) + \dfrac{(1+q)\gamma_i}{1+\gamma_i} s_1(\ell_i),$

(ii) $h_1 s_1(\ell_i) = \dfrac{1+q}{1+\gamma_i} s_0(\ell_i) + \dfrac{\gamma_i - q}{1+\gamma_i} s_1(\ell_i).$

Proof. We can write

$$h_1 s_0(\ell_i) = c_i s_0(\ell_i) + \tilde{c}_i s_1(\ell_i) \text{ and } h_1 s_1(\ell_i) = b_i s_0(\ell_i) + \tilde{b}_i s_1(\ell_i)$$

where b_i, $c_i \in [0,1]$, $\tilde{b}_i = 1 - b_i$ and $\tilde{c}_i = 1 - c_i$. Applying Lemma 3.2 we have

$$\begin{aligned}(h_1 s_0(\ell_i))s_1(\ell_i)^* &= c_i s_0(\ell_i)s_1(\ell_i)^* + \tilde{c}_i s_1(\ell_i)s_1(\ell_i)^* \\ &= c_i h_1 + \tilde{c}_i(q(1+\gamma_i^{-1})(1+q)^{-1}h_0 \\ &\quad +(1-q\gamma_i^{-1})(1+q)^{-1}h_1) \\ &= q(1+\gamma_i^{-1})(1+q)^{-1}\tilde{c}_i h_0 \\ &\quad + \left(c_i + (1-q\gamma_i^{-1})(1+q)^{-1}\tilde{c}_i\right) h_1\end{aligned}$$

and

$$h_1(s_0(\ell_i)s_1(\ell_i)^*) = h_1^2 = qh_0 + (1-q)h_1.$$

The coefficients of h_0 in each are $q(1+\gamma_i^{-1})(1+q)^{-1}\tilde{c}_i$ and q respectively. Since the both must be equal, we have

$$c_i = (1-q\gamma_i)(1+\gamma_i)^{-1} \text{ and } \tilde{c}_i = (1+q)\gamma_i(1+\gamma_i)^{-1}.$$

Moreover, we also see that

$$b_i = (1+q)(1+\gamma_i)^{-1} \text{ and } \tilde{b}_i = (\gamma_i - q)(1+\gamma_i)^{-1}$$

by comparing the coefficients of h_0 in $h_1(s_1(\ell_i)s_1(\ell_i)^*)$ and $(h_1 s_1(\ell_i))s_1(\ell_i)^*$. $\square$

The extension hypergroup $\mathcal{K}$ of $\mathcal{L}$ by $\mathcal{H}$ is written as $\mathcal{K} = \{s_0(\ell_i),\ s_1(\ell_i),\ s(\ell_j)\ ;\ \ell_i \in L_0,\ \ell_j \in \mathcal{L} \cap L_0^c\}$. For $\ell_i, \ell_j \in L_0$ we write the constants a_{ij}^k, b_{ij}^k and c_{ij}^k given by

$$\begin{aligned} s_0(\ell_i)s_0(\ell_j) &= a_{ij}^k s_0(\ell_k^*) + \tilde{a}_{ij}^k s_1(\ell_k^*), \\ s_1(\ell_i)s_1(\ell_j) &= b_{ij}^k s_0(\ell_k^*) + \tilde{b}_{ij}^k s_1(\ell_k^*), \\ s_0(\ell_i)s_1(\ell_j) &= c_{ij}^k s_0(\ell_k^*) + \tilde{c}_{ij}^k s_1(\ell_k^*) \end{aligned} \tag{3.2}$$

where $\ell_k^* = \ell_i\ell_j$, $\tilde{a}_{ij}^k = 1 - a_{ij}^k$, $\tilde{b}_{ij}^k = 1 - b_{ij}^k$ and $\tilde{c}_{ij}^k = 1 - c_{ij}^k$. We note that $0 \le a_{ij}^k \le 1$, $0 \le b_{ij}^k \le 1$ and $0 \le c_{ij}^k \le 1$ by Axiom of a finite commutative hypergroup. We use the functions f_+ and f_- defined by $f_+(x,y,z) := (1 + \sqrt{qxyz})/(1+z)$ and $f_-(x,y,z) := (1 - \sqrt{qxyz})/(1+z)$ for $x, y, z > 0$.

We give the necessary condition that the extension $\mathcal{K}$ of $\mathcal{L}$ by $\mathcal{H}$ is hypergroup.

Theorem 3.7. *Let $\mathcal{H} = \mathbb{Z}_q(2) = \{h_0, h_1\}$ be a hypergroup of order two and $\mathcal{L} = \{\ell_0, \ell_1, \ldots, \ell_n\}$ an Abelian group with unit ℓ_0. Let $\mathcal{K}$ be an extension hypergroup of $\mathcal{L}$ by $\mathcal{H}$ such that $|S_i| = 2$ for all $i = 1, 2, \ldots, n$. For $\ell_i, \ell_j, \ell_k \in \mathcal{L}$ with $\ell_k^* = \ell_i\ell_j$, each coefficient a_{ij}^k, b_{ij}^k and c_{ij}^k in (3.2) has the following values:*

(i) *In the case of $\ell_k \in F$, either (1) or (2) occurs.*

(1) $a_{ij}^k = f_+(\gamma_i, \gamma_j, \gamma_k)$, $b_{ij}^k = f_+(\gamma_i^{-1}, \gamma_j^{-1}, \gamma_k)$, $c_{ij}^k = f_-(\gamma_i, \gamma_j^{-1}, \gamma_k)$ *where* $\gamma_i^{-1}\gamma_j^{-1}\gamma_k \ge q$, $\gamma_i\gamma_j\gamma_k \ge q$, $\gamma_i^{-1}\gamma_j\gamma_k^{-1} \ge q$, $\gamma_i\gamma_j^{-1}\gamma_k^{-1} \ge q$.

(2) $a_{ij}^k = f_-(\gamma_i, \gamma_j, \gamma_k)$, $b_{ij}^k = f_-(\gamma_i^{-1}, \gamma_j^{-1}, \gamma_k)$, $c_{ij}^k = f_+(\gamma_i, \gamma_j^{-1}, \gamma_k)$ *where* $\gamma_i^{-1}\gamma_j^{-1}\gamma_k^{-1} \ge q$, $\gamma_i\gamma_j\gamma_k^{-1} \ge q$, $\gamma_i^{-1}\gamma_j\gamma_k \ge q$, $\gamma_i\gamma_j^{-1}\gamma_k \ge q$.

(ii) *In the case of $\ell_k \in C$, either (1) or (2) occurs.*

(1) $a_{ij}^k = f_+(\gamma_i, \gamma_j, \gamma_k^{-1})$, $b_{ij}^k = f_+(\gamma_i^{-1}, \gamma_j^{-1}, \gamma_k^{-1})$, $c_{ij}^k = f_-(\gamma_i, \gamma_j^{-1}, \gamma_k^{-1})$ *where* $\gamma_i^{-1}\gamma_j^{-1}\gamma_k^{-1} \ge q$, $\gamma_i\gamma_j\gamma_k^{-1} \ge q$, $\gamma_i^{-1}\gamma_j\gamma_k \ge q$, $\gamma_i\gamma_j^{-1}\gamma_k \ge q$.

(2) $a_{ij}^k = f_-(\gamma_i, \gamma_j, \gamma_k^{-1})$, $b_{ij}^k = f_-(\gamma_i^{-1}, \gamma_j^{-1}, \gamma_k^{-1})$, $c_{ij}^k = f_+(\gamma_i, \gamma_j^{-1}, \gamma_k^{-1})$, *where* $\gamma_i^{-1}\gamma_j^{-1}\gamma_k \ge q$, $\gamma_i\gamma_j\gamma_k \ge q$, $\gamma_i^{-1}\gamma_j\gamma_k^{-1} \ge q$, $\gamma_i\gamma_j^{-1}\gamma_k^{-1} \ge q$.

Proof. (i) Case of $\ell_k \in F$.
Consider the product $s_0(\ell_i)s_0(\ell_j)s_0(\ell_i)^*s_0(\ell_j)^*$ for $\ell_i, \ell_j \in \mathcal{L}$. We have

$$\begin{aligned} & s_0(\ell_i)s_0(\ell_j)s_0(\ell_i)^*s_0(\ell_j)^* \\ &= (s_0(\ell_i)s_0(\ell_j))(s_0(\ell_i)s_0(\ell_j))^* \\ &= (a_{ij}^k s_0(\ell_k^*) + \tilde{a}_{ij}^k s_1(\ell_k^*))(a_{ij}^k s_0(\ell_k^*)^* + \tilde{a}_{ij}^k s_1(\ell_k^*)^*) \\ &= (a_{ij}^k s_0(\ell_k)^* + \tilde{a}_{ij}^k s_1(\ell_k)^*)(a_{ij}^k s_0(\ell_k) + \tilde{a}_{ij}^k s_1(\ell_k)) \\ &= (a_{ij}^k)^2 s_0(\ell_k)s_0(\ell_k)^* + (\tilde{a}_{ij}^k)^2 s_1(\ell_k)s_1(\ell_k)^* \\ &\quad + a_{ij}^k \tilde{a}_{ij}^k (s_0(\ell_k)^* s_1(\ell_k) + s_1(\ell_k)^* s_0(\ell_k)) \end{aligned}$$

$$
\begin{aligned}
&= (a_{ij}^k)^2(1+q)^{-1}\left(q(1+\gamma_k)h_0+(1-q\gamma_k)h_1\right) \\
&\quad +(\tilde{a}_{ij}^k)^2(1+q)^{-1}\left(q(1+\gamma_k^{-1})h_0+(1-q\gamma_k^{-1})h_1\right)+2a_{ij}^k\tilde{a}_{ij}^k h_1 \\
&\quad \text{(by Lemma 3.2)} \\
&= q(1+q)^{-1}\left((1+\gamma_k)(a_{ij}^k)^2+(1+\gamma_k^{-1})(\tilde{a}_{ij}^k)^2\right)h_0 \\
&\quad +(1+q)^{-1}\left((1-q\gamma_k)(a_{ij}^k)^2+(1-q\gamma_k^{-1})(\tilde{a}_{ij}^k)^2\right)h_1+2a_{ij}^k\tilde{a}_{ij}^k h_1 \\
&= \alpha((1+\gamma_k)(a_{ij}^k)^2-2a_{ij}^k+1)h_0+\alpha\left(-(1+\gamma_k)(a_{ij}^k)^2+2a_{ij}^k\right)h_1 \\
&\quad +\left((\gamma_k-q)(1+q)^{-1}\gamma_k^{-1}\right)h_1,
\end{aligned}
$$

where $\alpha = q(1+q)^{-1}\gamma_k^{-1}(1+\gamma_k)$ and

$$
\begin{aligned}
&s_0(\ell_i)s_0(\ell_j)s_0(\ell_i)^*s_0(\ell_j)^* \\
&= (s_0(\ell_i)s_0(\ell_i)^*)(s_0(\ell_j)s_0(\ell_j)^*) \\
&= (1+q)^{-2}\left(q(1+\gamma_i)h_0+(1-q\gamma_i)h_1\right)\left(q(1+\gamma_j)h_0+(1-q\gamma_j)h_1\right) \\
&\quad \text{(by Lemma 3.2)} \\
&= q(1+q)^{-1}\left((1+q\gamma_i\gamma_j)h_0+(1-q^2\gamma_i\gamma_j)h_1\right) \\
&= q(1+q)^{-1}(1+q\gamma_i\gamma_j)h_0+q(1+q)^{-1}(1-q^2\gamma_i\gamma_j)h_1.
\end{aligned}
$$

