Nihonkai Math. J.
Vol.7 (1996), 29-39

COMPOSITION OPERATORS ON SOME F-ALGEBRAS OF HOLOMORPHIC FUNCTIONS

JUN SOO CHOA AND HONG OH KIM

ABSTRACT. We let N^p, $p > 1$, be the $F-$algebra of holomorphic functions f on the unit disc $\mathbb{D}$ which satisfy

$$\lim_{r\nearrow 1}\int_0^{2\pi}\left(\log(1+|f(re^{i\theta})|^2)\right)^p d\theta < \infty.$$

In this paper we prove that the composition operator induced by a holomorphic self-map of the unit disc is compact on N^p, $p > 1$, if and only if it is compact on the Hardy space H^2.

1. INTRODUCTION

For $p \geq 1$, we let N^p denote the class of all functions f holomorphic in the unit disc $\mathbb{D}$ which satisfy the growth condition

$$\lim_{r\nearrow 1}\int_0^{2\pi}(\log^+|f(re^{i\theta})|)^p d\theta < \infty.$$

If $p \geq 1$, the inequalities

$$(1) \qquad (\log^+ x)^p \leq \left(\log(1+x^2)\right)^p \leq 2^{p-1}\left(1+(\log^+ x)^p\right) \quad \text{for all } x \geq 0$$

1991 *Mathematics Subject Classification.* Primary 47B05, 47B38; Secondary 30D55.
Key words and phrases. Compact composition operator, Nevanlinna counting function, Carleson measure.
Supported in part by TGRC, and Korean ministry of Education: BSRI-95-1420

imply that

$$f \in N^p \quad \text{if and only if} \quad \|f\|_{N^p}^p := \lim_{r \nearrow 1} \int_0^{2\pi} \left(\log(1 + |f(re^{i\theta})|^2)\right)^p \frac{d\theta}{2\pi} < \infty$$

for f holomorphic in $\mathbb{D}$. Note that N^1 is the classical Nevanlinna class N. It was known that

$$H^p \subset N^p \subset N \quad \text{for } p > 1$$

and these containments are proper([5], [20]). Under the metric d_p, defined for $f, g \in N^p$ by $d_p(f, g) = \|f - g\|_{N^p}$, N^p becomes an algebra if $p \geq 1$ and moreover N^p is an F-algebra(i.e., a topological vector space which is an algebra) if $p > 1$. See [10] and [20] for this and more information on N^p.

If φ is a holomorphic self-map of the unit disc $\mathbb{D}$, then such map φ induces a linear operator C_φ on the space of holomorphic functions on $\mathbb{D}$ by means of the equation $C_\varphi(f) = f \circ \varphi$. This C_φ is called the composition operator induced by φ. The study of composition operators began in 1968 with the work of E. Nordgren [12]. From then on, most work was on the properties of composition operators on the Hardy space H^p(see for example [2], [4], [16], [17], [18] and [19]) although there were various results obtained in other function spaces (see [1], [8], [9], [10], [13], [14] and [21]). In 1987, J. Shapiro [17] has obtained a prevailing result on the compactness of C_φ on the Hardy space H^p. In fact he gave a complete characterization of φ, in terms of Nevanlinna counting function, for which C_φ is compact on H^p. However, as far as we know, the operator C_φ as an operator on the class N^p, $p \geq 1$, was first studied by Masri in his thesis [10], where he obtained several necessary conditions and sufficient conditions on φ for the operator C_φ to be compact on the class N^p, but he could not find necessary and sufficient conditions for the compactness of C_φ on N^p except the sequential one(see, Lemma 1 of Section 2), and indeed the conditions are in the same spirit as conditions developed in [19] for studying the compactness of C_φ on H^p. In this paper we find, however, a necessary and sufficient condition(which is not a sequential one) for the compactness of C_φ on the algebra N^p when $p > 1$. More precisely we prove:

A composition operator C_φ is compact on the F-algebra N^p, $p > 1$, if and only if it is compact on H^2.

Appealing to Shapiro[17], this result gives a complete characterization of φ for which C_φ is compact on N^p for the case $p > 1$. Recently, the authors [3] verified that the compactness of C_φ on the Nevanlinna class N is equivalent to its compactness on H^2. From this viewpoint, later on, we will consider the compactness of C_φ on N^p in the case $p > 1$. The result of this paper relies on the Shapiro's Nevanlinna counting function criterion [17] and MacCluer's Carleson-measure criterion[7](where the setting was more general) for the compactness of C_φ on H^2.

Throughout this paper, the symbol φ will be used to denote a holomorphic self-map of the unit disc $\mathbb{D}$.

