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ON A SAITO'S PROBLEM FOR THE GENERATIONS OF VON NEUMANN ALGEBRAS BY POWER PARTIAL ISOMETRIES

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Abstract

Saito raised a problem whether any properly infinite von Neumann algebra acting on a separable Hilbert space is generated by a power partial isometry. We give the complete answer for this problem in this paper.

In this paper, we discuss the generation of von Neumann algebras by the power partial isometries. These were discussed by many authors, for example, Behncke, Wogen and Saito. In particular, Saito showed the following result in [4].

Let M be a properly infinite von Neumann algebra. Then, for any positive integer n, there exists a generator V of M satisfying the following properties;

(1) $V, V^2, \ldots, V^n$ are non-zero partial isometries and $V^k (k \geq n+1)$ are not.

(2) V is a nilpotent operator of index $n+3$.

Furthermore, when Saito showed the above result, he raised the following problem in [4;Question in page 489]:

Let M be a properly infinite von Neumann algebra. Then, can we choose a generator T of M such that $T, T^2, \ldots, T^n, \ldots$ are all non-zero partial isometries ?

For the above mentioned problem by Saito, in this paper we shall give the complete answer that the von Neumann algebra generated by the power partial isometry is the von Neumann algebra of type I and is not the von Neumann algebra of type II_∞ and type III.

We thank to the refree for pointng the error in our proof.

Definition 1 *We have the following notations used in this paper.*

(1) An operator T on a Hilbert space H is called a power partial isometry if T^n is a partial isometry for $n = 1, 2, \ldots$.

(2) An operator T on H is called a truncated shift of index n ($n = 1, 2, \ldots$) if T is the operator such taht H is the n-fold direct sum

$$H = \overbrace{H_0 \oplus H_0 \oplus \cdots \oplus H_0}^{n}$$

and T=0 if $n = 1$ and

$$T < f_1, f_2, \cdots, f_n > = < 0, f_1, f_2, \cdots, f_{n-1} >$$

if $n > 1$.

(3) An operator T on H is called quasinormal if $T(T^*T) = (T^*T)T$.

For an operator T acting on a separable Hilbert space H, let M(T) be the von Neumann algebra generated by T and the identity operator I and M(T)is called the von Neumann algebra generated by (simply) T. In this note, we shall show that the von Neumann algebra M(U) generated by a power partial isometry U is a von Neumann algebra of type I and so not a von Neumann algebra of type II and type III.

For the power partial isometries, we have the following decomposition by Saito [5;Theorem 7.5] (and also Halmos and Wallen [2]).

Theorem 2 (*Saito*[5]) *Let U be a power partial isometry, then U can be decomposed into the central direct sum of the following form*

$$U = U_1 \oplus U_2 \oplus U_3 \oplus U_4$$

where U_1 is the unitary operator, U_2 is the unilateral shift operator, U_3 is the adjoint of the unilateral shift operator and U_4 is the direct sum of truncated shif operators.

We denoted the definition of quasi-normal operator. The unitary operators and the unilateral shift operators are the quasi-normal operators. Thus, if U is a unitary operator or a unilateral shift operator, then the von Neumann algebra $\mathrm{M}(U)$ generated by U is of type I by the following theorem (see Saito [5;Theorem 7.4]). Furthermore, if U is the adjoint of a unilateral shift operator, then $\mathrm{M}(U)$ is also a von Neumann algebra of type I.

Theorem 3 (*Saito*[5; *Theorem*7.4]) *If U is a quasi-normal operator on a Hilbert space H, then the von Neumann algebra M(U) is of type I.*

Considering Theorem 2 and Theorem 3, then we want to show the following result:

If T is an operator acting on a Hilbert space H represented by the finite or infinite direct sum of the truncated shifts

$$T = \sum_{k=1}^{K} \oplus U_{n(k)} \qquad (1 \leq n(1) < n(2) < \cdots < n(k) < \cdots)$$

where $U_{n(k)}$ is a truncated shifts of index $n(k)$ and K is a positive integer or ∞, then the von Neumann algera $\mathrm{M}(T)$ generated by T is of type I.

The above result gave the complete answer for another Saito's problem raised in [3]. Then, we gave the proof of the above consideration in [6;Theorem 4], but we did not know the Saito's problem when we gave the proof of the above mentioned consideration. Thus, we give the reproof of [6;Theorem 4] for this consideration here and have the main theorem in which we shall give the complete answer for the Saito's problem.

