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PARAMETERIZED KANTOROVICH INEQUALITY FOR POSITIVE OPERATORS

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Abstract. The Kantorovich inequality says that if A is a positive operator on H such that $0 < m \le A \le M$ for some $M \ge m > 0$, then

$$(Ax,x)(A^{-1}x,x) \le \frac{(M+m)^2}{4Mm}$$

for all unit vectors $x \in H$. We generalize it by the use of a family of power means, which gives us a parameterization of the Kantorovich inequality. Moreover we give a parameterization of the Pólya-Szegö inequality.

1. Introduction. Let a, g and h be the arithmetic, geometric and harmonic mean respectively. It is known that these means are unified by the family of power means $\{m_r; -1 \le r \le 1\}$, i.e.,

(1) $$\alpha\, m_r\, \beta = \left(\frac{\alpha^r + \beta^r}{2}\right)^{\frac{1}{r}} \quad \text{for } \alpha, \beta > 0.$$

It is easily seen that $m_1 = a, m_0 = g$ and $m_{-1} = h$. The family of power means plays an interesting role, e.g., [1,3,5,7]. We refer to [6] for the theory of operator means.

Now Kantorovich established the following inequality in his study on applications of functional analysis to numerical analysis, cf. [2] : If $\{a_k\}$ is a sequence in $\mathbb{R}$ such that $0 < m \le a_k \le M$ for some m and M, then

$$\sum_k a_k x_k^2 \sum_k \frac{1}{a_k} x_k^2 \le \frac{(M+m)^2}{4Mm}\left(\sum_k x_k^2\right)^2$$

holds for all $x = \{x_k\}$ in $l^2(\mathbb{N})$.

If we define the diagonal operator A by $A = \text{diag}\,(a_k)$, then we have

$$(Ax,x)(A^{-1}x,x) \le \frac{(M+m)^2}{4Mm}\|x\|^4 \quad \text{for } x \in l^2(\mathbb{N})$$

if $0 < m \le A \le M$. As a matter of fact, the following inequality is proved by Greub and Rheinboldt [2], which we call the Kantorovich inequality.

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The Kantorovich inequality. *If A is a positive operator on a Hilbert space H such that $0 < m \le A \le M$ for some $M \ge m > 0$, then*

$$(2) \qquad (Ax,x)(A^{-1}x,x) \le \frac{(M+m)^2}{4Mm}$$

for all unit vectors $x \in H$.

From the mean theoretic view, the Kantorovich inequality (2) is seen as follows :

$$(3) \qquad (Ax,x)\ m_0\ (A^{-1}x,x) \le \frac{M+m}{2\sqrt{Mm}}$$

for all unit vectors $x \in H$.

In this note, we give a parameterization of the Kantorovich inequality by the use of power means which includes (3) as the case $r = 0$. In the proof, the convexity of the function t^{-1} on $(0,\infty)$ is effective. Moreover we parameterise the Pólya-Szegö inequality [2 ; Theorem 2] which is equivalent to the Kantorovich inequality.

2. Parameterised Kantorovich inequality. The Kantorovich inequality has the following parameterization by power means.

Theorem 1. *Let A be a positive operator on a Hilbert space H such that $0 < m \le A \le M$ for some $M \ge m > 0$. Then, for power means $m_r (-1 \le r \le 1)$*

$$(4) \qquad (Ax,x)\ m_r\ (A^{-1}x,x) \le \begin{cases} 2^{-\frac{1}{r}}(M^r + M^{-r})^{\frac{1}{r}} & \text{if } M^{1-2r} \le m \\ 2^{-\frac{1}{r}}(M+m)(1+(Mm)^{\frac{r}{r-1}})^{\frac{1-r}{r}} & \text{if } m^2 \le (Mm)^{\frac{1}{1-r}} \le M^2 \\ 2^{-\frac{1}{r}}(m^r + m^{-r})^{\frac{1}{r}} & \text{if } M \le m^{1-2r} \end{cases}$$

for all unit vectors $x \in H$. The bound is optimal.