2. Preliminaries

As is shown in [10], the boundedness of C_φ on the algebra N^p follows from Harnack's inequality. (This can also be proved by Littlewood's subordination principle.) So, from now on, we confine ourselves to the compactness of C_φ on N^p. Following [10], we say that the operator C_φ is *compact* on N^p if the closure of the image, under C_φ, of each bounded set is compact. We recall that a subset E of N^p is bounded if there exists a finite constant M such that $\|f - g\|_{N^p} \le M$ for all $f, g \in E$.

Now we collect some material that will be used later. Recall that the exponent p which appears in the rest of this paper is bigger than 1.

The first one is the following characterization of compactness of C_φ on N^p expressed in terms of sequential convergence, which is taken from [10, Theorem 2.4.2].

Lemma 1. *Let φ be a holomorphic self-map of $\mathbb{D}$. Then C_φ is a compact operator on N^p if and only if for every sequence $\{f_n\}$ which is bounded in N^p and converges to zero uniformly on compact subsets of $\mathbb{D}$, we have $\|f_n \circ \varphi\|_{N^p} \to 0$.*

The lemma below is a Littlewood and Paley-type identity, Since the proof which is based on the Green's formula [6, page 236] can be obtained by a slight modification of that of [3, Lemma1], we just state it without proof. In what follows, dA denotes the normalized Lebesegue area measure on $\mathbb{D}$.

Lemma 2. *Suppose f is holomorphic in $\mathbb{D}$. Then*

$$(2)\qquad \|f\|_{N^p}^p = \big(\log(1+|f(0)|^2)\big)^p + 2\int_{\mathbb{D}} \Big\{ p(p-1)(\log(1+|f(z)|^2))^{p-2} \frac{|f(z)|^2|f'(z)|^2}{(1+|f(z)|^2)^2} + p(\log(1+|f(z)|^2))^{p-1} \frac{|f'(z)|^2}{(1+|f(z)|^2)^2} \Big\} \log\frac{1}{|z|} dA(z)$$

where, as always "$\|\ \ \|_{N^p}$" denotes the quasi-norm as defined in Section 1, and "$\|f\|_{N^p} = \infty$" means "$f \notin N^p$".

The next lemma is a well-known change of variable formula for the integral means, and it can be found in [18, page 186].

Lemma 3. *If g is a non-negative measurable function on $\mathbb{D}$ and φ is a holomorphic self-map of $\mathbb{D}$, then*

$$(3)\qquad \int_{\mathbb{D}} g(\varphi(z))|\varphi'(z)|^2 \log\frac{1}{|z|} dA(z) = \int_{\mathbb{D}} g(w) N_\varphi(w)\, dA(w),$$

where $N_\varphi(w)$ is the (usual) Nevanlinna counting function defined by

$$N_\varphi(w) = \begin{cases} \sum_{z\in\varphi^{-1}(w)} \log\frac{1}{|z|} & \text{if } w \in \varphi(\mathbb{D}), \\ 0 & \text{if } w \notin \varphi(\mathbb{D}). \end{cases}$$

The following result is an immediate consequence of the above two formulas (2) and (3).

Corollary 4. *Suppose φ is a holomorphic self-map of $\mathbb{D}$. Then*

$$\|C_\varphi f\|_{N^p}^p = \left(\log(1+|f(\varphi(0))|^2)\right)^p + 2\int_{\mathbb{D}} \left\{ p(p-1)(\log(1+|f(w)|^2))^{p-2} \frac{|f(w)|^2|f'(w)|^2}{(1+|f(w)|^2)^2} + p(\log(1+|f(w)|^2))^{p-1} \frac{|f'(w)|^2}{(1+|f(w)|^2)^2} \right\} N_\varphi(w)\, dA(w) \tag{4}$$

for all f holomorphic in $\mathbb{D}$.

The above corollary suggests that the Nevanlinna counting function is closely related to composition operators on the algebra N^p.

The next criteria of compactness of C_φ, which are due to Shapiro [17] and MacCluer [7], play crucial roles in the proof of the main result of this paper. In the followings, we say that a positive measure μ on $\overline{\mathbb{D}}$ is a *little Carleson measure* if

$$\lim_{\delta\to 0} \frac{\mu(S_\delta(\zeta))}{\delta} = 0 \quad \text{uniformly in} \quad \zeta \in \partial\mathbb{D},$$

where $S_\delta(\zeta) = \{re^{it} \in \overline{\mathbb{D}} : 1-\delta < r \le 1 \text{ and } |e^{it}-\zeta| < \delta\}$.

Lemma 5. *For φ a holomorphic self-map of $\mathbb{D}$, the following conditions are equivalent:*

(a) C_φ is compact on H^2.

