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# Another Proof of decomposability of Nambu-Poisson tensors

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**Abstract.** Although Nambu-Poisson bracket is a natural generalization of Poisson bracket, a very distinguished property of Nambu-Poisson bracket comparing Poisson bracket is decomposability of its tensor. This is first conjectured in [1] and is given affirmative answers by [2] and [4] independently. In this paper, we shall show another proof to decomposability of Nambu-Poisson tensor, which is more elementary and more direct to the property of decomposability comparing that of [2] or [4].

## 1 Introduction

In contrast to Poisson bracket being a binary operation, Nambu-Poisson is a multi-fold operation provided with the same properties of Poisson bracket and the fundamental identity which is a natural generalization of Jacobi identity. We recall the precise definition of Nambu-Poisson bracket. Let $M$ be a $n$-dimensional $C^\infty$-manifold. An order $p$ Nambu-Poisson bracket on $M$ is a $p$-fold skew-symmetric $\mathbf{R}$-multilinear operation

$$\{\dots\} : C^\infty(M)^p := \underbrace{C^\infty(M) \times \cdots \times C^\infty(M)}_{p\text{-times}} \longrightarrow C^\infty(M)$$

provided with Leibniz rule for each argument, and the fundamental identity (or generalized Jacobi identity):

$$\{\mathcal{F}, \{\mathcal{G}\}\} = \sum_{\ell=1}^{p} \{g_1, \dots, \{\mathcal{F}, g_\ell\}, \dots, g_p\}$$

where $\mathcal{F} = (f_1, \dots, f_{p-1}) \in C^\infty(M)^{p-1}$, $\mathcal{G} = (g_1, \dots, g_p) \in C^\infty(M)^p$.

If order $p = 2$, then the fundamental identity is just Jacobi identity and order 2 Nambu-Poisson brackets are Poisson brackets. Like as Poisson brackets, every order $p$ Nambu-Poisson bracket

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is defined by the unique $p$-multivector field $\pi$ as

$$\{f_1, \cdots, f_p\} = \langle \pi, df_1 \wedge \cdots \wedge df_p \rangle \ .$$

For a $p$-fold skew-symmetric bracket $\{\cdots\}$ defined from a $p$-multivector field $\pi$, let

$$Jac(\mathcal{F}; \mathcal{G}) := \{\mathcal{F}, \{\mathcal{G}\}\} - \sum_{\ell=1}^{p} \{g_1, \cdots, \{\mathcal{F}, g_\ell\}, \cdots, g_p\} = \{\mathcal{F}, \{\mathcal{G}\}\} + \sum_{\ell=1}^{p} (-1)^\ell \{\{\mathcal{F}, g_\ell\}, \mathcal{G}[\ell]\}$$

where $\mathcal{F} \in C^\infty(M)^{p-1}$, $\mathcal{G} = (g_1, \cdots, g_p) \in C^\infty(M)^p$, $\mathcal{G}[\ell] := (g_1, \cdots, \widehat{g_\ell}, \cdots, g_p)$ for $\ell = 1, \cdots, p$, and the symbol $\widehat{\ }$ means that the term is omitted. Then, $Jac = 0$ if and only if the $p$-fold skew-symmetric bracket $\{\cdots\}$ defined from a $p$-multivector field $\pi$ satisfies the fundamental identity. Analogy of Hamiltonian vector fields, we can consider the vector field $H_{\mathcal{F}} := \{\mathcal{F}, \cdot\}$ for each $\mathcal{F} \in C^\infty(M)^{p-1}$ for a given Nambu-Poisson bracket. It is known that the distribution spanned by $H_{\mathcal{F}}$'s is involutive from the fundamental identity of Nambu-Poisson bracket.

For a given $p$-fold skew-symmetric bracket defined by a $p$-multivector field (does not necessarily satisfy the fundamental identity), we plug the product of $f_{p-1}$ and $f_p$ into the $(p-1)$-th entry of $Jac$. Then we have

$$\begin{aligned} Jac(\mathcal{E}, f_{p-1}f_p; \mathcal{G}) &= Jac(\mathcal{E}, f_{p-1}; \mathcal{G})f_p + f_{p-1}Jac(\mathcal{E}, f_{p-1}; \mathcal{G}) \\ &\quad + \sum_{\ell=1}^{p} (-1)^\ell \{\mathcal{E}, f_{p-1}, g_\ell\}\{f_p, \mathcal{G}[\ell]\} + \sum_{\ell=1}^{p} (-1)^\ell \{\mathcal{E}, f_p, g_\ell\}\{f_{p-1}, \mathcal{G}[\ell]\} \end{aligned}$$

for $\mathcal{E} \in C^\infty(M)^{p-2}$ and $\mathcal{G} \in C^\infty(M)^p$ (cf. [1]).