Theorem 4 *If T is an operator acting on a Hilbert space H represented by the finite or infinite direct sum of the truncated shifts*

$$T = \sum_{k=1}^{K} \oplus U_{n(k)} \qquad (1 \leq n(1) < n(2) < \cdots < n(k) < \cdots)$$

where $U_{n(k)}$ is a truncated shift of index $n(k)$ and K is a positive integer or ∞, then the von Neumann algebra M(T) generated by T is of type I.

Proof. We shall show the theorem in the case of $K = \infty$. Even if $K < \infty$, we can show the theorem by using a similar argument. Thus, let T be the infinite direct sum of truncated shifts

$$T = \sum_{k=1}^{\infty} \oplus U_{n(k)} \qquad \textit{acting on } H = \sum_{k=1}^{\infty} \oplus H_{n(k)}$$

where each $H_{n(k)}$ is the $n(k)$-fold direct sum

$$H_{n(k)} = H_0^{(k)} \oplus H_0^{(k)} \oplus \cdots \oplus H_0^{(k)}.$$

Then, every $H_{n(k)}$ reduces T $(k = 1, 2, \ldots)$. Thus the projection $E^{(k)}$ of H onto $H_{n(k)}$ is an element of the commutant M(T)' of M(T). Since

$$T^{*n(1)}T^{n(1)} = O^{(1)} \oplus (E_{11}^{(2)} \oplus O_{11}^{(2)}) \oplus \cdots \oplus (E_{11}^{(k)} \oplus O_{11}^{(k)}) \oplus \cdots ,$$

where $O^{(k)}$ is the zero operator on $H_{n(k)}$ and each $O_{11}^{(k)}$ is the zero operator on the $n(1)$-fold direct sum

$$(O_0^{(k)} \oplus O_0^{(k)} \oplus \cdots \oplus O_0^{(k)} \oplus) \overbrace{H_0^{(k)} \oplus \cdots \oplus H_0^{(k)}}^{n(1)}$$

and $E_{11}^{(k)}$ is the projection on the $(n(k) - n(1))$-fold direct sum

$$\overbrace{H_0^{(k)} \oplus \cdots \oplus H_0^{(k)}}^{n(k)-n(1)} (\oplus O_0^{(k)} \oplus O_0^{(k)} \oplus \cdots \oplus O_0^{(k)}) \qquad (k = 2, 3, \cdots),$$

$$T^{*n(1)}T^{n(1)} = O^{(1)} \oplus (E_{11}^{(2)} \oplus O_{11}^{(2)}) \oplus \cdots \oplus (E_{11}^{(k)} \oplus O_{11}^{(k)}) \oplus \cdots$$

is an element of M(T). Furthermore, since

$$TT^* = (E^{(1)} - E_1^{(1)}) \oplus (E^{(2)} - E_1^{(2)}) \oplus \cdots \oplus (E^{(k)} - E_1^{(k)}) \oplus \cdots$$

where $E_1^{(k)}$ is the projection on

$$H_0^{(k)} \oplus \overbrace{O_0^{(k)} \oplus O_0^{(k)} \oplus \cdots \oplus O_0^{(k)}}^{n(k)-1} \qquad (k = 1, 2, 3, \cdots),$$

$$I - TT^* = E_1^{(1)} \oplus E_1^{(2)} \oplus \cdots \oplus E_1^{(k)} \oplus \cdots$$

is an element of M(T). Then, by considering the relation $n(k)-n(1)\geq 1$ we have the relation

$$(T^{*n(1)}T^{n(1)})(I-TT^{*})$$

$$=O^{(1)}\oplus E_1^{(2)}\oplus\cdots\oplus E_1^{(k)}\oplus\cdots .$$

Therefore, the element

$$(I-TT^{*})(I-(T^{*n(1)}T^{n(1)})(I-TT^{*}))$$

$$=(E_1^{(1)}\oplus E_1^{(2)}\oplus\cdots\ \oplus E_1^{(k)}\oplus\cdots)$$

$$\cdot\,(E^{(1)}\oplus(E^{(2)}-E_1^{(2)})\oplus\cdots\ (E^{(k)}-E_1^{(k)})\oplus\cdots)$$

$$=E_1^{(1)}$$

is an element of M(T). Apply a similar argument for

$$T^{*n(1)-1}T^{n(1)-1} \qquad \textit{and} \qquad I-T^2T^{*2},$$

$$T^{*n(1)-1}T^{n(1)-1}$$

$$=E_1^{(1)}\oplus(E_{12}^{(2)}\oplus O_{12}^{(2)})\oplus\cdots\ \oplus(E_{12}^{(k)}\oplus O_{12}^{(k)})\oplus\cdots$$

and

$$T^2T^{*2}$$

$$= (E^{(1)} - (E_1^{(1)} + E_2^{(1)})) \oplus (E^{(2)} - (E_1^{(2)} + E_2^{(2)})) \oplus \cdots \oplus (E^{(k)} - (E_1^{(k)} + E_2^{(k)})) \oplus \cdots$$

where for each k, $E_s^{(k)}$ is the projection on

$$O_0^{(k)} \oplus \cdots \oplus O_0^{(k)} \oplus \overbrace{H_0^{(k)}}^{s} \oplus O_0^{(k)} \oplus \cdots \oplus O_0^{(k)} \qquad (k = 1, 2, 3, \cdots),$$