Remark. In the case $r = 0$, i.e., m_0 is the geometric mean, the right hand side in the above (4) is regarded as the limit by taking $r \to 0$; namely

$$\lim_{r \to 0} 2^{-\frac{1}{r}}(M+m)(1+(Mm)^{\frac{r}{r-1}})^{\frac{1-r}{r}} = \frac{M+m}{2\sqrt{Mm}}.$$

It is clear that the second case in (4) only happens and so it is the Kantorovich inequality (3). On the other hand, if $r = 1$, i.e., $m_1 = a$, then the second case happens if and only if $Mm = 1$. Therefore we have

$$(Ax,x)\ a\ (A^{-1}x,x) \le \frac{1}{2}\ \max\{m + \frac{1}{m}, M + \frac{1}{M}\}.$$

for all unit vectors $x \in H$. As a matter of fact, we can directly compute it. Finally, if $r = -1$, i,e., $m_{-1} = h$, then the mixed type ieqaulity (4) happens ;

$$(Ax,x)\ h\ (A^{-1}x,x) \le \begin{cases} 2(M+M^{-1})^{-1} & \text{if } M^3 \le m \\ \dfrac{2(M+m)}{(1+\sqrt{Mm})^2} & \text{if } m^4 \le Mm \le M^4 \\ 2(m+m^{-1})^{-1} & \text{if } M \le m^3 \end{cases}$$

for all unit vectors $x \in H$.

Now the computational part of the proof is concentrated to the following lemma. For this, we prepare the functions f_r on $[0, M+m]$ for $-1 \le r \le 1$;

$$\begin{aligned} f_r(t) &= t \; m_r \; g(t) \\ &= \frac{1}{Mm} 2^{-\frac{1}{r}}((Mmt)^r + (M+m-t)^r)^{\frac{1}{r}}, \end{aligned}$$

where

$$g(t) = \frac{M+m-t}{Mm}.$$

Lemma. *Let f_r be as in above and put $\alpha_r = \frac{M+m}{1+(Mm)^{r/(r-1)}}$. Then*

$$\max_{m \le t \le M} f_r(t) = \begin{cases} f_r(M) & \text{if } \; M^{1-2r} \le m \\ f_r(\alpha_r) & \text{if } \; m^2 \le (Mm)^{\frac{1}{1-r}} \le M^2 \\ f_r(m) & \text{if } \; M \le m^{1-2r}. \end{cases}$$

Incidentally,

$$\max f_1(t) = \max\{f_1(m), f_1(M)\}.$$

Proof. Since

$$f_r'(t) = \frac{1}{Mm} 2^{-\frac{1}{r}}((Mmt)^r + (M+m-t)^r)^{\frac{1-r}{r}}((Mm)^r t^{r-1} - (M+m-t)^{r-1}),$$

it follows that $f_r'(t) > 0$ for $0 \le t < \alpha_r$, $f_r'(\alpha_r) = 0$ and $f_r'(t) < 0$ for $\alpha_r < t \le M+m$. Therefore we have

$$\max f_r(t) = \begin{cases} f_r(M) & \text{if } \; M < \alpha_r \\ f_r(\alpha_r) & \text{if } \; m \le \alpha_r \le M \\ f_r(m) & \text{if } \; \alpha_r < m. \end{cases}$$

Finally we remark that $m \le \alpha_r \le M$ if and only if $m^2 \le (Mm)^{\frac{1}{1-r}} \le M^2$. Actually the former is rephrased that

$$M(Mm)^{\frac{r}{r-1}} \ge m \quad \text{and} \quad M \ge m(Mm)^{\frac{r}{r-1}},$$

or equivalently

$$M^2(Mm)^{\frac{1}{1-r}} \ge 1 \quad \text{and} \quad 1 \ge m^2(Mm)^{\frac{1}{1-r}}.$$

Furthermore it is equivalent to the desired inequality. In addition, the other cases are easily checked.