Comparing each coefficient of h_0, we obtain the following quadratic equation for a_{ij}^k:

$$(1+\gamma_k)^2(a_{ij}^k)^2-2(1+\gamma_k)a_{ij}^k+1-q\gamma_i\gamma_j\gamma_k=0.$$

Then

$$a_{ij}^k=\frac{1\pm\sqrt{q\gamma_i\gamma_j\gamma_k}}{1+\gamma_k}=f_\pm(\gamma_i,\gamma_j,\gamma_k).$$

We show the relation of c_{ij}^k and a_{ij}^k by associativity $(h_1s_0(\ell_j))s_0(\ell_i)=h_1(s_0(\ell_j)s_0(\ell_i))$. The left hand side is

$$
\begin{aligned}
(h_1s_0(\ell_j))s_0(\ell_i) &= (s_1(\ell_0)s_0(\ell_j))s_0(\ell_i) \\
&= (c_{j0}^{j^*}s_0(\ell_j)+\tilde{c}_{j0}^{j^*}s_1(\ell_j))s_0(\ell_i) \\
&= c_{j0}^{j^*}(a_{ij}^k s_0(\ell_k^*)+\tilde{a}_{ij}^k s_1(\ell_k^*))+\tilde{c}_{j0}^{j^*}(c_{ij}^k s_0(\ell_k^*)+\tilde{c}_{ij}^k s_1(\ell_k^*)) \\
&= (c_{j0}^{j^*}a_{ij}^k+\tilde{c}_{j0}^{j^*}c_{ij}^k)s_0(\ell_k^*)+(c_{j0}^{j^*}\tilde{a}_{ij}^k+\tilde{c}_{j0}^{j^*}\tilde{c}_{ij}^k)s_1(\ell_k^*),
\end{aligned}
$$

where $c_{j0}^{j^*}=(1-q\gamma_j)/(1+\gamma_j)$ by Proposition 3.6 and the right hand side is

$$h_1(s_0(\ell_j)s_0(\ell_i)) = (c_{k^*0}^k a_{ij}^k+b_{k^*0}^k\tilde{a}_{ij}^k)s_0(\ell_k^*)+(\tilde{c}_{k^*0}^k a_{ij}^k+\tilde{b}_{k^*0}^k\tilde{a}_{ij}^k)s_1(\ell_k^*),$$

where

$$c_{k^*0}^k=(1-q\gamma_{k^*})/(1+\gamma_{k^*})=(1-q\gamma_k)/(1+\gamma_k)$$

and

$$b_{k^*0}^k=(1+q)/(1+\gamma_{k^*})=(1+q)/(1+\gamma_k)$$

by Proposition 3.6 and Lemma 3.4. By comparing coefficients of $s_0(\ell_k^*)$, applying Proposition 3.6 and Lemma 3.4, we have the following equalities:

$$c_{ij}^k = \frac{1+\gamma_j}{\gamma_j(1+\gamma_k)} - \gamma_j^{-1} a_{ij}^k = \frac{1 \mp \sqrt{q\gamma_i\gamma_j^{-1}\gamma_k}}{1+\gamma_k} = f_{\mp}(\gamma_i, \gamma_j^{-1}, \gamma_k).$$

In a similar way to the above, we have the following equalities:

$$b_{ij}^k = \frac{1+\gamma_i}{\gamma_i(1+\gamma_k)} - \gamma_i^{-1} c_{ij}^k = \frac{1 \pm \sqrt{q\gamma_i^{-1}\gamma_j^{-1}\gamma_k}}{1+\gamma_k} = f_{\pm}(\gamma_i^{-1}, \gamma_j^{-1}, \gamma_k).$$

We note that $0 \le a_{ij}^k \le 1$, $0 \le b_{ij}^k \le 1$ and $0 \le c_{ij}^k \le 1$ by Axiom of a finite commutative hypergroup. In the case that $a_{ij}^k = f_+(\gamma_i, \gamma_j, \gamma_k)$, $b_{ij}^k = f_+(\gamma_i^{-1}, \gamma_j^{-1}, \gamma_k)$, $c_{ij}^k = f_-(\gamma_i, \gamma_j^{-1}, \gamma_k)$ and $c_{ji}^k = f_-(\gamma_j, \gamma_i^{-1}, \gamma_k)$, the real numbers γ_i, γ_j and γ_k satisfy $\gamma_i^{-1}\gamma_j^{-1}\gamma_k \ge q$, $\gamma_i\gamma_j\gamma_k \ge q$, $\gamma_i^{-1}\gamma_j\gamma_k^{-1} \ge q$ and $\gamma_i\gamma_j^{-1}\gamma_k^{-1} \ge q$ by $a_{ij}^k \le 1$, $b_{ij}^k \le 1$, $c_{ij}^k \ge 0$ and $c_{ji}^k \ge 0$. In the case that $a_{ij}^k = f_-(\gamma_i, \gamma_j, \gamma_k)$, $b_{ij}^k = f_-(\gamma_i^{-1}, \gamma_j^{-1}, \gamma_k)$, $c_{ij}^k = f_+(\gamma_i, \gamma_j^{-1}, \gamma_k)$ and $c_{ji}^k = f_+(\gamma_j, \gamma_i^{-1}, \gamma_k)$, we obtain that $\gamma_i^{-1}\gamma_j^{-1}\gamma_k^{-1} \ge q$, $\gamma_i\gamma_j\gamma_k^{-1} \ge q$, $\gamma_i^{-1}\gamma_j\gamma_k \ge q$ and $\gamma_i\gamma_j^{-1}\gamma_k \ge q$ in a similar way to the above.

(ii) Case of $\ell_k \in C$.
Since $s_0(\ell_k)^* = s_1(\ell_k^*)$, we obtain the following quadratic equation in a similar way to the case of (i):

$$(1+\gamma_k)^2(a_{ij}^k)^2 - 2\gamma_k(1+\gamma_k)a_{ij}^k + \gamma_k^2 - q\gamma_i\gamma_j\gamma_k = 0.$$

The solution is

$$a_{ij}^k = \frac{\gamma_k \pm \sqrt{q\gamma_i\gamma_j\gamma_k}}{1+\gamma_k} = f_{\pm}(\gamma_i, \gamma_j, \gamma_k^{-1}).$$

We have the following equalities in a similar computation to the case of (i):

$$c_{ij}^k = \frac{(1+\gamma_j)\gamma_k}{\gamma_j(1+\gamma_k)} - \gamma_j^{-1} a_{ij}^k = \frac{\gamma_k \mp \sqrt{q\gamma_i\gamma_j^{-1}\gamma_k}}{1+\gamma_k} = f_{\mp}(\gamma_i, \gamma_j^{-1}, \gamma_k^{-1})$$

and

$$b_{ij}^k = \frac{(1+\gamma_i)\gamma_k}{\gamma_i(1+\gamma_k)} - \gamma_i^{-1} c_{ij}^k = \frac{\gamma_k \pm \sqrt{q\gamma_i^{-1}\gamma_j^{-1}\gamma_k}}{1+\gamma_k} = f_{\pm}(\gamma_i^{-1}, \gamma_j^{-1}, \gamma_k^{-1}).$$

It is easy to see that the desired conditions on γ_i, γ_j and γ_k in a similar way to the case of (i). □

Corollary 3.8. *If $L_0 = \{\ell_i \in \mathcal{L};\ |S_i| = 2\}$ is not necessary to be $\mathcal{L}$, the extension $\mathcal{K} = \{s_0(\ell_i), s_1(\ell_i), s(\ell_j)\ ;\ \ell_i \in L_0,\ \ell_j \in \mathcal{L} \cap L_0^c\}$ has the same structure equations for $\ell_i \in L_0$ as described in Theorem 3.7. Moreover, for $\ell_j \in \mathcal{L} \cap L_0^c$, $\mathcal{K}$ has the following structure equations:*

(i) *If* $\ell_i \in L_0,\ \ell_j \in \mathcal{L} \cap L_0^c$, *then*

$$\ell_k^* = \ell_i \ell_j \in \mathcal{L} \cap L_0^c \textit{ and } s_0(\ell_i)s(\ell_j) = s_1(\ell_i)s(\ell_j) = s(\ell_k^*).$$

(ii) *If* $\ell_i, \ell_j \in \mathcal{L} \cap L_0^c$, *then*

$$s(\ell_i)s(\ell_j) = \begin{cases} \dfrac{1}{1+\gamma_k} s_0(\ell_k^*) + \dfrac{\gamma_k}{1+\gamma_k} s_1(\ell_k^*) & \textit{if } \ell_k^* \in F, \\ \dfrac{\gamma_k}{1+\gamma_k} s_0(\ell_k^*) + \dfrac{1}{1+\gamma_k} s_1(\ell_k^*) & \textit{if } \ell_k^* \in C \cap C_0^c, \\ \dfrac{1}{2} s_0(\ell_k^*) + \dfrac{1}{2} s_1(\ell_k^*) & \textit{if } \ell_k^* \in C_0, \\ s(\ell_k^*) & \textit{if } \ell_k^* \in \mathcal{L} \cap L_0^c. \end{cases}$$

Proof. (i) It is easy to see the desired equality. So we omit the details.

(ii) For $\ell_i, \ell_j \in \mathcal{L} \cap L_0^c$ let $\ell_k^* \in L_0$. Then there exists $0 \le a \le 1$ such that

$$s(\ell_i)s(\ell_j) = a s_0(\ell_k^*) + (1-a) s_1(\ell_k^*).$$

We have

$$(h_1 s(\ell_i))s(\ell_j) = s(\ell_i)s(\ell_j) = a s_0(\ell_k^*) + (1-a) s_1(\ell_k^*)$$

and by Proposition 3.6

$$\begin{aligned} h_1(s(\ell_i)s(\ell_j)) &= h_1(a s_0(\ell_k^*) + \tilde{a} s_1(\ell_k^*)) \\ &= a h_1 s_0(\ell_k^*) + \tilde{a} h_1 s_1(\ell_k^*) \\ &= a\left(\frac{1 - q\gamma_{k^*}}{1+\gamma_{k^*}} s_0(\ell_k^*) + \frac{(1+q)\gamma_{k^*}}{1+\gamma_{k^*}} s_1(\ell_k^*)\right) \\ &\quad + \tilde{a}\left(\frac{1+q}{1+\gamma_{k^*}} s_0(\ell_k^*) + \frac{\gamma_{k^*} - q}{1+\gamma_{k^*}} s_1(\ell_k^*)\right) \\ &= \frac{1 + q - q(1+\gamma_{k^*})a}{1+\gamma_{k^*}} s_0(\ell_k^*) + \frac{\gamma_{k^*} - q + q(1+\gamma_{k^*})a}{1+\gamma_{k^*}} s_1(\ell_k^*). \end{aligned}$$

The coefficients of $s_0(\ell_k^*)$ in each are a and $(1 + q - q(1+q)\gamma_{k^*} a)(1+\gamma_{k^*})^{-1}$, respectively. Since the both must be equal, we obtain

$$a = \frac{1}{1+\gamma_{k^*}}.$$

Therefore, by this equality and Lemma 3.4 we have the product of $s(\ell_i)s(\ell_j)$.