If the $p$-fold skew-symmetric bracket $\{\dots\}$ satisfies the fundamental identity, then

$$\sum_{\ell=1}^{p} (-1)^\ell \left( \{\mathcal{E}, f_{p-1}, g_\ell\}\{f_p, \mathcal{G}[\ell]\} + \{\mathcal{E}, f_p, g_\ell\}\{f_{p-1}, \mathcal{G}[\ell]\} \right)$$

must vanish. We put the above by $\Phi(\mathcal{E}; f_{p-1}, f_p; \mathcal{G})$. Gautheron ([2]) adds the trivial term

$$\{\mathcal{E}, f_{p-1}, f_p\}\{\mathcal{G}\} + \{\mathcal{E}, f_p, f_{p-1}\}\{\mathcal{G}\}$$

to $\Phi$ above and gets

$$\Phi(\mathcal{E}; f_{p-1}, f_p; \mathcal{G}) = \mathcal{B}(\mathcal{E}, f_{p-1}, f_p, \mathcal{G}) + \mathcal{B}(\mathcal{E}, f_p, f_{p-1}, \mathcal{G}) \ ,$$

where $\mathcal{B}$ is equal to the symbol $B$ in ([2]). By using the relation

$$\begin{aligned} \mathcal{B}(\mathcal{F}, \mathcal{G}) &= (\pi(d\mathcal{F}[p], \cdot) \wedge \pi)(df_p, d\mathcal{G}) \\ &= \pi(d\mathcal{F})\pi(\mathcal{G}) + \sum_{\ell=1}^{p} \pi(d\mathcal{F}[p], g_\ell)\pi(f_p, \mathcal{G}[\ell]) \end{aligned}$$

where $\mathcal{F},\mathcal{G} \in C^\infty(M)^p$ and $d\mathcal{F} = (df_1,\dots,df_p)$ for $\mathcal{F} = (f_1,\dots,f_p)$, he shows if $\Phi = 0$ then $\mathcal{B}$ is full skew-symmetric and gets $\mathcal{B} = 0$ by restricting $\mathcal{B}$ on each $2p$-dimensional subspace of $T^*(M)$. Then it turns out $\pi$ is decomposable. "Decomposability of $\pi$" at a point, say $x$ means $\pi_x = v_1 \wedge \cdots \wedge v_p$ for some $v_j \in \mathrm{T}_x M$ $(j = 1,\dots,p)$. By using this fact, the following result is obtained.

**Theorem 1 ([2] or [4])** *If $\pi$ is an order $p(> 2)$ Nambu-Poisson structure, and $\pi$ is not zero at a point $x$, then $\pi_x$ is decomposable and the characteristic distribution of $\pi$ at $x$ has dimension $p$ whose basis gives decomposition of $\pi$.*

We state a little remark about the discussion above.

**Remark 1.1** If $p = 2$, then $\Phi = 0$ automatically.
For any $p$, we do not get any new relation even if we deal with $Jac(\mathcal{F}; g_1,\dots,g_{p-1}, g_p g_{p+1})$.

## 2 Another proof of decomposability

Since the discussion hereafter is local, we can take a local coordinate system $(x^1,\dots,x^n)$ around each point of $M$.
We abbreviate $\Phi(x^{i_1},\dots,x^{i_{p-2}}; x^{i_{p-1}}, x^{i_p}; x^{j_1},\dots,x^{j_p})$ by $\Phi(i_1,\dots,i_{p-2}; i_{p-1}, i_p; j_1,\dots,j_p)$.
It was observed in [1] that $\Phi$ is related to decomposability conditions of $\pi$.
We also use multi-index notation $\pi^I$ for each multi-index $I = (i_1,\cdots,i_p) \in \{1,\dots,n\}^p$, namely, $\pi^I = \pi^{i_1\cdots i_p} =< \pi, dx^{i_1} \wedge \cdots \wedge dx^{i_p} >= \{x^{i_1},\dots,x^{i_p}\}$.

### 2.1 Decomposability of multi-vectors

We recall here the decomposability condition of multi-vector at the tangent space $T_x(M)$ for a fixed $x \in M$. For a $p$-vector $(p > 2)$ $\pi$, we define the following tensor $\Psi$ by

$$\Psi(i_1,\dots,i_{p-1}; j_0, j_1,\dots,j_p) := \sum_{\ell=0}^{p} (-1)^\ell \pi^{I j_\ell} \pi^{J[\ell]}$$

where $I = (i_1,\dots,i_{p-1})$, $J = (j_0, j_1,\dots,j_p)$, and $J[\ell] = (j_0, j_1,\dots,\widehat{j_\ell},\dots,j_p)$. Of course, $\Psi(I, J)$ is skew-symmetric in $I$ or $J$.
The following result is well-known.