Proof of Theorem 1. Let $A = \int t \, dE_t$ be the spectral decomposition of A. Then, for a fixed unit vector $x \in H$,

$$\begin{aligned} (Ax, x) \; m_r \; (A^{-1}x, x) &= \int t \, d(E_t x, x) \; m_r \; \int t^{-1} \, d(E_t x, x) \\ &\le t_0 \; m_r \; g(t_0) \end{aligned}$$

for some $t_0 \in [m, M]$ because the function t^{-1} is convex and g is the straight line through the points (m, m^{-1}) and (M, M^{-1}). Recalling that $f_r(t) = t\, m_r\, g(t)$, we have the required inequality (4) by combining with Lemma.

The following theorem is another direct generalization of the Kantorovich inequality as $r = 1/2$, which is pointed out by the referee.

Theorem 2. *Let A be a positive operator on a Hilbert space H such that $0 < m \le A \le M$ for some $M \ge m > 0$ and $0 < r < 1$. Then*

$$(Ax, x)^r (A^{-1}x, x)^{1-r} \le \begin{cases} m^{2r-1} & \text{if } 0 < r < \frac{m}{M+m} \\ (M+m)(Mm)^{r-1} r^r (1-r)^{1-r} & \text{if } \frac{m}{M+m} \le r \le \frac{M}{M+m} \\ M^{2r-1} & \text{if } \frac{M}{M+m} \le r < 1 \end{cases}$$

for all unit vectors $x \in H$,. The bound is optimal.

The proof of Theorem 2 can be done similarly to that of Theorem 1 by putting $f_r(t) = t^r g(t)^{1-r}$.

3. Parameterized Pólya-Szegö inequality. The Kantorovich inequality is equivalent to the following inequality [2; Theorem 2]. Since it is an operator version of an inequality due to Pólya and Szegö, we may call it the Pólya-Szegö inequality.

The Pólya-Szegö inequality. *Let A and B be commuting positive operators on H such that*

$$0 < m_1 \le A \le M_1 \quad \text{and} \quad 0 < m_2 \le B \le M_2. \tag{5}$$

Then

$$(A^2x, x)(B^2x, x) \le \frac{(M_1M_2 + m_1m_2)^2}{4M_1M_2m_1m_2}(Ax, Bx)^2 \tag{6}$$

for all $x \in H$.

The Pólya-Szegö inequality will be parameterized as well as the Kantorovich one. In the below, we suppose that A and B satisfy the condition (5) for some m_i and $M_i (i = 1, 2)$. For the sake of convenience, we put the constant K_r for $-1 \le r \le 1$;

$$K_r = \begin{cases} 2^{-\frac{1}{r}}\left(\left(\frac{M_1}{m_2}\right)^r + \left(\frac{m_2}{M_1}\right)^r\right)^{\frac{1}{r}} & \text{if } M_1^{1-2r}M_2 \le m_1 m_2^{1-2r} \\ 2^{-\frac{1}{r}}\frac{M_1M_2 + m_1m_2}{M_2m_2}\left(1 + \left(\frac{M_1m_1}{M_2m_2}\right)^{\frac{r}{r-1}}\right)^{\frac{1-r}{r}} & \\ & \text{if } m_1m_2 \le M_2m_2\left(\frac{M_1m_1}{M_2m_2}\right)^{\frac{1}{2(1-r)}} \le M_1M_2 \\ 2^{-\frac{1}{r}}\left(\left(\frac{m_1}{M_2}\right)^r + \left(\frac{M_2}{m_1}\right)^r\right)^{\frac{1}{r}} & \text{if } M_1M_2^{1-2r} \le m_1^{1-2r}m_2. \end{cases}$$

Theorem 3. *Let A and B be commuting positive operators satisfying (5). Then*

$$(A^2x,x)\ m_r\ (B^2x,x) \le K_r(Ax,Bx)^2 \tag{7}$$

for all $x \in H$.