For $\ell_i, \ell_j \in \mathcal{L} \cap L_0^c$, let $\ell_k^* \in \mathcal{L} \cap L_0^c$. Then it is easy to see that $s(\ell_i)s(\ell_j) = s(\ell_k^*)$. So we omit the details. □

If $L_0 = \{\ell_0\}$, then the extension $\mathcal{K}$ of $\mathcal{L}$ by $\mathcal{H}$ is the join $\mathcal{H} \vee \mathcal{L}$ of $\mathcal{H}$ by $\mathcal{L}$. Theorem 3.7 and Corollary 3.8 are the necessary condition that the extension $\mathcal{K}$ of $\mathcal{L}$ by $\mathcal{H}$ is a hypergroup where $\mathcal{L}$ is a finite Abelian group and $\mathcal{H} = \mathbb{Z}_q(2)$ is a hypergroup of order two. We give a condition that the associativity law of the extension $\mathcal{K}$ holds. For ℓ_i, ℓ_j, $\ell_k \in L_0$ with $\ell_k^* = \ell_i \ell_j$, let θ be a mapping from $L_0 \times L_0$ to $Z_2 = \{-1, 1\}$ such that for $\ell_j \neq \ell_i^*$

$$\theta(\ell_i, \ell_j) = \begin{cases} \;\;\,1 & \text{if } a_{ij}^k = f_+(\gamma_i, \gamma_j, \gamma_k) \;\text{ or }\; a_{ij}^k = f_+(\gamma_i, \gamma_j, \gamma_k^{-1}) \\ -1 & \text{if } a_{ij}^k = f_-(\gamma_i, \gamma_j, \gamma_k) \;\text{ or }\; a_{ij}^k = f_-(\gamma_i, \gamma_j, \gamma_k^{-1}) \end{cases}$$

and for $\ell_j = \ell_i^*$

$$\theta(\ell_i, \ell_i^*) = \begin{cases} \;\;\,1 & \text{if } a_{ii^*}^0 \neq 0 \\ -1 & \text{if } a_{ii^*}^0 = 0 \end{cases} \tag{3.3}$$

where a_{ij}^k is the coefficient in (3.2). Note that $\theta(\ell_0, \ell_i) = 1$.

Proposition 3.9. *Let $\mathcal{L} = \{\ell_0, \ell_1, \ldots, \ell_n\}$ be an Abelian group with unit ℓ_0. Then the associativity $(s_{\sigma(i)}(\ell_i)s_{\sigma(j)}(\ell_j))s_{\sigma(r)}(\ell_r) = s_{\sigma(i)}(\ell_i)(s_{\sigma(j)}(\ell_j)\ s_{\sigma(r)}(\ell_r))$ holds for $\sigma(i)$, $\sigma(j)$, $\sigma(r) \in \{0, 1\}$ if and only if $\theta(\ell_i, \ell_j)\theta(\ell_i\ell_j, \ell_r) = \theta(\ell_j, \ell_r)\theta(\ell_i, \ell_j\ell_r)$ for ℓ_i, ℓ_j, $\ell_r \in L_0$, namely θ is a Z_2-valued 2-cocycle on L_0.*

Proof. We can establish the following conditions between the value of $\theta(\ell_i, \ell_j)\theta(\ell_i\ell_j, \ell_r)$ and the product of $s_0(\ell_i)$, $s_0(\ell_j)$ and $s_0(\ell_r)$ and between the value of $\theta(\ell_j, \ell_r)\theta(\ell_i, \ell_j\ell_r)$ and the product of them by straightforward computation:

(i) $\theta(\ell_i, \ell_j)\theta(\ell_i\ell_j, \ell_r) = 1 \iff$

$$(s_0(\ell_i)s_0(\ell_j))s_0(\ell_r) = \begin{cases} f_+(q\gamma_i\gamma_j, \gamma_r, \gamma_t)s_0(\ell_t^*) + f_-(q\gamma_i\gamma_j, \gamma_r, \gamma_t^{-1})s_1(\ell_t^*) \\ \hfill \text{if } \ell_t^* \in F, \\ f_+(q\gamma_i\gamma_j, \gamma_r, \gamma_t^{-1})s_0(\ell_t^*) + f_-(q\gamma_i\gamma_j, \gamma_r, \gamma_t)s_1(\ell_t^*) \\ \hfill \text{if } \ell_t^* \in C, \end{cases}$$

(ii) $\theta(\ell_j, \ell_r)\theta(\ell_i, \ell_j\ell_r) = 1 \iff$

$$s_0(\ell_i)(s_0(\ell_j)s_0(\ell_r)) = \begin{cases} f_+(q\gamma_j\gamma_r, \gamma_i, \gamma_t)s_0(\ell_t^*) + f_-(q\gamma_j\gamma_r, \gamma_i, \gamma_t^{-1})s_1(\ell_t^*) \\ \hfill \text{if } \ell_t^* \in F, \\ f_+(q\gamma_j\gamma_r, \gamma_i, \gamma_t^{-1})s_0(\ell_t^*) + f_-(q\gamma_j\gamma_r, \gamma_i, \gamma_t)s_1(\ell_t^*) \\ \hfill \text{if } \ell_t^* \in C, \end{cases}$$

for $\ell_i, \ell_j, \ell_r \in L_0$ where $\ell_t^* = \ell_i\ell_j\ell_r$. Since $f_+(qxy, v, z) = f_+(qyv, x, z)$, it is easy to see that

$$\begin{aligned} &\theta(\ell_i, \ell_j)\theta(\ell_i\ell_j, \ell_r) = \theta(\ell_j, \ell_r)\theta(\ell_i, \ell_j\ell_r) \\ &\iff \quad (s_0(\ell_i)s_0(\ell_j))s_0(\ell_r) = s_0(\ell_i)(s_0(\ell_j)s_0(\ell_r)) \end{aligned}$$

from the above computation. In the case that

$$\theta(\ell_i,\ell_j)\theta(\ell_i\ell_j,\ell_r)=\theta(\ell_j,\ell_r)\theta(\ell_i,\ell_j\ell_r)=-1,$$

we obtain the above condition in a similar way to the above since $f_-(qxy,v,z)=f_-(qyv,x,z)$. Consider in the case that

$$\theta(\ell_i,\ell_j)\theta(\ell_i\ell_j,\ell_r)\neq\theta(\ell_j,\ell_r)\theta(\ell_i,\ell_j\ell_r).$$

Since $q>0$, $x>0$, $y>0$ and $z>0$, we have $f_+(qxy,v,z)\neq f_-(qyv,x,z)$. So if $\theta(\ell_i,\ell_j)\theta(\ell_i\ell_j,\ell_r)\neq\theta(\ell_j,\ell_r)\theta(\ell_i,\ell_j\ell_r)$, then $(s_0(\ell_i)s_0(\ell_j))s_0(\ell_r)\neq s_0(\ell_i)(s_0(\ell_j)s_0(\ell_r))$.

We can obtain the same results in the case of $\sigma(i)=1$, $\sigma(j)=1$ or $\sigma(r)=1$ in a similar computation to the above.

Therefore, $(s_{\sigma(i)}(\ell_i)s_{\sigma(j)}(\ell_j))s_{\sigma(r)}(\ell_r)=s_{\sigma(i)}(\ell_i)(s_{\sigma(j)}(\ell_j)s_{\sigma(r)}(\ell_r))$ if and only if $\theta(\ell_i,\ell_j)\theta(\ell_i\ell_j,\ell_r)=\theta(\ell_j,\ell_r)\theta(\ell_i,\ell_j\ell_r)$ for ℓ_i, ℓ_j, $\ell_r\in L_0$. □

Next we will give the sufficient condition that the extension $\mathcal{K}$ of finite Abelian groups $\mathcal{L}$ by hypergroups $\mathcal{H}$ of order two is a commutative hypergroup.

Theorem 3.10. *Let $\mathcal{H}=\{h_0,h_1\}=\mathbb{Z}_q(2)$ be a hypergroup of order two and $\mathcal{L}=\{\ell_0,\ell_1,\dots,\ell_n\}$ be a finite Abelian group with unit ℓ_0. Let $\mathcal{K}$ be the disjoint union of the sets $S_i=\{s_0(\ell_i),s_1(\ell_i)\}$ for $\ell_i\in\mathcal{L}$, namely $\mathcal{K}=\bigcup_{i=0}^n S_i$. For $1\le i\le n$, γ_i is the real number such that $q\le\gamma_i\le 1/q$, $\gamma_{i^*}=\gamma_i$ and $\gamma_0=1/q$. For $\ell_i,\ell_j,\ell_k\in\mathcal{L}$ with $\ell_i\ell_j\ell_k=\ell_0$, the real numbers γ_i, γ_j and γ_k satisfy that $q\le\gamma_i\gamma_j\gamma_k$ and $q\le\gamma_i^{-1}\gamma_j^{-1}\gamma_k$. If the structure equations of $\mathcal{K}$ is given by the following:*

$$\begin{aligned}
s_0(\ell_i)s_0(\ell_j)&=f_+(\gamma_i,\gamma_j,\gamma_k)s_0(\ell_k^*)+f_-(\gamma_i,\gamma_j,\gamma_k^{-1})s_1(\ell_k^*),\\
s_1(\ell_i)s_1(\ell_j)&=f_+(\gamma_i^{-1},\gamma_j^{-1},\gamma_k)s_0(\ell_k^*)+f_-(\gamma_i^{-1},\gamma_j^{-1},\gamma_k^{-1})s_1(\ell_k^*),\\
s_0(\ell_i)s_1(\ell_j)&=f_-(\gamma_i,\gamma_j^{-1},\gamma_k)s_0(\ell_k^*)+f_+(\gamma_i,\gamma_j^{-1},\gamma_k^{-1})s_1(\ell_k^*),
\end{aligned}$$

then $\mathcal{K}$ is a commutative hypergroup such that $s_0(\ell_i)^*=s_0(\ell_i^*)$ for all $\ell_i\in\mathcal{L}$ and $h_0=s_0(\ell_0)$ and $\mathcal{K}$ is an extension of $\mathcal{L}$ by $\mathcal{H}$.