(b) $\displaystyle\lim_{|w|\nearrow 1} \frac{N_\varphi(w)}{\log\frac{1}{|w|}} = 0.$

(c) The pull-back measure μ_φ defined by $\mu_\varphi = \sigma\circ\varphi^{-1}$ is a little Carleson measure on $\overline{\mathbb{D}}$, here $\sigma = d\theta/2\pi$.

3. Proof of the result

We proceed now to prove the main result of this paper. As stated in the introduction, what we want to prove is:

Main Theorem. *Suppose φ is a holomorphic self-map of $\mathbb{D}$. Then C_φ is compact on the F-algebra N^p, $p > 1$, if and only if it is compact on H^2.*

Proof. First we assume that C_φ is compact on H^2 and will show that C_φ is compact on N^p. The argument to prove this part is very similar to that of [18, Section 10.5]. For this, fix a sequence $\{f_n\}$ with $\|f_n\|_{N^p} \leq M$ that converges to zero uniformly on compact subsets of $\mathbb{D}$. By Lemma 1, it is enough to prove that $\|f_n \circ \varphi\|_{N^p} \longrightarrow 0$. Before proving this result, to simplify some writing, let us introduce the notation $I_p(f)$ for

$$p(p-1)\big(\log(1+|f(w)|^2)\big)^{p-2}\frac{|f(w)|^2|f'(w)|^2}{(1+|f(w)|^2)^2}+p\big(\log(1+|f(w)|^2)\big)^{p-1}\frac{|f'(w)|^2}{|1+|f(w)|^2)^2}$$

whenever f is a function holomorphic in $\mathbb{D}$ and $p > 1$.

Now let $\varepsilon > 0$ be given. Then it follows from Lemma 5 that we can choose $0 < r < 1$ such that

$$N_\varphi(w) < \varepsilon \log\frac{1}{|w|} \quad \text{whenever} \quad r \leq |w| < 1.$$

Since $f_n \longrightarrow 0$ uniformly on compact subsets of $\mathbb{D}$, so is f_n'. Thus we can choose n_ε so that

$$|f_n| \quad \text{and} \quad |f_n'| < \sqrt{\varepsilon}$$

on $r\mathbb{D}\cup\{\varphi(0)\}$ whenever $n > n_\varepsilon$. Hence for each such n we have from formula (4) and the elementary inequalities $x/(1+x) \le \log(1+x) \le x \quad (x \ge 0)$,

$$\text{(5)} \qquad \|f_n\circ\varphi\|_{N^p}^p = \big(\log(1+|f_n(\varphi(0))|^2)\big)^p + 2\left[\int_{r\mathbf{D}} + \int_{\mathbf{D}\setminus r\mathbf{D}} \{I_p(f)\} N_\varphi(w)\, dA(w)\right]$$

$$\le \varepsilon^p + 2(p(p-1)\varepsilon^p + p\varepsilon^p)\int_{r\mathbf{D}} N_\varphi(w)\, dA(w) + 2\varepsilon \int_{\mathbf{D}\setminus r\mathbf{D}} \{I_p(f)\}\log\frac{1}{|w|}\, dA(w).$$

The quantity in the inequality of the above (5) is at most

$$\varepsilon^p + 2p^2\varepsilon^p \int_{\mathbf{D}} N_\varphi(w)\, dA(w) + 2\varepsilon\int_{\mathbf{D}} \{I_p(f)\}\log\frac{1}{|w|}\, dA(w)$$

$$\le \varepsilon^p + p^2\varepsilon^p(1-|\varphi(0)|^2) + \varepsilon\left[\|f_n\|_{N^p}^p - \big(\log(1+|f_n(0)|^2)\big)^p\right]$$

$$\le \varepsilon^p + p^2\varepsilon^p + \varepsilon\|f_n\|_{N_p}^p$$

$$\le \varepsilon^p(1+p^2) + \varepsilon M^p,$$

where in the first inequality we have used the estimate

$$\int_{\mathbf{D}} N_\varphi(w)\, dA(w) \le \frac{1-|\varphi(0)|^2}{2}$$

of [17, Section 4.5] and Lemma 2, and in the last inequality we used the fact that $\|f_n\|_{N^p} \le M$ for each n. Thus $\|f_n\circ\varphi\|_{N^p} \to 0$, which establishes the compactness of C_φ on N^p.

For the converse direction, we assume C_φ is compact on N^p. Because of Lemma 5, we only need to verify that the pull-back measure $\mu_\varphi = \sigma\circ\varphi^{-1}$ is a little Carleson on $\overline{\mathbb{D}}$.

