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SOME GENERALIZED THEOREMS ON p-QUASIHYPONORMAL OPERATORS FOR $0 < p < 1$

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ABSTRACT. Let $T = U|T|$ be the polar decomposition of p-quasihyponormal for $0 < p < 1$. Then the operator $\tilde{T}_\epsilon = |T|^\epsilon U|T|^{1-\epsilon}$, $0 < \epsilon \leq \frac{1}{2}$, is $(p+\epsilon)$-quasihyponormal if $p+\epsilon < 1$ and is quasihyponormal if $p+\epsilon \geq 1$. And we will prove that every p-quasihyponormal operator is paranormal and give an example to show that the converse is not true.

1. Introduction. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *p-hyponormal* if $(T^*T)^p \geq (TT^*)^p$ for $p > 0$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *p-quasihyponormal* if $T^*((T^*T)^p - (TT^*)^p)T \geq 0$ for $p > 0$. If $p = 1$, then T is quasihyponormal and if $p = \frac{1}{2}$, then T is semi-quasihyponormal. It is well known that a p-quasihyponormal operator is a q-quasihyponormal operator for $q \leq p$. But the converse is not true in general. Also, it is immediate that every p-hyponormal operator is p-quasihyponormal but not necessarily conversely(see [4]). Hyponormal operators and quasihyponormal operators have been studied by many authors(see [9] and [11]). The p-hyponormal operator was first introduced by A. Aluthge and he studied basic properties of p-hyponormal operators(see [1] and [2]). Recently, S. C. Arora and P. Arora [4] introduced p-quasihyponormal as a generalization of quasihyponormal, and the operator $\tilde{T} = |T|^{\frac{1}{2}} U|T|^{\frac{1}{2}}$, where $T = U|T|$ is the polar decomposition of T. And they studied some properties of p-quasihyponormal using the operator $\tilde{T} = |T|^{\frac{1}{2}} U|T|^{\frac{1}{2}}$.

For a p-quasihyponormal operator $T = U|T|$, in section 2 we will introduce the operator $\tilde{T}_\epsilon = |T|^\epsilon U|T|^{1-\epsilon}$, $0 < \epsilon \leq \frac{1}{2}$, which is $(p+\epsilon)$-quasihyponormal if $p+\epsilon < 1$ and is quasihyponormal if $p+\epsilon \geq 1$. In section 3, we will prove

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that every p-quasihyponormal operator is paranormal operator and give an example to show that the converse is not true.

2. The operator $\tilde{T}_\epsilon = |T|^\epsilon U|T|^{1-\epsilon}$, $0 < \epsilon \leq \frac{1}{2}$. Throughout this paper $0 < p < 1$. In this section we will generalize some results of [4].

We begin with the following lemma which is a generalization of [4]. Let $T = U|T|$ be the polar decomposition of T and let $\tilde{T}_\epsilon = |T|^\epsilon U|T|^{1-\epsilon}$ and $0 < \epsilon \leq \frac{1}{2}$. We denote $R(T)$ the range of T.

Lemma 1. *Let $T = U|T|$ be the polar decomposition of T. Then $R(\tilde{T}_\epsilon) \subset R(|T|)$.*

Proof. Since $|T|^r$ is positive for all $0 < r < \frac{1}{2}$, the proof is similar to Lemma 1 in [4].

The key to the proof of the following theorem is Furuta's Inequality. We start that result without a proof(see [7]).

Furuta's Inequality. *Let A and B be self-adjoint such that $A \geq B \geq 0$. Then for each $r \geq 0$,*

$$(B^r A^s B^r)^{\frac{1}{q}} \geq B^{\frac{s+2r}{q}}$$

and

$$A^{\frac{s+2r}{q}} \geq (A^r B^s A^r)^{\frac{1}{q}}$$

for each s and q such that $s \geq 0, q \geq 1$ and $(1+2r)q \geq s+2r$.

Theorem 2. *Let $T = U|T|$ be the polar decomposition of p-quasihyponormal for $0 < p < \frac{1}{2}$ and U be unitary. Then the operator $\tilde{T}_\epsilon = |T|^\epsilon U|T|^{1-\epsilon}, 0 < \epsilon \leq \frac{1}{2}$, is $(p+\epsilon)$-quasihyponormal.*

Proof. Since T is p-quasihyponormal, we have

$$\begin{aligned} T^*((T^*T)^p - (TT^*)^p)T &= |T|U^*(|T|^{2p} - |T^*|^{2p})U|T| \\ &= |T|(U^*|T|^{2p}U - |T|^{2p})|T| \\ &\geq 0. \end{aligned}$$

Thus

$$U^*|T|^{2p}U \geq |T|^{2p} \geq U|T|^{2p}U^*$$

on $R(|T|)$. Let $A = U^*|T|^{2p}U, B = |T|^{2p}, C = U|T|^{2p}U^*, r = \frac{\epsilon}{2p}, s = \frac{1-\epsilon}{p}$ and $q = \frac{1}{p+\epsilon}$ in Furuta's Inequality. Then, on $R(|T|)$,

$$\begin{aligned}
(\tilde{T}_\epsilon\tilde{T}_\epsilon^*)^{p+\epsilon} &= (|T|^\epsilon U|T|^{2(1-\epsilon)}U^*|T|^\epsilon)^{p+\epsilon} \\
&= (B^{\frac{\epsilon}{2p}}C^{\frac{1-\epsilon}{p}}B^{\frac{\epsilon}{2p}})^{p+\epsilon} \\
&= (B^rC^sB^r)^{\frac{1}{q}} \\
&\leq B^{\frac{s+2r}{q}} \\
&= |T|^{2(p+\epsilon)}.
\end{aligned}$$