Proof. To show that $\mathcal{K}$ is a finite commutative hypergroup, we will check that $\mathcal{K}$ satisfies Axiom of a finite commutative hypergroup. Since $a_{ij}^k=f_+(\gamma_i,\gamma_j,\gamma_k)$ for $\ell_i,\ell_j,\ell_k\in\mathcal{L}$ such that $\ell_k^*=\ell_i\ell_j$, we obtain $\theta(\ell_i,\ell_j)=1$ and so $\theta(\ell_i,\ell_j)\theta(\ell_i\ell_j,\ell_r)=\theta(\ell_j,\ell_r)\theta(\ell_i,\ell_j\ell_r)$. Hence the associativity law in $\mathcal{K}$ holds by Proposition 3.9. Since it is easy to see that $f_+(\gamma_i,\gamma_{i^*},\gamma_0)=q(1+\gamma_i)(1+q)^{-1}$, it must be $s_0(\ell_i)^*=s_0(\ell_i^*)$ and $s_1(\ell_i)^*=s_1(\ell_i^*)$ for all $\ell_i\in\mathcal{L}$ by Lemma 3.2, i.e., $\mathcal{K}^*=\mathcal{K}$. Hence $\mathcal{K}$ satisfies the conditions (i) and (ii) in Axiom of a finite commutative hypergroup. Observe that $f_\pm(x,y,z)$ for $x,y,z>0$. Thus the other conditions (iii), (iv) and (v) are automatically satisfied. Therefore $\mathcal{K}$ is a finite commutative hypergroup.

Let φ be a mapping from $\mathcal{K}$ onto $\mathcal{L}$ such that $\varphi(s_0(\ell_i))=\varphi(s_1(\ell_i))=\ell_i$ for $\ell_i\in\mathcal{L}$. It is easy to see that φ becomes a homomorphism from $\mathcal{K}$ onto $\mathcal{L}$ such that $\operatorname{Ker}\varphi=\mathcal{H}$ and $\mathcal{H}$ is a subhypergroup of $\mathcal{K}$.

Therefore $\mathcal{K}$ is an extension hypergroup of $\mathcal{L}$ by $\mathcal{H}$. □

Corollary 3.11. *Let L_0 be a subgroup of a finite Abelian group $\mathcal{L}$ and $\mathcal{K} = \{s_0(\ell_i), s_1(\ell_i),\ s(\ell_j);\ \ell_i \in L_0,\ \ell_j \in \mathcal{L} \cap L_0^c\}$ have the same structure equations as described in Theorem 3.10 for $\ell_i \in L_0$. For $\ell_j \in \mathcal{L} \cap L_0^c$, let $\mathcal{K}$ have the following structure equations:*

(i) *If $\ell_i \in L_0$, $\ell_j \in \mathcal{L} \cap L_0^c$, then*

$$\ell_k^* = \ell_i \ell_j \in \mathcal{L} \cap L_0^c \text{ and } s_0(\ell_i)s(\ell_j) = s_1(\ell_i)s(\ell_j) = s(\ell_k^*).$$

(ii) *If ℓ_i, $\ell_j \in \mathcal{L} \cap L_0^c$, then*

$$s(\ell_i)s(\ell_j) = \begin{cases} \dfrac{1}{1+\gamma_k} s_0(\ell_k^*) + \dfrac{\gamma_k}{1+\gamma_k} s_1(\ell_k^*) & \text{if } \ell_k^* \in L_0, \\ s(\ell_k^*) & \text{if } \ell_k^* \in \mathcal{L} \cap L_0^c. \end{cases}$$

Then $\mathcal{K}$ becomes a commutative hypergroup which is an extension of $\mathcal{L}$ by $\mathcal{H}$.

Proof. Since $s(\ell_j)s(\ell_j^*) = \omega(\mathcal{H})$, we obtain $s(\ell_j)^* = s(\ell_j^*)$ for $\ell_j \in \mathcal{L} \cap L_0^c$ by Lemma 3.2. We already showed that $s_0(\ell_i)^* = s_0(\ell_i^*)$ for $\ell_i \in \mathcal{L}$ in the proof of Theorem 3.10. Hence $\mathcal{K}^* = \mathcal{K}$. We will check that the associativity law holds for $\ell_i, \ell_j, \ell_r \in \mathcal{L}$. If $\ell_i, \ell_j \in \mathcal{L} \cap L_0^c$ and $\ell_r \in L_0$, then we have

$$\begin{aligned} (s(\ell_i)s(\ell_j))s_{\sigma(r)}(\ell_r) &= s(\ell_i)(s(\ell_j)s_{\sigma(r)}(\ell_r)) \\ &= \begin{cases} \dfrac{1}{1+\gamma_t} s_0(\ell_t^*) + \dfrac{\gamma_t}{1+\gamma_t} s_1(\ell_t^*) & \text{if } \ell_t^* \in L_0, \\ s(\ell_t^*) & \text{if } \ell_t^* \in \mathcal{L} \cap L_0^c \end{cases} \end{aligned}$$

by straightforward computation for $\sigma(r) \in \{0,1\}$ where t is a number such that $\ell_i \ell_j \ell_r \ell_t = \ell_0$. If ℓ_i, ℓ_j, $\ell_r \in \mathcal{L} \cap L_0^c$, then we have $(s(\ell_i)s(\ell_j))s(\ell_r) = s(\ell_i)(s(\ell_j)s(\ell_r))$ in a similar way to the above. If $\ell_i \in \mathcal{L} \cap L_0^c$ and ℓ_j, $\ell_r \in L_0$, then $\ell_t^* \in \mathcal{L} \cap L_0^c$ and we have $(s(\ell_i)s_{\sigma(j)}(\ell_j))s_{\sigma(r)}(\ell_r) = s(\ell_i)(s_{\sigma(j)}(\ell_j)s_{\sigma(r)}(\ell_r)) = s(\ell_t^*)$ for $\sigma(j), \sigma(r) \in \{0,1\}$. The other conditions of Axiom of a finite commutative hypergroup can be established in $\mathcal{K}$ in a similar way to the proof of Theorem 3.10. Therefore $\mathcal{K}$ is a finite commutative hypergroup.

Let φ be a mapping from $\mathcal{K}$ onto $\mathcal{L}$ such that $\varphi(s_0(\ell_i)) = \varphi(s_1(\ell_i)) = \ell_i$ for $\ell_i \in L_0$ and $\varphi(s(\ell_j)) = \ell_j$ for $\ell_j \in \mathcal{L} \cap L_0^c$. It is easy to see that φ becomes a homomorphism from $\mathcal{K}$ onto $\mathcal{L}$ such that $\operatorname{Ker}\varphi = \mathcal{H}$ and $\mathcal{H}$ is a subhypergroup of $\mathcal{K}$. Therefore $\mathcal{K}$ is an extension hypergroup of $\mathcal{L}$ by $\mathcal{H}$. □

Next we will give another sufficient condition that the extension $\mathcal{K}$ of cyclic groups $\mathcal{L}$ by hypergroups $\mathcal{H}$ of order two is a commutative hypergroup.

Theorem 3.12. *Let $\mathcal{H} = \{h_0, h_1\} = \mathbb{Z}_q(2)$ be a hypergroup of order two and $\mathcal{L} = \{\ell_0, \ell_1, \ldots, \ell_{2m-1}\}$ be a cyclic group of order $2m$ with unit ℓ_0. Let $\mathcal{K}$ be the disjoint union of the sets $S_i = \{s_0(\ell_i), s_1(\ell_i)\}$, where $s_0(\ell_0) = h_0$ and $s_1(\ell_0) = h_1$ for $\ell_i \in \mathcal{L}$, namely $\mathcal{K} = \bigcup_{i=0}^{2m-1} S_i$. For $1 \le i \le 2m-1$, γ_i is the real number such that $q \le \gamma_i \le 1/q$, $\gamma_{i^*} = \gamma_i^{-1}$ $(i \ne m)$, $\gamma_m = 1$ and $\gamma_0 = 1/q$. For ℓ_i, ℓ_j, $\ell_k \in \mathcal{L}$ with $\ell_i\ell_j\ell_k = \ell_0$, the real numbers γ_i, γ_j and γ_k satisfy that $q \le \gamma_i\gamma_j\gamma_k \le 1/q$ and $q \le \gamma_i^{-1}\gamma_j^{-1}\gamma_k \le 1/q$. Let $\mathcal{K}$ be the set which is given by the following structure equations:*

(i) *Case of $0 < i + j < 2m$ where $i \ne 0$ or $j \ne 0$.*

$$\begin{aligned} s_0(\ell_i)s_0(\ell_j) &= f_+(\gamma_i, \gamma_j, \gamma_k^{-1})s_0(\ell_k^*) + f_-(\gamma_i, \gamma_j, \gamma_k)s_1(\ell_k^*), \\ s_1(\ell_i)s_1(\ell_j) &= f_+(\gamma_i^{-1}, \gamma_j^{-1}, \gamma_k^{-1})s_0(\ell_k^*) + f_-(\gamma_i^{-1}, \gamma_j^{-1}, \gamma_k)s_1(\ell_k^*), \\ s_0(\ell_i)s_1(\ell_j) &= f_-(\gamma_i, \gamma_j^{-1}, \gamma_k^{-1})s_0(\ell_k^*) + f_+(\gamma_i, \gamma_j^{-1}, \gamma_k)s_1(\ell_k^*). \end{aligned}$$

(ii) *Case of $2m < i + j < 4m$.*

$$\begin{aligned} s_0(\ell_i)s_0(\ell_j) &= f_-(\gamma_i, \gamma_j, \gamma_k^{-1})s_0(\ell_k^*) + f_+(\gamma_i, \gamma_j, \gamma_k)s_1(\ell_k^*), \\ s_1(\ell_i)s_1(\ell_j) &= f_-(\gamma_i^{-1}, \gamma_j^{-1}, \gamma_k^{-1})s_0(\ell_k^*) + f_+(\gamma_i^{-1}, \gamma_j^{-1}, \gamma_k)s_1(\ell_k^*), \\ s_0(\ell_i)s_1(\ell_j) &= f_+(\gamma_i, \gamma_j^{-1}, \gamma_k^{-1})s_0(\ell_k^*) + f_-(\gamma_i, \gamma_j^{-1}, \gamma_k)s_1(\ell_k^*). \end{aligned}$$

(iii) *Case of $i + j = 2m$.*

$$\begin{aligned} s_0(\ell_i)s_0(\ell_j) &= s_0(\ell_i)s_0(\ell_i^*) = h_1, \quad s_1(\ell_i)s_1(\ell_j) = s_1(\ell_i)s_1(\ell_i^*) = h_1, \\ s_0(\ell_i)s_1(\ell_j) &= s_0(\ell_i)s_1(\ell_i^*) = f_+(\gamma_i, \gamma_{i^*}^{-1}, 1/q)h_0 + f_-(\gamma_i, \gamma_{i^*}^{-1}, q)h_1. \end{aligned}$$

Then $\mathcal{K}$ is a commutative hypergroup such that $s_0(\ell_i)^* = s_1(\ell_i^*)$ for $\ell_i \in \mathcal{L}\backslash\{\ell_0\}$ and $\mathcal{K}$ is an extension of $\mathcal{L}$ by $\mathcal{H}$.