To prove this, we let $a = (1-\delta)\zeta$ where $\zeta \in \partial\mathbb{D}$ and $0 < \delta < 1$, and define

$$f_a(z) = (1-|a|)^{1/p}\exp\Big(\frac{(1-|a|)^\alpha}{1-\bar{a}z}\Big)$$

where $\alpha = 1 - \frac{1}{p}(> 0)$. Then clearly $f_a \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $|a| \nearrow 1$. By simple calculations, together with the trivial inequalities

$$\log^+ xy \leq \log^+ x + \log^+ y \qquad \text{for } x, y \geq 0$$

and

$$\log^+ \exp t \leq |t| \quad \text{for } t \text{ real},$$

we have

$$\begin{aligned}
\int_0^{2\pi} \left(\log^+ |f_a(re^{i\theta})|\right)^p \frac{d\theta}{2\pi} &\leq \int_0^{2\pi} \left[\log^+ \left(\exp\left(\mathrm{Re}\left(\frac{(1-|a|)^\alpha}{1-\bar{a}re^{i\theta}}\right)\right)\right)\right]^p \frac{d\theta}{2\pi} \\
&\leq \int_0^{2\pi} \left|\mathrm{Re}\left(\frac{(1-|a|)^\alpha}{1-\bar{a}re^{i\theta}}\right)\right|^p \frac{d\theta}{2\pi} \\
&= (1-|a|)^{\alpha p} \int_0^{2\pi} \left\{\frac{1-\mathrm{Re}(\bar{a}re^{i\theta})}{|1-\bar{a}re^{i\theta}|^2}\right\}^p \frac{d\theta}{2\pi} \\
&\leq (1-|a|)^{\alpha p} \int_0^{2\pi} \frac{1}{|1-\bar{a}re^{i\theta}|^p} \frac{d\theta}{2\pi}.
\end{aligned}$$

In the above, the last step follows from the inequality $1 - \mathrm{Re}\, w \leq |1-w|$ for $|w| < 1$. By Proposition 1.4.10 of [15] (recall that $p > 1$), there is an absolute constant $\widetilde{M} > 0$ such that

$$\int_0^{2\pi} \frac{1}{|1-\bar{a}re^{i\theta}|^p} \frac{d\theta}{2\pi} \leq \widetilde{M}(1-|a|)^{1-p},$$

so that

$$\int_0^{2\pi} \left(\log^+ |f_a(re^{i\theta})|\right)^p \frac{d\theta}{2\pi} \leq \widetilde{M},$$

and thus we have from (1) that

$$\|f_a\|_{N^p}^p = \lim_{r \nearrow 1} \int_0^{2\pi} \left(\log(1+|f_a(re^{i\theta})|^2)\right)^p \frac{d\theta}{2\pi} \leq 2^{p-1}(1+\widetilde{M}).$$

It now follows from the compactness of C_φ on N^p and Lemma 1 that

$$\lim_{|a|\nearrow 1} \|f_a \circ \varphi\|_{N^p} = 0.$$

On the other hand, if $z \in S_\delta(\zeta)$ then

$$\frac{1-|a|}{|1-\overline{a}z|^2} \geq \frac{C}{\delta}$$

for some absoulte constant $C > 0$. Thus, for $z \in S_\delta(\zeta)$, we have

$$\begin{aligned}
\log^+ |f_a(z)| &\geq \log^+ \left[(1-|a|)^{1/p} \exp\left((1-|a|)^\alpha \frac{1-\mathrm{Re}(\overline{a}z)}{|1-\overline{a}z|^2}\right)\right] \\
&\geq \log^+ \left[(1-|a|)^{1/p} \exp\left((1-|a|)^\alpha \frac{(1-|a||z|)}{|1-\overline{a}z|^2}\right)\right] \\
&\geq \log^+ \left[(1-|a|)^{1/p} \exp\left((1-|a|)^\alpha \frac{1-|a|}{|1-\overline{a}z|^2}\right)\right] \\
&\geq \log^+ \left[\delta^{1/p} \exp(C\delta^{-1/p})\right].
\end{aligned}$$