Proof. The proof is quite similar to [2 ; Theorem 2]. We put $C = AB^{-1}$; $m = \frac{m_1}{M_2}$ and $M = \frac{M_1}{m_2}$. Then we have $0 < m \le C \le M$. Hence Theorem 1 implies that

$$\frac{(Cx,x)\ m_r\ (C^{-1}x,x)}{\|x\|^4} \le \begin{cases} 2^{-\frac{1}{r}}(M^r + M^{-r})^{\frac{1}{r}} & \text{if } M^{1-2r} \le m \\ 2^{-\frac{1}{r}}(M+m)(1+(Mm)^{\frac{r}{r-1}})^{\frac{1-r}{r}} & \text{if } m^2 \le (Mm)^{\frac{1}{1-r}} \le M^2 \\ 2^{-\frac{1}{r}}(m^r + m^{-r})^{\frac{1}{r}} & \text{if } M \le m^{1-2r} \end{cases}$$

for all $x \in H$. It is easily checked that the right hand side of the above is just K_r and the left hand side becomes

$$\frac{(A^2x,x)\ m_r\ (B^2x,x)}{(Ax,Bx)^2}$$

by replacing x to $(AB)^{\frac{1}{2}}x$, which completes the proof.

Remark. Theorem 3 is implied by Theorem 1, as seen in the proof of it. Conversely Theorem 1 follows from Theorem 2. In fact, for a given C with $0 < m \le C \le M$, we take

$$A = C^{\frac{1}{2}}, B = C^{-\frac{1}{2}}; m_1 = m^{\frac{1}{2}}, M_1 = M^{\frac{1}{2}}, m_2 = M^{-\frac{1}{2}}, M_2 = m^{-\frac{1}{2}},$$

and apply it to Theorem 3.

Finally we consider a noncommutative generalization of the Pólya-Szegö inequality and Theorem 3.

Theorem 4. *Let A and B be positive operators satisfying (5). Then*

$$\|B^{-\frac{1}{2}}AB^{\frac{1}{2}}x\|\|Bx\| \le \frac{M_1M_2 + m_1m_2}{2\sqrt{M_1M_2m_1m_2}}\|A^{\frac{1}{2}}B^{\frac{1}{2}}x\|^2$$

for all $x \in H$.

Proof. We put $C = A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}$. Then we have

$$0 < m = \frac{m_1}{M_2} \le C \le M = \frac{M_1}{m_2}. \tag{8}$$

The Kantorovich inequality implies that

$$(Cx,x)(C^{-1}x,x) \le \frac{(M+m)^2}{4Mm}\|x\|^4 \tag{9}$$

for all $x \in H$. If we replace x in (9) by $A^{\frac{1}{2}}B^{\frac{1}{2}}x$ and M, m by $M_i, m_i (i = 1,2)$, then the desired inequality is obtained.

Theorem 5. *Let A and B be positive operators satisfying (5). Then*

$$\|B^{-\frac{1}{2}}AB^{\frac{1}{2}}x\|^2 \ m_r \ \|Bx\|^2 \leq K_r\|A^{\frac{1}{2}}B^{\frac{1}{2}}x\|^4$$

for all $x \in H$.

Proof. We also put $C = A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}$ and so we have (8). Hence it follows from Theorem 1 that

$$(Cx, x) \ m_r \ (C^{-1}x, x) \leq K_r\|x\|^4$$

for all $x \in H$. Replacing x in the above by $A^{\frac{1}{2}}B^{\frac{1}{2}}x$ and M, m by $M_i, m_i (i = 1, 2)$, we have the desired inequality, as in the proof of Theorem 3.

4. A concluding remark. Generalizations of the Kantorovich inequality are discussed by several authors, for which we refer to [8] and [4]. Though the former is somewhat complicated, the latter is simple as follows :

Theorem K. (Kijima) *Let A and B be positive operators satisfying (5). Then*

$$M_1m_1(A^{-1}x, x)(By, y) + M_2m_2(Ax, x)(B^{-1}y, y) \leq M_1M_2 + m_1m_2$$

for all unit vectors $x, y \in H$.

The proof of Theorem K is reduced to the following elementary inequality : If $0 < m_1 \leq a \leq M_1$ and $0 < m_2 \leq b \leq M_2$, then

$$\frac{M_2a}{m_1b} + \frac{M_1b}{m_2a} \leq 1 + \frac{M_1M_2}{m_1m_2}.$$

He also gave a path of results whose starting point is Theorem K and final one is the Pólya-Szegö inequality.

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