Proof. First we will check that the associativity law under $\mathcal{K}$-multiplication. For ℓ_i, ℓ_j, $\ell_r \in \mathcal{L}$, let k, u and t be numbers such that $\ell_i\ell_j\ell_k = \ell_0$, $\ell_j\ell_r\ell_u = \ell_0$ and $\ell_i\ell_j\ell_r\ell_t = \ell_0$. The associativity law holds if and only if $\theta(\ell_i, \ell_j)\theta(\ell_i\ell_j, \ell_r) = \theta(\ell_j, \ell_r)\theta(\ell_i, \ell_j\ell_r)$ for ℓ_i, ℓ_j, $\ell_r \in \mathcal{L}$ by Proposition 3.9. The values of $\theta(\ell_i, \ell_j)\theta(\ell_i\ell_j, \ell_r)$ and $\theta(\ell_j, \ell_r)\theta(\ell_i, \ell_j\ell_r)$ depend on the values of $i + j$ and $j + r$. So we check the following case:

(i) $0 < i + j < 2m$. Since $a_{ij}^k = f_+(\gamma_i, \gamma_j, \gamma_k^{-1})$ in (3.3), $\theta(\ell_i, \ell_j) = 1$.

(1) $0 < j + r < 2m$.
It is obvious that $\theta(\ell_j, \ell_r) = 1$. Since $k^* + r = i + j + r = i + u^*$, we obtain $\theta(\ell_i\ell_j, \ell_r) = \theta(\ell_i, \ell_j\ell_r)$. Hence $\theta(\ell_i, \ell_j)\theta(\ell_i\ell_j, \ell_r) = \theta(\ell_j, \ell_r)\theta(\ell_i, \ell_j\ell_r)$.

(2) $2m < j + r < 4m$.
It is obvious that $\theta(\ell_j, \ell_r) = -1$. Since $k^* + r = i + j + r > 2m$, we obtain $\theta(\ell_i\ell_j, \ell_r) = -1$. Since $i + u^* = i + (j + r - 2m) = (i + j - 2m) + r < r < 2m$, we obtain $\theta(\ell_i, \ell_j\ell_r) = 1$. Hence $\theta(\ell_i, \ell_j)\theta(\ell_i\ell_j, \ell_r) = \theta(\ell_j, \ell_r)\theta(\ell_i, \ell_j\ell_r)$.

(3) $j + r = 2m$.
Since $r = j^*$ and $t^* \equiv i + j + r \pmod{2m} = i$, we will check the values of $\theta(\ell_i, \ell_j)\theta(\ell_i\ell_j, \ell_j^*)$ and $\theta(\ell_j, \ell_j^*)\theta(\ell_0, \ell_i)$. Since $j^* + k^* = j^* + i + j = i + 2m \geq 2m$, we obtain $\theta(\ell_i\ell_j, \ell_j^*) = -1$. Hence we have $\theta(\ell_i, \ell_j)\theta(\ell_i\ell_j, \ell_j^*) = -1$. Since $a_{jj^*}^0 = 0$ in (3.3), $\theta(\ell_j, \ell_j^*) = -1$. The value of $\theta(\ell_0, \ell_i)$ is always equal to 1. Hence we have $\theta(\ell_j, \ell_j^*)\theta(\ell_0, \ell_i) = -1$. Therefore $\theta(\ell_i, \ell_j)\theta(\ell_i\ell_j, \ell_j^*) = \theta(\ell_j, \ell_j^*)\theta(\ell_0, \ell_i)$.

(ii) $2m \leq i+j < 4m$. We obtain $\theta(\ell_i, \ell_j)\theta(\ell_i\ell_j, \ell_r) = \theta(\ell_j, \ell_r)\theta(\ell_i, \ell_j\ell_r)$ in a similar way to the case of (i). So we omit details.

The associativity holds by Proposition 3.9. The other conditions of Axiom of a finite commutative hypergroup can be established in $\mathcal{K}$ in a similar way to the proof of Theorem 3.10. Therefore $\mathcal{K}$ is a finite commutative hypergroup.

Let φ be a mapping from $\mathcal{K}$ onto $\mathcal{L}$ such that $\varphi(s_0(\ell_i)) = \varphi(s_1(\ell_i)) = \ell_i$ for $\ell_i \in \mathcal{L}$. It is easy to see that φ becomes a homomorphism from $\mathcal{K}$ onto $\mathcal{L}$ such that Ker $\varphi = \mathcal{H}$ and $\mathcal{H}$ is a subhypergroup of $\mathcal{K}$.

Therefore $\mathcal{K}$ is an extension of hypergroup of $\mathcal{L}$ by $\mathcal{H}$. □

Corollary 3.13. *Let L_0 be a subgroup of a cyclic group $\mathcal{L}$ such that $|L_0| = 2m$ and $\mathcal{K} = \{s_0(\ell_i), s_1(\ell_i), s(\ell_j);\ \ell_i \in L_0,\ \ell_j \in \mathcal{L} \cap L_0^c\}$ have the same structure equations as described in Theorem 3.12 for $\ell_i \in L_0$. For $\ell_j \in \mathcal{L} \cap L_0^c$, let $\mathcal{K}$ have the following structure equations:*

(i) *If $\ell_i \in L_0$, $\ell_j \in \mathcal{L} \cap L_0^c$, then*

$$\ell_{k^*} = \ell_i\ell_j \in \mathcal{L} \cap L_0^c \text{ and } s_0(\ell_i)s(\ell_j) = s_1(\ell_i)s(\ell_j) = s(\ell_k^*).$$

(ii) *If ℓ_i, $\ell_j \in \mathcal{L} \cap L_0^c$, then*

$$s(\ell_i)s(\ell_j) = \begin{cases} \dfrac{\gamma_k}{1+\gamma_k}s_0(\ell_k^*) + \dfrac{1}{1+\gamma_k}s_1(\ell_k^*) & \text{if } \ell_k^* \in L_0, \\ s(\ell_k^*) & \text{if } \ell_k^* \in \mathcal{L} \cap L_0^c. \end{cases}$$

Then $\mathcal{K}$ becomes a commutative hypergroup which an extension of $\mathcal{L}$ by $\mathcal{H}$.

Proof. We can show in a similar way to the proof of Corollary 3.11 so we omit the details. □

4. Applications and Examples

Under these preparations one can determine the extensions $\mathcal{K}$ of $\mathcal{L}$ by $\mathcal{H}$ for concrete Abelian groups $\mathcal{L} = \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_5$ and a hypergroup $\mathcal{H} = \{h_0, h_1\} = \mathbb{Z}_q(2)$ of order two.

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two extensions of $\mathcal{L}$ by $\mathcal{H}$ and φ_1 [resp., φ_2] be a hypergroup homomorphism from $\mathcal{K}_1$ [resp., $\mathcal{K}_2$] onto $\mathcal{L}$. Then $\mathcal{K}_1$ is called to be equivalent to $\mathcal{K}_2$ as extensions if there exists a hypergroup isomorphism ψ from $\mathcal{K}_1$ onto $\mathcal{K}_2$ such that $\psi(h) = h$ for all $h \in \mathcal{H}$ and $\varphi_2 \circ \psi = \varphi_1$. A hypergroup isomorphism means that a bijective hypergroup homomorphism. Let L_0 be a subgroup of $\mathcal{L}$ such that $L_0 = \{\ell \in \mathcal{L}\ ;\ |\varphi^{-1}(\ell)| = 2\}$.

We have already calculated all extensions of the hypergroup of order two by concrete Abelian groups in our paper [7]. The following examples are dual versions of such extension.

Example 4.1. Let $\mathcal{L} = \{\ell_0, \ell_1;\ \ell_1^2 = \ell_0\} \cong \mathbb{Z}_2$ and $\mathcal{H} = \mathbb{Z}_q(2) = \{h_0, h_1;\ h_1^2 = qh_0 + (1-q)h_1,\ 0 < q \leq 1\}$. Since the subgroup L_0 of $\mathcal{L}$ is $\mathcal{L}$ or $\{\ell_0\}$, one has the extensions such that $|\mathcal{K}| = 4$ and $|\mathcal{K}| = 3$ respectively.

(i) Case of $|\mathcal{K}| = 4$.

(1) Hermitian case, i.e., $s_0(\ell_1)^* = s_0(\ell_1)$.
Let γ be a real number such that $q \leq \gamma \leq 1/q$. We denote $\mathcal{K}_a(\gamma) = \{h_0,\ h_1,\ s_0(\ell_1),\ s_1(\ell_1)\}$. The structure equations of $\mathcal{K}_a(\gamma)$ is given by the following:

(a) $h_1 s_0(\ell_1) = \dfrac{1-q\gamma}{1+\gamma} s_0(\ell_1) + \dfrac{(1+q)\gamma}{1+\gamma} s_1(\ell_1)$,

(b) $h_1 s_1(\ell_1) = \dfrac{(1+q)\gamma^{-1}}{1+\gamma^{-1}} s_0(\ell_1) + \dfrac{1-q\gamma^{-1}}{1+\gamma^{-1}} s_1(\ell_1)$,

(c) $s_0(\ell_1)^2 = \dfrac{q(1+\gamma)}{1+q} h_0 + \dfrac{1-q\gamma}{1+q} h_1$,

(d) $s_1(\ell_1)^2 = \dfrac{q(1+\gamma^{-1})}{1+q} h_0 + \dfrac{1-q\gamma^{-1}}{1+q} h_1$,

(e) $s_0(\ell_1)s_1(\ell_1) = h_1$.

Next we give a non Hermitian hypergroup extension.

(2) Non Hermitian case, i.e., $s_0(\ell_1)^* = s_1(\ell_1)$.
We denote $\mathcal{K}_b = \{h_0, h_1, s_0(\ell_1), s_1(\ell_1)\}$. The structure equations of $\mathcal{K}_b$ is given by the following:

(a) $h_1 s_0(\ell_1) = \dfrac{1-q}{2} s_0(\ell_1) + \dfrac{1+q}{2} s_1(\ell_1)$,

(b) $h_1 s_1(\ell_1) = \dfrac{1+q}{2} s_0(\ell_1) + \dfrac{1-q}{2} s_1(\ell_1)$,

(c) $s_0(\ell_1)^2 = s_1(\ell_1)^2 = h_1$,

(d) $s_0(\ell_1)s_1(\ell_1) = \dfrac{2q}{1+q} h_0 + \dfrac{1-q}{1+q} h_1$.

(ii) Case of $|\mathcal{K}| = 3$.
$\mathcal{K}_2 = \{h_0, h_1, s(\ell_1)\}$ is the join $\mathcal{H} \vee \mathcal{L}$ of $\mathcal{H}$ by $\mathcal{L}$.