Hence, for all $\zeta \in \partial\mathbb{D}$ and $0 < \delta < 1$,

$$\begin{aligned}
\left[\log^+ \left(\delta^{1/p} \exp(C\delta^{-1/p})\right)\right]^p \mu_\varphi(S_\delta(\zeta)) &\leq \int_{S_\delta(\zeta)} \left(\log^+ |f_a(z)|\right)^p d\mu_\varphi(z) \\
&\leq \int_{S_\delta(\zeta)} \left(\log(1+|f_a(z)|^2)\right)^p d\mu_\varphi(z) \\
&\leq \int_{\overline{\mathbb{D}}} \left(\log(1+|f_a(z)|^2)\right)^p d\mu_\varphi(z) \\
&\leq \lim_{r\nearrow 1} \int_0^{2\pi} \left(\log(1+|f_a \circ \varphi(re^{i\theta})|^2)\right)^p \frac{d\theta}{2\pi} \\
&= \|f_a \circ \varphi\|_{N^p}^p,
\end{aligned}$$

where the last inequality follows from the Fatou's lemma. As we saw above, the compactness of C_φ on N^p forces $\|f_a \circ \varphi\|_{N^p}$ to zero as $|a| \nearrow 1$, which implies

$$\lim_{\delta \to 0} \left[\log^+ \left(\delta^{1/p} \exp(C\delta^{-1/p})\right)\right]^p \mu_\varphi(S_\delta(\zeta)) = 0,$$

uniformly in $\zeta \in \partial\mathbb{D}$. Therefore the desired conclusion follows since

$$\lim_{\delta \to 0} \delta \left[\log^+ \left(\delta^{1/p} \exp(C\delta^{-1/p})\right)\right]^p = \lim_{t \to \infty} \frac{\left(Ct^{1/p} - \frac{1}{p}\log t\right)^p}{t} = C^p.$$

Acknowledgement. The first author wishes to his sincere gratitude to Professor Patrick Ahern for the helpful discussions during his visit to the University of Wisconsin last year.

REFERENCES

1. D. M. Boyd, *Composition operators on the Bergman space*, Colloq. Math. **34** (1975), 127–136.

2. J. G. Caughran and H. J. Schwartz, *Spectra of compact composition operators*, Proc. Amer. Math. Soc. **51** (1975), 127–130.

3. J. S. Choa and H. O. Kim, *Compact composition operators on the Nevanlinna class*, Proc. Amer. Math. Soc. (to appear).

4. C. C. Cowen, *Composition operators on H^2*, J. Operator Th. **9** (1983), 77-106.

5. C. M. Eoff, *Fréchet envelopes of certain algebras of analytic functions*, Michigan Math. J. **35** (1988), 413–426.

6. J. B. Garnett, *Bounded analytic functions*, Academic Press, New York, 1981.

7. B. D. MacCluer, *Compact composition operators on $H^p(B_N)$*, Michigan Math. J. **32** (1985), 237–248.

8. ———, *Composition operators on S^p*, Houston J. Math. **13** (1987), 245–254.

9. B. D. MacCluer and J. H. Shapiro, *Angular derivatives and compact composition operators on the Hardy and Bergman spaces*, Canadian J. Math. **38** (1986), 878–906.

10. M. I. Masri, *Compact composition operators on the Nevanlinna and Smirnov classes*, Thesis, University of North Carolina, Chapel Hill, 1985.

11. N. Mochizuki, *Algebras of holomoprhic functions between H^p and N_**, Proc. Amer. Math. Soc. **105** (1989), 898–902.

12. E. Nordgren, *Composition operators*, Canadian J. Math. **20** (1968), 442–449.

13. R. C. Roan, *Composition operators on the space of functions with H^p-derivative*, Houston J. Math. **4** (1978), 423–438.

14. J. W. Roberts and M. Stoll, *Composition operators on F^+*, Studia Math. **57** (1976), 217–258.

15. W. Rudin, *Function theory in the unit ball of $\mathbb{C}^n$*, Springer-Verlag, New York, 1980.

16. H. J. Schwartz, *Composition operators on H^p*, Thesis, University of Toledo, 1969.

17. J. H. Shapiro, *The essential norm of a composition operator*, Annals of Math. **125** (1987), 375–404.

18. ———, *Composition operators and classical function theory*, Springer-Verlag, 1993.

19. J. H. Shapiro and P. D. Taylor, *Compact, nuclear, and Hilbert-Schmidt composition operators on H^2*, Indiana Univ. Math. J. **125** (1973), 471–496.

20. M. Stoll, *Mean growth and Taylor coefficients of some topological algebras of analytic functions*, Ann. Polon. Math. **35** (1977), 139–158.

21. N. Yanagihara and Y. Nakamura, *Composition operators on N^+*, TRU Math. **14** (1978), 9–16.

Department of Mathematics Education
Sung Kyun Kwan University
Jongro-gu, Seoul 110-745, Korea
E-mail address: jschoa@yurim.skku.ac.kr

Department of Mathematics
Korea Advanced Institute of Science and Technology
373–1 Yusung-Gu Gusung-Dong, Taejun 305–701, Korea

Received July 10, 1995