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SPECTRAL MAPPING THEOREMS AND WEYL SPECTRA FOR HYPONORMAL OPERATORS

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Abstract

In this paper, we will give an elementary proof of Lemma VI.4.2 of [6] and show that the spectral mapping theorem holds for Weyl spectra of this mapping.

1. Introduction.

Let $\mathcal{H}$ be a complex Hilbert space and $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in B(\mathcal{H})$ is said to be hyponormal if $T^*T \geq TT^*$. For an operator T, we denote the spectrum and the approximate point spectrum by $\sigma(T)$ and $\sigma_a(T)$, respectively. A point $z \in \mathbf{C}$ is in the joint approximate point spectrum $\sigma_{ja}(T)$ if there exists a sequence of unit vectors $\{x_n\}$ in $\mathcal{H}$ such that $(T - z)x_n \to 0$ and $(T - z)^* x_n \to 0$. In [6] D. Xia proved the following result:

THEOREM A (Lemma VI.4.2 of [6]). *Let $T = H + iK$ be hyponormal and f, g be bounded real-valued, continuous functions and $f(x) \neq 0$. Take a mapping in the complex plane*

$$\tau(x + iy) = x + i(f(x)^2 y + g(x))$$

and denote $\tau(T) = H + i(f(H)Kf(H) + g(H))$. Then

$$\sigma(\tau(T)) = \tau(\sigma(T)).$$

This proof needs the singular integral model of a hyponormal operator. In this paper we will give an elementary proof of the following theorem without the singular integral model.

THEOREM 1. *Let $T = H + iK$ be hyponormal and f, g be bounded real-valued, continuous functions and $f(x) \neq 0$ at $x \in \sigma(H)$. Take a mapping in the complex plane*

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$$\tau(x+iy) = x + i(f(x)^2y + g(x))$$

and denote $\tau(T) = H + i(f(H)Kf(H) + g(H))$. *Then*

$$\sigma(\tau(T)) = \tau(\sigma(T)).$$

2. Proof.

We need the following theorem:

THEOREM B (Lemma I.3.1 of [6]). *Let $\mathcal{R}$ be a set of the complex plane* $\mathbf{C}$, $T(t)$ *be an operator-valued function of* $t \in [0,1]$ *which is continuous in the norm topology,* τ_t, $t \in [0,1]$, *be a family of bijective mappings from $\mathcal{R}$ onto* $\tau_t(\mathcal{R}) \subset \mathbf{C}$ *and for any fixed* $z \in \mathcal{R}$, $\tau_t(z)$ *be continuous function of* $t \in [0,1]$ *such that* τ_o *is the identity function. Suppose*

$$\sigma_a(T(t)) \cap \tau_t(\mathcal{R}) = \tau_t(\sigma_a(T(0)) \cap \mathcal{R})$$

for all $t \in [0,1]$. *Then, for all* $t \in [0,1]$,

$$\sigma(T(t)) \cap \tau_t(\mathcal{R}) = \tau_t(\sigma(T(0)) \cap \mathcal{R}).$$

PROOF OF THEOREM 1. First we assume that $f(x) > 0$ ($x \in \sigma(H)$). For any $t \in [0,1]$, set

$$T(t) = H + i\{(tf(H) + 1-t)K(tf(H) + 1-t) + tg(H)\}$$

and

$$\tau_t(x + iy) = x + i((tf(x) + 1-t)^2y + tg(x)).$$

Then it holds that $T(0) = T$ and $\tau_o(x+iy) = x+iy$. It is clear that $T(t)$ and τ_t satisfy the condition of Theorem B. Let $A = tf(H) + 1 - t$. Since A commutes with H, we have

$$T(t)^*T(t) - T(t)T(t)^* = 2i(HAKA - AKAH)$$

$$= A(2i(HK - KH))A \geq 0.$$

Hence $T(t)$ is hyponormal. Since, for every $t \in [0,1]$,

$$Re(\sigma(T(t))) = \sigma(H),$$

let $\mathcal{R}$ be a set of $\mathbf{C}$ such that

$$f(Re(z)) > 0 \text{ at } z \in \mathcal{R} \text{ and } \sigma(T(t)) \subset \tau_t(\mathcal{R}) \text{ for every } t \in [0,1].$$

For any $a+ib \in \sigma_a(T(t))$, since $T(t)$ is hyponormal and

$$\sigma_a(T(t)) = \sigma_{ja}(T(t)),$$

there exists a sequence $\{x_n\}$ of unit vectors such that

$$\lim_{n\to\infty}(H - a)x_n = 0 \text{ and } \lim_{n\to\infty}(AKA + tg(H) - b)x_n = 0. \quad (1)$$

Let $c = \dfrac{b-tg(a)}{(tf(a)+1-t)^2}$. Then we have

$$\tau_t(a+ic) = a+i(b-tg(a)+tg(a)) = a+ib.$$

And

$$(K-c)x_n = \{K - \frac{b-tg(a)}{(tf(a)+1-t)^2}\}x_n$$

$$= \frac{1}{(tf(a)+1-t)^2}\{(tf(a)+1-t)^2K - b + tg(a)\}x_n.$$

Since

$$\{(tf(a)+1-t)^2K - b + tg(a)\}x_n = \{AKA + tg(H) - b\}x_n$$

$$- \{AKA + tg(H) - (tf(a)+1-t)^2K - tg(a)\}x_n, \quad (2)$$

from (1) it is clear that the first part of (2) tends to 0. About the second part: Since $T = H+iK$ is hyponormal,

$$i(HK-KH) = i((H-a)K - K(H-a)) \geq 0.$$

Therefore we have $\lim_{n\to\infty}(H-a)Kx_n = 0$. Hence, for every polynomial $p(\cdot)$, we have