**Proposition 2 (cf. [3])** Let $\pi^J \neq 0$ for some $J$. Then $\pi$ is decomposable if and only if $\Psi(I; j_0, J) = 0$ for each $(p-1)$-tuple $I$ and $j_0$.

The following observation is obtained in [1].

**Theorem 3** ([1]) *The relation* $\Phi(I;k,\ell;J) = \Psi(I,\ell;k,J) + \Psi(I,k;\ell,J)$ *holds for each* $(p-2)$*-tuple* $I$*,* $p$*-tuple* $J$*,* $k$*, and* $\ell$*.*

Thus, if $\Psi = 0$, namely if $\pi$ is decomposable, then $\Phi = 0$, namely the second order property of the fundamental identity holds for the bracket defined by $\pi$.

## 2.2 Decomposability of Nambu-Poisson tensors

The conjecture stated in [1] is that $\Phi = 0$ implies $\Psi = 0$. We shall show it, namely, the fundamental identity yields the decomposability of Nambu-Poisson tensor.

**Theorem 4** *Let* $\pi$ *be a* $p$*-multi vector satisfying* $\Phi(I;k,\ell;J) = 0$ *for each* $(p-2)$*-tuple* $I$*,* $p$*-tuple* $J$*,*$k$*, and* $\ell$*. Then* $\Psi(I,k,J) = 0$ *for each* $(p-1)$*-tuple* $I$*,* $p$*-tuple* $J$*, and* $k$*.*

Proof: Let us find and fix some multi-index $B$ so that $\pi^B \neq 0$. Put $B = (b_1, \ldots, b_p)$ as an ordered set. We assume that the indices $u_i$, $v_j$ run between 1 to $n$ and $\lambda_j$, $\mu_k$ run outside of $B$. Our final goal is to see that $\Psi(U; u_p, B) = 0$ for each $U = (u_1, \ldots, u_{p-1})$ and $u_p$ under the condition $\Phi = 0$. Hereafter the abbreviation "*something* $\equiv 0 \pmod{\Phi}$" means that *something* $= 0$ holds if $\Phi = 0$. Since

$$\begin{aligned}\Phi(u_1, \ldots, u_{p-2}; u_{p-1}, b_1; B) &= \Psi(u_1, \ldots, u_{p-2}, u_{p-1}; b_1, B) + \Psi(u_1, \ldots, u_{p-2}, b_1; u_{p-1}, B) \\ &= \Psi(u_1, \ldots, u_{p-2}, b_1; u_{p-1}, B)\ ,\end{aligned}$$

we see that

$$\Psi(b_1, u_2, \ldots, u_{p-2}; u_{p-1}, B) \equiv 0 \pmod{\Phi}\ .$$

Thus, the relation we have to see is

$$\Psi(\lambda_1, \ldots, \lambda_{p-1}; \lambda_p, B) \equiv 0 \pmod{\Phi}\ .$$

We observe that

$$\begin{aligned}&\Phi(\lambda_1, \ldots, \lambda_{p-2}; \lambda_{p-1}, b_p; B[p], \lambda_p) \\ =\ &\Psi(\lambda_1, \ldots, \lambda_{p-2}, \lambda_{p-1}; b_p, B[p], \lambda_p) + \Psi(\lambda_1, \ldots, \lambda_{p-2}, b_p; \lambda_{p-1}, B[p], \lambda_p) \\ =\ &-\Psi(\Lambda[p]; \lambda_p B) + (-1)^{p+1}\Psi(\Lambda[p-1,p], b_p; B[p], \lambda_{p-1}, \lambda_p)\end{aligned}$$

where $\Lambda[\ell-1,\ell]$ means the multi-index which first was omitted the $\ell$-th entry, and then omitted the $(\ell-1)$-th entry from $\Lambda$.