Remark 1. The set $\mathcal{K}_a(\gamma)$ is a commutative Hermitian hypergroup and the extension of $\mathcal{L}$ by $\mathcal{H}$ by Theorem 3.10. This $\mathcal{K}_a(\gamma)$ is also the extensions such that $a_{11}^0 = f_+(\gamma, \gamma, 1/q)$ in (i)–(1) of Theorem 3.7. By Theorem 3.7 there is another possibility that $a_{11}^0 = f_-(\gamma, \gamma, 1/q) = q(1-\gamma)/(1+q)$. Since the case of $a_{11}^0 = f_-(\gamma, \gamma, 1/q)$ does not satisfy Lemma 3.2, the extensions $\mathcal{K}_a(\gamma)$ and $\mathcal{K}_2$ are all extensions of $\mathcal{L}$ by $\mathcal{H}$ in Hermitian case. The set $\mathcal{K}_b$ is a commutative non Hermitian hypergroup and the extension of $\mathcal{L}$ by $\mathcal{H}$ by Theorem 3.12. This $\mathcal{K}_b$ is also the extension such that $a_{11}^0 = f_-(1, 1, 1/q) = 0$ in (i)–(2) of Theorem 3.7. In a similar discussion to the above $\mathcal{K}_b$ is all extensions in non Hermitian case. Therefore all extensions $\mathcal{K}$ of $\mathcal{L}$ by $\mathcal{H}$ are $\mathcal{K}_a(\gamma)$ $(q \le \gamma \le 1/q)$, $\mathcal{K}_b$ and $\mathcal{K}_2$. Moreover, $\mathcal{K}_a(\gamma)$ is equivalent to $\mathcal{K}_a(\gamma')$ as extensions if and only if $\gamma' = \gamma$ or $\gamma' = \gamma^{-1}$.

Example 4.2. Let $\mathcal{L} = \{\ell_0, \ell_1, \ell_2;\ \ell_1^2 = \ell_2,\ \ell_1^* = \ell_2\} \cong \mathbb{Z}_3$ and $\mathcal{H} = \mathbb{Z}_q(2) = \{h_0, h_1;\ h_1^2 = qh_0 + (1-q)h_1,\ 0 < q \le 1\}$. Since the subgroup L_0 of $\mathcal{L}$ is $\mathcal{L}$ or $\{\ell_0\}$, one has the extensions such that $|\mathcal{K}| = 6$ and $|\mathcal{K}| = 4$ respectively.

(i) Case of $|\mathcal{K}| = 6$.
Let γ be a real number such that $q^{1/3} \le \gamma \le 1/q$. We denote

$$\mathcal{K}_a(\gamma) = \{h_0, h_1, s_0(\ell_1), s_1(\ell_1), s_0(\ell_2), s_1(\ell_2)\}.$$

The structure equations of $\mathcal{K}_a(\gamma)$ is given by the following:

(1) $$h_1 s_0(\ell_1) = \frac{1-q\gamma}{1+\gamma} s_0(\ell_1) + \frac{(1+q)\gamma}{1+\gamma} s_1(\ell_1),$$
$$h_1 s_0(\ell_2) = \frac{1-q\gamma}{1+\gamma} s_0(\ell_2) + \frac{(1+q)\gamma}{1+\gamma} s_1(\ell_2),$$

(2) $$h_1 s_1(\ell_1) = \frac{(1+q)\gamma^{-1}}{1+\gamma^{-1}} s_0(\ell_1) + \frac{1-q\gamma^{-1}}{1+\gamma^{-1}} s_1(\ell_1),$$
$$h_1 s_1(\ell_2) = \frac{(1+q)\gamma^{-1}}{1+\gamma^{-1}} s_0(\ell_2) + \frac{1-q\gamma^{-1}}{1+\gamma^{-1}} s_1(\ell_2),$$

(3) $$s_0(\ell_1) s_0(\ell_2) = \frac{q(1+\gamma)}{1+q} h_0 + \frac{1-q\gamma}{1+q} h_1,$$
$$s_1(\ell_1) s_1(\ell_2) = \frac{q(1+\gamma^{-1})}{1+q} h_0 + \frac{1-q\gamma^{-1}}{1+q} h_1,$$
$$s_0(\ell_1) s_1(\ell_2) = s_0(\ell_2) s_1(\ell_1) = h_1,$$

(4) $$s_0(\ell_1)^2 = \frac{1+\gamma\sqrt{q\gamma}}{1+\gamma} s_0(\ell_2) + \frac{\gamma-\gamma\sqrt{q\gamma}}{1+\gamma} s_1(\ell_2),$$
$$s_0(\ell_2)^2 = \frac{1+\gamma\sqrt{q\gamma}}{1+\gamma} s_0(\ell_1) + \frac{\gamma-\gamma\sqrt{q\gamma}}{1+\gamma} s_1(\ell_1),$$

(5) $s_1(\ell_1)^2 = \dfrac{1+\sqrt{q\gamma^{-1}}}{1+\gamma}s_0(\ell_2) + \dfrac{\gamma-\sqrt{q\gamma^{-1}}}{1+\gamma}s_1(\ell_2),$

$s_1(\ell_2)^2 = \dfrac{1+\sqrt{q\gamma^{-1}}}{1+\gamma}s_0(\ell_1) + \dfrac{\gamma-\sqrt{q\gamma^{-1}}}{1+\gamma}s_1(\ell_1),$

(6) $s_0(\ell_1)s_1(\ell_1) = \dfrac{1-\sqrt{q\gamma}}{1+\gamma}s_0(\ell_2) + \dfrac{\gamma+\sqrt{q\gamma}}{1+\gamma}s_1(\ell_2),$

$s_0(\ell_2)s_1(\ell_2) = \dfrac{1-\sqrt{q\gamma}}{1+\gamma}s_0(\ell_1) + \dfrac{\gamma+\sqrt{q\gamma}}{1+\gamma}s_1(\ell_1).$

(ii) Case of $|\mathcal{K}| = 4$.
$\mathcal{K}_2 = \{h_0, h_1, s(\ell_1), s(\ell_2)\}$ is the join $\mathcal{H} \vee \mathcal{L}$ of $\mathcal{H}$ by $\mathcal{L}$.

Remark 2. The set $\mathcal{K}_a(\gamma)$ is a commutative hypergroup such that $s_0(\ell_i)^* = s_0(\ell_i^*)$ for $i = 1, 2$ and the extension of $\mathcal{L}$ by $\mathcal{H}$ by Theorem 3.10. The real numbers γ_1 and γ_2 in Theorem 3.10 satisfy $\gamma_1^2\gamma_2 \geq q$ and $\gamma_1^{-2}\gamma_2 \geq q$. Since $s_0(\ell_1)^* = s_0(\ell_2)$, we obtain $\gamma_2 = \gamma_{1^*} = \gamma_1$ by Lemma 3.4. We write $\gamma = \gamma_1$ simply. Hence $q^{1/3} \leq \gamma \leq 1/q$. This $\mathcal{K}_a(\gamma)$ is also the extension such that $a_{11}^1 = f_+(\gamma, \gamma, \gamma)$ in (i)–(1) of Theorem 3.7. There are other extensions by Theorem 3.7 and Proposition 3.9. However it is easy to see that other extensions are equivalent to $\mathcal{K}_a(\gamma)$ as extensions by transposing $s_0(\ell_1)$ to $s_1(\ell_1)$ or $s_0(\ell_2)$ to $s_1(\ell_2)$. Therefore all extensions $\mathcal{K}$ of $\mathcal{L}$ by $\mathcal{H}$ are equivalent to one of $\mathcal{K}_a(\gamma)$ and $\mathcal{K}_2$ as extensions.

Example 4.3. Let $\mathcal{L} = \left\{\ell_0, \ell_1, \ell_2, \ell_3;\ \ell_1^k = \ell_k\ (k = 2, 3), \ell_1^* = \ell_3, \ell_2^* = \ell_2\right\} \cong \mathbb{Z}_4$ and $\mathcal{H} = \mathbb{Z}_q(2) = \{h_0, h_1;\ h_1^2 = qh_0 + (1-q)h_1, 0 < q \leq 1\}$. Since the subgroup L_0 of $\mathcal{L}$ is $\mathcal{L}$, $\{\ell_0, \ell_2\}$ or $\{\ell_0\}$, one has the extensions such that $|\mathcal{K}| = 8$, $|\mathcal{K}| = 6$ and $|\mathcal{K}| = 5$.

(i) Case of $|\mathcal{K}| = 8$.

(1) Case of $s_0(\ell)^* = s_0(\ell^*)$ for all $\ell \in \mathcal{L}$.
Let γ_1 and γ_2 be real numbers such that $q \leq \gamma_i \leq 1/q$ for $i = 1, 2$, $q \leq \gamma_1^2\gamma_2$ and $q \leq \gamma_1^{-2}\gamma_2$. We denote $\mathcal{K}_{1\text{-}a}(\gamma_1, \gamma_2) = \{h_0,\ h_1,\ s_0(\ell_1),\ s_1(\ell_1),\ s_0(\ell_2),\ s_1(\ell_2),\ s_0(\ell_3),\ s_1(\ell_3)\}$. The structure equations of $\mathcal{K}_{1\text{-}a}(\gamma_1, \gamma_2)$ is given by the following:

(a) $h_1s_0(\ell_1) = \dfrac{1-q\gamma_1}{1+\gamma_1}s_0(\ell_1) + \dfrac{(1+q)\gamma_1}{1+\gamma_1}s_1(\ell_1),$

$h_1s_0(\ell_2) = \dfrac{1-q\gamma_2}{1+\gamma_2}s_0(\ell_2) + \dfrac{(1+q)\gamma_2}{1+\gamma_2}s_1(\ell_2),$

$h_1s_0(\ell_3) = \dfrac{1-q\gamma_1}{1+\gamma_1}s_0(\ell_3) + \dfrac{(1+q)\gamma_1}{1+\gamma_1}s_1(\ell_3),$

(b) $h_1s_1(\ell_1) = \dfrac{(1+q)\gamma_1^{-1}}{1+\gamma_1^{-1}}s_0(\ell_1) + \dfrac{1-q\gamma_1^{-1}}{1+\gamma_1^{-1}}s_1(\ell_1),$

$h_1s_1(\ell_2) = \dfrac{(1+q)\gamma_2^{-1}}{1+\gamma_2^{-1}}s_0(\ell_2) + \dfrac{1-q\gamma_2^{-1}}{1+\gamma_2^{-1}}s_1(\ell_2),$

$h_1s_1(\ell_3) = \dfrac{(1+q)\gamma_1^{-1}}{1+\gamma_1^{-1}}s_0(\ell_3) + \dfrac{1-q\gamma_1^{-1}}{1+\gamma_1^{-1}}s_1(\ell_3),$

(c) $s_0(\ell_1)s_0(\ell_3) = \dfrac{q(1+\gamma_1)}{1+q}h_0 + \dfrac{1-q\gamma_1}{1+q}h_1,$