$E_{12}^{(k)}$ is thé projection on the $(n(k) - n(1) + 1)$-fold direct sum

$$\overbrace{H_0^{(k)} \oplus \cdots \oplus H_0^{(k)}}^{n(k)-n(1)+1} (\oplus O_0(k) \oplus O_0(k) \oplus \cdots \oplus O_0^{(k)})$$

and $O_{12}^{(k)}$ is the zero operator on the $(n(1) - 1)$-fold direct sum

$$(O_0^{(k)} \oplus O_0(k) \oplus \cdots \oplus O_0^{(k)} \oplus) \overbrace{H_0^{(k)} \oplus \cdots \oplus H_0^{(k)}}^{n(1)-1}.$$

Thus, the operator

$$(I - T^2 T^{*2})(T^{*n(1)-1} T^{n(1)-1})$$

$$= E_1^{(1)} \oplus (E_1^{(2)} + E_2^{(2)}) \oplus \cdots \oplus (E_1^{(k)} + E_2^{(k)}) \oplus \cdots$$

is an element of M(T). Furthermore, since the operator

$$E_1^{(1)} \oplus O^{(2)} \oplus \cdots \oplus O^{(k)} \oplus \cdots$$

is an element of M(T), the operator

$$O^{(1)} \oplus (E_1^{(2)} + E_2^{(2)}) \oplus \cdots \oplus (E_1^{(k)} + E_2^{(k)}) \oplus \cdots$$

is an element of M(T). Therefore, the operator

$$(I - T^2 T^{*2}) - (O^{(1)} \oplus (E_1^{(2)} + E_2^{(2)}) \oplus \cdots \oplus (E_1^{(k)} + E_2^{(k)}) \oplus \cdots)$$

$$= E_1^{(1)} + E_2^{(1)}$$

is an element of M(T). Thus, since $E_1^{(1)}$ is an element of M(T), $E_2^{(1)}$ is an element of M(T) too.

Continuiting this process, we can show that $E_1^{(1)}$, $E_2^{(1)}$, $\cdots$,$E_{n(1)}^{(1)}$ are also the elements of M(T) and so $E^{(1)}$ is an element of M(T). Thus, the projection $E^{(1)}$ is a central projection of M(T).

Next, apply the above process for $T(I - E^{(1)})$, we can show that $E^{(2)}$ is a central projection of M(T).

By considering this process, we can show that all $E^{(k)}$ ($k = 1, 2, \cdots$) are central projections of M(T). Therefore, since every $U_{n(k)}$ is a truncated shift of index n(k), M($U_{n(k)}$) is a von Neumann algebra of type $\mathrm{I}_{n(k)}$ and so M(T) is a von Neumann algebra of type I. Thus, we have the complete proof of Theorem 4.

By Theorem 2, Theorem 3 and Theorem 4, we have the following main result.

Theorem 5 *Let T be a power partial isometry acting on a Hilbert space H, then the von Neumann algebra M(T) generated by T is of type I.*

Proof. Since T is a power partial isometry, by Theorem 2, T can be docomposed into the central direct sum

$$T = T_1 \oplus T_2 \oplus T_3 \oplus T_4$$

where T_1 is the unitary operator, T_2 is the unilateral shift operator, T_3 is the adjoint of the unilateral shift operator and T_4 is the direct sum of truncated shift operators. Then, by Theorem 3, the von Neumann algebras M(T_1), M(T_2) and M(T_3) are of type I. Furthermore, by Theorem 4, the von Neumann algebra M(T_4) is also of type I. So, since the direct sum

$$T = T_1 \oplus T_2 \oplus T_3 \oplus T_4$$

is the central direct sum, the von Neumann algebra M(T) generated by T can be decomposed into the central direct sum

$$M(T) = M(T_1) \oplus M(T_2) \oplus M(T_3) \oplus M(T_4).$$

Therefore, the von Neumann algebra M(T) generated by the power partial isometry T is of type I. This completes the proof of Theorem 5.

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