After proving

$$\Psi(\Lambda[p-1,p], b_p; B[p], \lambda_{p-1}, \lambda_p) \equiv 0 \pmod{\Phi}$$

in the next Lemma, we get

$$\Psi(\Lambda[p]; \lambda_p B) \equiv 0 \pmod{\Phi} .$$

**Lemma 1**

$$\Psi(\Lambda[p-1,p], b_p; B[p], \lambda_{p-1}, \lambda_p) \equiv 0 \pmod{\Phi}$$

<u>Proof of Lemma:</u>

For each $C = (c_1, \dots, c_k)$ where $k \geq 1$ and $c_j \in B$ and $U = (u_{k+1}, \dots, u_p)$ with $U_0 = (u_{k+1}, \dots, u_{p-1})$, we have already seen that $\Psi(C, U_0; u_p, B) \equiv 0 \pmod{\Phi}$

We write down the definition of $\Psi$ and get a recursive relations as follows:

$$\pi^{CU} \equiv -\frac{1}{\pi^B} \sum_{\ell}^{p} (-1)^{\ell} \pi^{CU_0 b_s} \pi^{u_p B[s]} \pmod{\Phi} .$$

We finally get the required property as follows.

$$\begin{aligned}
& \Psi(\Lambda[p-1,p], b_p; B[p], \lambda_{p-1}, \lambda_p) \\
= \; & \sum_{s=1}^{p-1} (-1)^{s+1} \pi^{\Lambda[p-1,p], b_p, b_s} \pi^{B[s,p], \lambda_{p-1}, \lambda_p} \\
& + (-1)^{p-1} \pi^{\Lambda[p-1,p], b_p, \lambda_{p-1}} \pi^{B[p], \lambda_p} + (-1)^{p} \pi^{\Lambda[p-1,p], b_p, \lambda_p} \pi^{B[p], \lambda_{p-1}} \\
= \; & \sum_{s=1}^{p-1} (-1)^{s+1} \pi^{b_p, b_s, \Lambda[p-1,p]} \pi^{B[s,p], \lambda_{p-1}, \lambda_p} + \pi^{b_p, \Lambda[p]} \pi^{B[p], \lambda_p} - \pi^{b_p, \Lambda[p-1]} \pi^{B[p], \lambda_{p-1}} \\
\equiv \; & \sum_{s=1}^{p-1} (-1)^{s+1} \pi^{b_p, b_s, \Lambda[p-1,p]} \frac{-1}{\pi^B} \sum_{t=1}^{p} (-1)^{t} \pi^{B[s,p], \lambda_{p-1}, b_t} \pi^{\lambda_p, B[t]} \\
& + \frac{-1}{\pi^B} \sum_{s=1}^{p} (-1)^{s} \pi^{b_p, \Lambda[p-1,p], b_s} \pi^{\lambda_{p-1}, B[s]} \pi^{B[p], \lambda_p} - \frac{-1}{\pi^B} \sum_{s=1}^{p} (-1)^{s} \pi^{b_p, \Lambda[p-1,p], b_s} \pi^{\lambda_p, B[s]} \pi^{B[p], \lambda_{p-1}} \\
= \; & \frac{-1}{\pi^B} \sum_{s=1}^{p} (-1)^{s+1} \pi^{b_p, b_s, \Lambda[p-1,p]} \Big( (-1)^{s} \pi^{B[s,p], \lambda_{p-1}, b_s} \pi^{\lambda_p, B[s]} + (-1)^{p} \pi^{B[s,p], \lambda_{p-1}, b_p} \pi^{\lambda_p, B[p]} \\
& - (-1)^{p-1} \pi^{\lambda_{p-1}, B[s]} \pi^{B[p], \lambda_p} + (-1)^{p-1} \pi^{\lambda_p, B[s]} \pi^{B[p], \lambda_{p-1}} \Big) \\
= \; & \frac{-1}{\pi^B} \sum_{s=1}^{p} (-1)^{s+1} \pi^{b_p, b_s, \Lambda[p-1,p]} \Big( -\pi^{\lambda_{p-1}, B[p]} \pi^{\lambda_p, B[s]} + \pi^{\lambda_{p-1}, B[s]} \pi^{\lambda_p, B[p]} \\
& - \pi^{\lambda_{p-1}, B[s]} \pi^{\lambda_p, B[p]} + \pi^{\lambda_p, B[s]} \pi^{\lambda_{p-1}, B[p-1]} \Big) \\
= \; & 0 \qquad \pmod{\Phi} .
\end{aligned}$$

## References

[1] Rupka Chatterjee and Leon Takhtajan. Aspects of classical and quantum Nambu mechanics. *Lett. Math. Phys.*, 37(4):475–482, August 1996.

[2] Philippe Gautheron. Some remarks concerning Nambu mechanics. *Lett. Math. Phys.*, 37(1):103–116, May 1996.

[3] S. Iyanaga and M Fukawa. *Algebra.* Iwanami, Tokyo, 1968. (in Japanese).

[4] Nobutada Nakanishi. On Nambu-Poisson manifolds. *Rev. Math. Phys.*, 1998. (to appear).

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