$s_1(\ell_1)s_1(\ell_3) = \dfrac{q(1+\gamma_1^{-1})}{1+q}h_0 + \dfrac{1-q\gamma_1^{-1}}{1+q}h_1,$

$s_0(\ell_1)s_1(\ell_3) = s_0(\ell_3)s_1(\ell_1) = h_1,$

(d) $s_0(\ell_2)^2 = \dfrac{q(1+\gamma_2)}{1+q}h_0 + \dfrac{1-q\gamma_2}{1+q}h_1,$

$s_1(\ell_2)^2 = \dfrac{q(1+\gamma_2^{-1})}{1+q}h_0 + \dfrac{1-q\gamma_2^{-1}}{1+q}h_1, \quad s_0(\ell_2)s_1(\ell_2) = h_1,$

(e) $s_0(\ell_1)^2 = \dfrac{1+\gamma_1\sqrt{q\gamma_2}}{1+\gamma_2}s_0(\ell_2) + \dfrac{\gamma_2-\gamma_1\sqrt{q\gamma_2}}{1+\gamma_2}s_1(\ell_2),$

$s_1(\ell_1)^2 = \dfrac{1+\gamma_1^{-1}\sqrt{q\gamma_2}}{1+\gamma_2}s_0(\ell_2) + \dfrac{\gamma_2-\gamma_1^{-1}\sqrt{q\gamma_2}}{1+\gamma_2}s_1(\ell_2),$

$s_0(\ell_1)s_1(\ell_1) = \dfrac{1-\sqrt{q\gamma_2}}{1+\gamma_2}s_0(\ell_2) + \dfrac{\gamma_2+\sqrt{q\gamma_2}}{1+\gamma_2}s_1(\ell_2),$

(f) $s_0(\ell_1)s_0(\ell_2) = \dfrac{1+\gamma_1\sqrt{q\gamma_2}}{1+\gamma_1}s_0(\ell_3) + \dfrac{\gamma_1-\gamma_1\sqrt{q\gamma_2}}{1+\gamma_1}s_1(\ell_3),$

$s_1(\ell_1)s_1(\ell_2) = \dfrac{1+\sqrt{q\gamma_2^{-1}}}{1+\gamma_1}s_0(\ell_3) + \dfrac{\gamma_1-\sqrt{q\gamma_2^{-1}}}{1+\gamma_1}s_1(\ell_3),$

$s_0(\ell_1)s_1(\ell_2) = \dfrac{1-\gamma_1\sqrt{q\gamma_2^{-1}}}{1+\gamma_1}s_0(\ell_3) + \dfrac{\gamma_1+\gamma_1\sqrt{q\gamma_2^{-1}}}{1+\gamma_1}s_1(\ell_3).$

(2) Case of $s_0(\ell)^* = s_1(\ell^*)$ for all $\ell \in \mathcal{L}$.
Let γ be a real number such that $q^{1/2} \le \gamma \le q^{-1/2}$. We put $\mathcal{K}_{1\text{-}b}(\gamma) = \{h_0, h_1, s_0(\ell_1), s_1(\ell_1), s_0(\ell_2), s_1(\ell_2), s_0(\ell_3), s_1(\ell_3)\}$. The structure equation of $\mathcal{K}_{1\text{-}b}(\gamma)$ is given by the following:

(a) $h_1 s_0(\ell_1) = \dfrac{1-q\gamma}{1+\gamma}s_0(\ell_1) + \dfrac{(1+q)\gamma}{1+\gamma}s_1(\ell_1),$

$h_1 s_0(\ell_2) = \dfrac{1-q}{2}s_0(\ell_2) + \dfrac{1+q}{2}s_1(\ell_2),$

$h_1 s_0(\ell_3) = \dfrac{1-q\gamma^{-1}}{1+\gamma^{-1}}s_0(\ell_3) + \dfrac{(1+q)\gamma^{-1}}{1+\gamma^{-1}}s_1(\ell_3),$

(b) $h_1 s_1(\ell_1) = \dfrac{(1+q)\gamma^{-1}}{1+\gamma^{-1}}s_0(\ell_1) + \dfrac{1-q\gamma^{-1}}{1+\gamma^{-1}}s_1(\ell_1),$

$h_1 s_1(\ell_2) = \dfrac{1+q}{2}s_0(\ell_2) + \dfrac{1-q}{2}s_1(\ell_2),$

$h_1 s_1(\ell_3) = \dfrac{(1+q)\gamma}{1+\gamma}s_0(\ell_3) + \dfrac{1-q\gamma}{1+\gamma}s_1(\ell_3),$

(c) $s_0(\ell_1)s_1(\ell_3) = \dfrac{q(1+\gamma)}{1+q}h_0 + \dfrac{1-q\gamma}{1+q}h_1,$

$$s_1(\ell_1)s_0(\ell_3) = \frac{q(1+\gamma^{-1})}{1+q}h_0 + \frac{1-q\gamma^{-1}}{1+q}h_1, \quad s_0(\ell_1)s_0(\ell_3) = h_1,$$

(d) $s_0(\ell_2)s_1(\ell_2) = \frac{2q}{1+q}h_0 + \frac{1-q}{1+q}h_1, \quad s_0(\ell_2)^2 = s_1(\ell_2)^2 = h_1,$

(e) $s_0(\ell_1)^2 = \frac{1+\gamma\sqrt{q}}{2}s_0(\ell_2) + \frac{1-\gamma\sqrt{q}}{2}s_1(\ell_2),$

$$s_1(\ell_1)^2 = \frac{1+\gamma^{-1}\sqrt{q}}{2}s_0(\ell_2) + \frac{1-\gamma^{-1}\sqrt{q}}{2}s_1(\ell_2),$$

$$s_0(\ell_1)s_1(\ell_1) = \frac{1-\sqrt{q}}{2}s_0(\ell_2) + \frac{1+\sqrt{q}}{2}s_1(\ell_2),$$

(f) $s_0(\ell_1)s_0(\ell_2) = \frac{1+\gamma\sqrt{q}}{1+\gamma}s_0(\ell_3) + \frac{\gamma-\gamma\sqrt{q}}{1+\gamma}s_1(\ell_3),$

$$s_1(\ell_1)s_1(\ell_2) = \frac{1+\sqrt{q}}{1+\gamma}s_0(\ell_3) + \frac{\gamma-\sqrt{q}}{1+\gamma}s_1(\ell_3),$$

$$s_0(\ell_1)s_1(\ell_2) = \frac{1-\gamma\sqrt{q}}{1+\gamma}s_0(\ell_3) + \frac{\gamma+\gamma\sqrt{q}}{1+\gamma}s_1(\ell_3).$$

(ii) Case of $|\mathcal{K}| = 6$.

(1) Case of $s_0(\ell_2)^* = s_0(\ell_2)$.
Let γ be a real number such that $q \leq \gamma \leq 1/q$. We denote $\mathcal{K}_{2\text{-}a}(\gamma) = \{h_0, h_1, s(\ell_1), s_0(\ell_2), s_1(\ell_2), s(\ell_3)\}$. The structure equations of $\mathcal{K}_{2\text{-}a}(\gamma)$ is given by the following:

(a) $h_1 s(\ell_1) = s(\ell_1),\ h_1 s(\ell_3) = s(\ell_3),$

$$h_1 s_0(\ell_2) = \frac{1-q\gamma}{1+\gamma}s_0(\ell_2) + \frac{(1+q)\gamma}{1+\gamma}s_1(\ell_2),$$

$$h_1 s_1(\ell_2) = \frac{(1+q)\gamma^{-1}}{1+\gamma^{-1}}s_0(\ell_2) + \frac{1-q\gamma^{-1}}{1+\gamma^{-1}}s_1(\ell_2),$$

(b) $s(\ell_1)s(\ell_3) = \frac{q}{1+q}h_0 + \frac{1}{1+q}h_1,$

(c) $s_0(\ell_2)s_0(\ell_2) = \frac{q(1+\gamma)}{1+q}h_0 + \frac{1-q\gamma}{1+q}h_1,$

$$s_1(\ell_2)s_1(\ell_2) = \frac{q(1+\gamma^{-1})}{1+q}h_0 + \frac{1-q\gamma^{-1}}{1+q}h_1, \quad s_0(\ell_2)s_1(\ell_2) = h_1,$$

(d) $s(\ell_1)^2 = s(\ell_3)^2 = \frac{1}{1+\gamma}s_0(\ell_2) + \frac{\gamma}{1+\gamma}s_1(\ell_2),$

(e) $s(\ell_1)s_0(\ell_2) = s(\ell_1)s_1(\ell_2) = s(\ell_3),\ s_0(\ell_2)s(\ell_3) = s(\ell_1),$

$$s_1(\ell_2)s(\ell_3) = s(\ell_1).$$

(2) Case of $s_0(\ell_2)^* = s_1(\ell_2)$.
We denote $\mathcal{K}_{2\text{-}b} = \{h_0,\ h_1,\ s(\ell_1),\ s_0(\ell_2),\ s_1(\ell_2),\ s(\ell_3)\}$. The structure equations of $\mathcal{K}_{2\text{-}b}$ is given by the following:

(a) $h_1 s(\ell_1) = s(\ell_1),\ h_1 s(\ell_3) = s(\ell_3),$
$$h_1 s_0(\ell_2) = \frac{1-q}{2} s_0(\ell_2) + \frac{1+q}{2} s_1(\ell_2),$$
$$h_1 s_1(\ell_2) = \frac{1+q}{2} s_0(\ell_2) + \frac{1-q}{2} s_1(\ell_2),$$

(b) $s(\ell_1)s(\ell_3) = \dfrac{q}{1+q} h_0 + \dfrac{1}{1+q} h_1,$

(c) $s_0(\ell_2)s_1(\ell_2) = \dfrac{2q}{1+q} h_0 + \dfrac{1-q}{1+q} h_1, \quad s_0(\ell_2)s_0(\ell_2) = s_1(\ell_2)s_1(\ell_2) = h_1,$

(d) $s(\ell_1)^2 = s(\ell_3)^2 = \dfrac{1}{2} s_0(\ell_2) + \dfrac{1}{2} s_1(\ell_2),$

(e) $s(\ell_1)s_0(\ell_2) = s(\ell_1)s_1(\ell_2) = s(\ell_3),\ s_0(\ell_2)s(\ell_3) = s_1(\ell_2)s(\ell_3) = s(\ell_1).$

(iii) Case of $|\mathcal{K}| = 5$.
$\mathcal{K}_3 = \{h_0, h_1, s(\ell_1), s(\ell_2), s(\ell_3)\}$ is the join $\mathcal{H} \vee \mathcal{L}$ of $\mathcal{H}$ by $\mathcal{L}$.