$$\lim_{n\to\infty}(p(H)-p(a))Kx_n = 0$$

Therefore, it holds that

$$\lim_{n\to\infty}(A-\alpha)Kx_n = 0,$$

where $\alpha = tf(a)+1-t$. Since

$$AKA - \alpha^2K = AK(A-\alpha) + \alpha(A-\alpha)K,$$

the second part of (2) also tends to 0. Therefore we have $(K-c)x_n \rightarrow 0$ as $n \rightarrow \infty$ and it follows that

$$a + ic \in \sigma_a(T).$$

The converse inclusion relation is easy. Since $\sigma(T(t)) \subset \tau_t(\mathcal{R})$, we have

$$\sigma_a(T(t)) = \tau_t(\sigma_a(T)) \text{ for every } t \in [0,1].$$

By Theorem B, we have

$$\sigma(T(t)) = \tau_t(\sigma(T(0))) \text{ for every } t \in [0,1].$$

Letting $t = 1$, we have

$$T(1) = H + i(f(H)Kf(H) + g(H)) = \tau(T)$$

and

$$\tau_1(x+iy) = x + i(f(x)^2 y + g(x)) = \tau(x+iy).$$

Therefore, we have $\sigma(\tau(T)) = \tau(\sigma(T))$. If $f(x) < 0$ $(x \in \sigma(H))$, then we let

$$T(t) = H + i\{(t(-f(H)) + 1 - t)K(t(-f(H)) + 1 - t) + tg(H)\}$$

and

$$\tau_t(x+iy) = x + i\{(t(-f(x)) + 1 - t)^2 y + tg(x)\}.$$

Therefore, we have $\sigma(\tau(T)) = \tau(\sigma(T))$.

Finally, in a general case we let

$$\mathcal{R}^+ = \{z \in \mathbf{C} : f(Re(z)) > 0 \ \ (Re(z) \in \sigma(H))\}$$

and

$$\mathcal{R}^- = \{z \in \mathbf{C} : f(Re(z)) < 0 \ \ (Re(z) \in \sigma(H))\}.$$

And let

$$\tau_t^+(x+iy) = x + i((tf(x) + 1 - t)^2 y + tg(x)) \text{ on } \mathcal{R}^+$$

and

$$\tau_t^-(x+iy) = x + i((-tf(x) + 1 - t)^2 y + tg(x)) \text{ on } \mathcal{R}^-.$$

Then we have $\tau_o^+ = \tau_o^- = id$ and $\tau_1^+ = \tau_1^- = \tau$. It holds that τ_t^+ and τ_t^- are one-to-one and onto on $\mathcal{R}^+$ and $\mathcal{R}^-$, respectively $(\forall t \in [0,1])$. Also we let

$$T^+(t) = H + i(AKA + tg(H)) \text{ and } T^-(t) = H + i(BKB + tg(H)),$$

where $A = tf(H) + 1 - t$ and $B = t(-f(H)) + 1 - t$. Then from the above it holds that

$$\sigma(T^+(t)) \bigcap \tau_t^+(\mathcal{R}^+) = \tau_t^+(\sigma(T^+(0)) \bigcap \mathcal{R}^+)$$

and

$$\sigma(T^-(t)) \bigcap \tau_t^-(\mathcal{R}^-) = \tau_t^-(\sigma(T^-(0)) \bigcap \mathcal{R}^-),$$

for every $t \in [0, 1]$. Hence we have

$$\sigma(\tau(T)) \bigcap (\mathcal{R}^+ \bigcup \mathcal{R}^-) = \tau(\sigma(T) \bigcap (\mathcal{R}^+ \bigcup \mathcal{R}^-)).$$

This completes the proof.

3. Application.

For an operator $T \in B(\mathcal{H})$, Weyl spectrum $\omega(T)$ of T is defined by

$$\omega(T) = \bigcap_{K:\text{compact}} \sigma(T + K).$$

In [2], Berberian showed that the spectral mapping theorem does not generally hold for Weyl spectra (Th.3.2 of [2]). But recently in [4] Duggal and in [5] Huruya showed the interesting results for the spectral mapping theorem of Weyl spectrum of p-hyponormal operator. Also we have

THEOREM 2. *Let $T = H + iK$ be hyponormal and f, g be bounded, real-valued, continuous functions and $f(x) \neq 0$ at $x \in \sigma(H)$. Take a mapping in the complex plane*

$$\tau(x + iy) = x + i(f(x)^2 y + g(x))$$

and denote $\tau(T) = H + i(f(H)Kf(H) + g(H))$. Then

$$\omega(\tau(T)) = \tau(\omega(T)).$$

For the proof of this theorem, let $\pi_{oo}(T)$ denote the set of all isolated eigenvalues of finite multiplicity of T. Then in [3] Coburn proved the following

THEOREM C. *Let T be hyponormal. Then*

$$\omega(T) = \sigma(T) - \pi_{oo}(T).$$

PROOF OF THEOREM 2. From the proof of Theorem 1, it holds that $Tx = \lambda x$ if and only if $\tau(T)x = \tau(\lambda)x$. Hence we have

$$\pi_{oo}(\tau(T)) \;=\; \tau(\pi_{oo}(T)).$$

Since T and $\tau(T)$ are hyponormal, from Theorem C it holds that

$$\omega(T) = \sigma(T) - \pi_{oo}(T) \quad \text{and} \quad \omega(\tau(T)) = \sigma(\tau(T)) - \pi_{oo}(\tau(T)).$$

Therefore, Theorem 2 follows from Theorem 1. This completes the proof.

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