Remark 3. The set $\mathcal{K}_{1\text{-}a}(\gamma_1, \gamma_2)$ is a commutative hypergroup such that $s_0(\ell_i)^* = s_0(\ell_i^*)$ for $i = 1, 2, 3$ and the extension of $\mathcal{L}$ by $\mathcal{H}$ by Theorem 3.10. This $\mathcal{K}_{1\text{-}a}(\gamma_1, \gamma_2)$ is also the extension such that $a_{11}^2 = f_+(\gamma_1, \gamma_1, \gamma_2)$ in (i)–(1) of Theorem 3.7. One has other extensions such that $|\mathcal{K}| = 8$ and $s_0(\ell_2)^* = s_0(\ell_2)$ by Theorem 3.7 and Proposition 3.9. It is easy to see that other extension such that $|\mathcal{K}| = 8$ and $s_0(\ell_2)^* = s_0(\ell_2)$ are equivalent to $\mathcal{K}_{1\text{-}a}(\gamma_1, \gamma_2)$ as extensions by transposing $s_0(\ell_1)$ to $s_1(\ell_1)$, $s_0(\ell_2)$ to $s_1(\ell_2)$ or $s_0(\ell_3)$ to $s_1(\ell_3)$.

The set $\mathcal{K}_{1\text{-}b}(\gamma)$ is a commutative hypergroup such that $s_0(\ell_i)^* = s_1(\ell_i^*)$ for $i = 1, 2, 3$ and the extension of $\mathcal{L}$ by $\mathcal{H}$ Theorem 3.12. This $\mathcal{K}_{1\text{-}b}(\gamma)$ is characterized by $a_{11}^2 = f_+(\gamma, \gamma, 1)$ and $s_0(\ell_i)^* = s_1(\ell_i^*)$ in (ii)–(1) of Theorem 3.7 where $\gamma_2 = 1$, $\gamma_1 = \gamma$ and $q^{1/2} \le \gamma \le q^{-1/2}$. In a similar discussion to the above, all extensions such that $|\mathcal{K}| = 8$ and $s_0(\ell_2)^* = s_1(\ell_2)$ are equivalent to $\mathcal{K}_{1\text{-}b}(\gamma)$ as extensions.

The set $\mathcal{K}_{2\text{-}a}(\gamma)$ is a commutative hypergroup such that $|\mathcal{K}| = 6$ and $s_0(\ell_2)^* = s_0(\ell_2)$ and the extension of $\mathcal{L}$ by $\mathcal{H}$ by Corollary 3.11. In a similar discussion to Example 4.2, $\mathcal{K}_{2\text{-}a}(\gamma)$ is all extensions such that $|\mathcal{K}| = 6$ and $s_0(\ell_2)^* = s_0(\ell_2)$.

The set $\mathcal{K}_{2\text{-}b}$ is a commutative hypergroup such that $|\mathcal{K}| = 6$ and $s_0(\ell_2)^* = s_1(\ell_2)$ and the extension of $\mathcal{L}$ by $\mathcal{H}$ by Corollary 3.13.

Therefore all extensions $\mathcal{K}$ of $\mathcal{L}$ by $\mathcal{H}$ are equivalent to one of $\mathcal{K}_{1\text{-}a}(\gamma_1, \gamma_2)$, $\mathcal{K}_{1\text{-}b}(\gamma)$, $\mathcal{K}_{2\text{-}a}(\gamma)$, $\mathcal{K}_{2\text{-}b}$ and $\mathcal{K}_3$.

Example 4.4. Let $\mathcal{L} = \{\ell_0, \ell_1, \ell_2, \ell_3, \ell_4;\ \ell_1^k = \ell_k, k = 2, 3, 4,\ \ell_2^* = \ell_3,\ \ell_1^* = \ell_4\} \cong \mathbb{Z}_5$ and $\mathcal{H} = \mathbb{Z}_q(2) = \{h_0, h_1;\ h_1^2 = qh_0 + (1-q)h_1,\ 0 < q \le 1\}$. Since the subgroup L_0 of $\mathcal{L}$ is $\mathcal{L}$ or $\{\ell_0\}$, one has the extensions such that $|\mathcal{K}| = 10$ and $|\mathcal{K}| = 6$ respectively.

(i) Case of $|\mathcal{K}| = 10$.
Let γ_1 and γ_2 be real numbers such that $q \leq \gamma_i \leq 1/q$ for $i = 1, 2$, $q \leq \gamma_1^2\gamma_2$, $q \leq \gamma_1^{-2}\gamma_2$, $q \leq \gamma_1\gamma_2^2$ and $q \leq \gamma_1\gamma_2^{-2}$. We put $\mathcal{K}_1(\gamma_1,\gamma_2) = \{h_0,\ h_1,\ s_0(\ell_1),\ s_1(\ell_1),\ s_0(\ell_2),\ s_1(\ell_2),\ s_0(\ell_3),\ s_1(\ell_3),\ s_0(\ell_4),\ s_1(\ell_4)\}$. The structure equations of $\mathcal{K}_1(\gamma_1,\gamma_2)$ is given by

$$
\begin{aligned}
(1)\ \ h_1 s_0(\ell_1) &= \frac{1-q\gamma_1}{1+\gamma_1} s_0(\ell_1) + \frac{(1+q)\gamma_1}{1+\gamma_1} s_1(\ell_1),\\
h_1 s_0(\ell_2) &= \frac{1-q\gamma_2}{1+\gamma_2} s_0(\ell_2) + \frac{(1+q)\gamma_2}{1+\gamma_2} s_1(\ell_2),\\
h_1 s_0(\ell_3) &= \frac{1-q\gamma_2}{1+\gamma_2} s_0(\ell_3) + \frac{(1+q)\gamma_2}{1+\gamma_2} s_1(\ell_3),\\
h_1 s_0(\ell_4) &= \frac{1-q\gamma_1}{1+\gamma_1} s_0(\ell_4) + \frac{(1+q)\gamma_1}{1+\gamma_1} s_1(\ell_4),
\end{aligned}
$$

$$
\begin{aligned}
(2)\ \ h_1 s_1(\ell_1) &= \frac{(1+q)\gamma_1^{-1}}{1+\gamma_1^{-1}} s_0(\ell_1) + \frac{1-q\gamma_1^{-1}}{1+\gamma_1^{-1}} s_1(\ell_1),\\
h_1 s_1(\ell_2) &= \frac{(1+q)\gamma_2^{-1}}{1+\gamma_2^{-1}} s_0(\ell_2) + \frac{1-q\gamma_2^{-1}}{1+\gamma_2^{-1}} s_1(\ell_2),\\
h_1 s_1(\ell_3) &= \frac{(1+q)\gamma_2^{-1}}{1+\gamma_2^{-1}} s_0(\ell_3) + \frac{1-q\gamma_2^{-1}}{1+\gamma_2^{-1}} s_1(\ell_3),\\
h_1 s_1(\ell_4) &= \frac{(1+q)\gamma_1^{-1}}{1+\gamma_1^{-1}} s_0(\ell_4) + \frac{1-q\gamma_1^{-1}}{1+\gamma_1^{-1}} s_1(\ell_4),
\end{aligned}
$$

$$
\begin{aligned}
(3)\ \ s_0(\ell_1)s_0(\ell_4) &= \frac{q(1+\gamma_1)}{1+q} h_0 + \frac{1-q\gamma_1}{1+q} h_1,\\
s_1(\ell_1)s_1(\ell_4) &= \frac{q(1+\gamma_1^{-1})}{1+q} h_0 + \frac{1-q\gamma_1^{-1}}{1+q} h_1,\\
s_0(\ell_1)s_1(\ell_4) &= s_0(\ell_4)s_1(\ell_1) = h_1,
\end{aligned}
$$

$$
\begin{aligned}
(4)\ \ s_0(\ell_2)s_0(\ell_3) &= \frac{q(1+\gamma_2)}{1+q} h_0 + \frac{1-q\gamma_2}{1+q} h_1,\\
s_1(\ell_2)s_1(\ell_3) &= \frac{q(1+\gamma_2^{-1})}{1+q} h_0 + \frac{1-q\gamma_2^{-1}}{1+q} h_1,\\
s_0(\ell_2)s_1(\ell_3) &= s_0(\ell_3)s_1(\ell_2) = h_1,
\end{aligned}
$$

$$
\begin{aligned}
(5)\ \ s_0(\ell_1)^2 &= \frac{1+\gamma_1\sqrt{q\gamma_2}}{1+\gamma_2} s_0(\ell_2) + \frac{\gamma_2-\gamma_1\sqrt{q\gamma_2}}{1+\gamma_2} s_1(\ell_2),\\
s_1(\ell_1)^2 &= \frac{1+\gamma_1^{-1}\sqrt{q\gamma_2}}{1+\gamma_2} s_0(\ell_2) + \frac{\gamma_2-\gamma_1^{-1}\sqrt{q\gamma_2}}{1+\gamma_2} s_1(\ell_2),\\
s_0(\ell_1)s_1(\ell_1) &= \frac{1-\sqrt{q\gamma_2}}{1+\gamma_2} s_0(\ell_2) + \frac{\gamma_2+\sqrt{q\gamma_2}}{1+\gamma_2} s_1(\ell_2),
\end{aligned}
$$

$$(6)\ \ s_0(\ell_1)s_0(\ell_2) = \frac{1+\gamma_2\sqrt{q\gamma_1}}{1+\gamma_2}s_0(\ell_3) + \frac{\gamma_2-\gamma_2\sqrt{q\gamma_1}}{1+\gamma_2}s_1(\ell_3),$$
$$s_1(\ell_1)s_1(\ell_2) = \frac{1+\sqrt{q\gamma_1^{-1}}}{1+\gamma_2}s_0(\ell_3) + \frac{\gamma_2-\sqrt{q\gamma_1^{-1}}}{1+\gamma_2}s_1(\ell_3),$$
$$s_0(\ell_1)s_1(\ell_2) = \frac{1-\sqrt{q\gamma_1}}{1+\gamma_2}s_0(\ell_3) + \frac{\gamma_2+\sqrt{q\gamma_1}}{1+\gamma_2}s_1(\ell_3).$$

(ii) Case of $|\mathcal{K}| = 6$.
$\mathcal{K}_2 = \{h_0, h_1, s(\ell_1), s(\ell_2), s(\ell_3), s(\ell_4)\}$ is the join $\mathcal{H} \vee \mathcal{L}$ of $\mathcal{H}$ by $\mathcal{L}$.

Remark 4. The set $\mathcal{K}_1(\gamma_1, \gamma_2)$ is a commutative hypergroup such that $s_0(\ell_i)^* = s_0(\ell_i^*)$ for $1 \le i \le 4$ and the extension of $\mathcal{L}$ by $\mathcal{H}$ by Theorem 3.10. This $\mathcal{K}_1(\gamma_1, \gamma_2)$ is also the extension such that $a_{11}^3 = f_+(\gamma_1, \gamma_1, \gamma_2)$ and $a_{12}^2 = f_+(\gamma_1, \gamma_2, \gamma_2)$ in (i)–(1) of Theorem 3.7.

In a similar discussion to Examples 4.2 and 4.3, all extensions $\mathcal{K}$ of $\mathcal{L}$ by $\mathcal{H}$ are equivalent to one of $\mathcal{K}_1(\gamma_1, \gamma_2)$ and $\mathcal{K}_2$.

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