Similarly, on $R(|T|)$,

$$\begin{aligned}
(\tilde{T}_\epsilon^*\tilde{T}_\epsilon)^{p+\epsilon} &= (|T|^{1-\epsilon}U^*|T|^{2\epsilon}U|T|^{1-\epsilon})^{p+\epsilon} \\
&= (B^{\frac{1-\epsilon}{2p}}A^{\frac{\epsilon}{p}}B^{\frac{1-\epsilon}{2p}})^{p+\epsilon} \\
&= (B^rA^sB^r)^{\frac{1}{q}} \\
&\geq B^{\frac{s+2r}{q}} \\
&= |T|^{2(p+\epsilon)}.
\end{aligned}$$

So,

$$(\tilde{T}_\epsilon^*\tilde{T}_\epsilon)^{p+\epsilon} \geq |T|^{2(p+\epsilon)} \geq (\tilde{T}_\epsilon\tilde{T}_\epsilon^*)^{p+\epsilon}$$

on $R(\tilde{T}_\epsilon)$. Hence

$$\tilde{T}_\epsilon^*((\tilde{T}_\epsilon^*\tilde{T}_\epsilon)^{p+\epsilon} - (\tilde{T}_\epsilon\tilde{T}_\epsilon^*)^{p+\epsilon})\tilde{T}_\epsilon \geq 0.$$

The proof is complete.

Theorem 3. *Let $T = U|T|$ be the polar decomposition of p-quasihyponormal for $\frac{1}{2} \leq p < 1$ and U be unitary. Then the operator $\tilde{T}_\epsilon = |T|^\epsilon U|T|^{1-\epsilon}, 0 < \epsilon \leq \frac{1}{2}$, is $(p+\epsilon)$-quasihyponormal if $p+\epsilon < 1$, and is quasihyponormal if $p+\epsilon > 1$.*

Proof. The proof for the case of $p+\epsilon < 1$ is similar to the proof of Theorem 2. So, we consider the case $p+\epsilon > 1$. Since $p > 1-\epsilon$, by Löwner's Theorem, the condition

$$U^*|T|^{2p}U \geq |T|^{2p} \geq U|T|^{2p}U^*$$

implies

$$U^*|T|^{2(1-\epsilon)}U \geq |T|^{2(1-\epsilon)} \geq U|T|^{2(1-\epsilon)}U^*$$

on $R(|T|)$. Thus

$$\begin{aligned}\tilde{T}_\epsilon \tilde{T}_\epsilon^* &= |T|^\epsilon U|T|^{2(1-\epsilon)}U^*|T|^\epsilon \\ &\leq |T|^\epsilon |T|^{2(1-\epsilon)}|T|^\epsilon \\ &= |T|^2\end{aligned}$$

on $R(|T|)$. Therefore

$$\tilde{T}_\epsilon^* \tilde{T}_\epsilon \geq |T|^2 \geq \tilde{T}_\epsilon \tilde{T}_\epsilon^*$$

on $R(\tilde{T}_\epsilon)$. So,

$$\tilde{T}_\epsilon^*(\tilde{T}_\epsilon^*\tilde{T}_\epsilon - \tilde{T}_\epsilon\tilde{T}_\epsilon^*)\tilde{T}_\epsilon \geq 0$$

and hence $\tilde{T}_\epsilon$ is quasihyponormal.

3. Some properties of p-quasihyponormal operators. In this section we will study the relationship between the class of p-quasihyponormal and the class of paranormal. It is well known that every p-hyponormal operator is paranormal by Ando(see [3]).

First, we will study some properties of p-quasihyponormal $T = U|T|$ using the operator $S = U|T|^p$. Ando[3] proved that $T = U|T|$ is p-hyponormal if and only if $S = U|T|^p$ is hyponormal.

Theorem 4. *Let $T = U|T|$ be a polar decomposition of T and let U be unitary. Then T is p-quasihyponormal if and only if the operator $S = U|T|^p$ is quasihyponormal.*

Proof. Let $T = U|T|$ be p-quasihyponormal and let $S = U|T|^p$. Then $S^*S = |T|^{2p}, SS^* = |T^*|^{2p}$ and $|S| = |T|^p$. Since T is p-quasihyponormal, by the proof of Theorem 2,

$$|T|(U^*|T|^{2p}U - |T|^{2p})|T| \geq 0.$$

Thus

$$U^*|T|^{2p}U \geq |T|^{2p} \geq U|T|^{2p}U^*$$

on $R(|T|) = R(|T|^p) = R(|S|)$. That is, $S^*(S^*S - SS^*)S \geq 0$ and so S is quasihyponormal.

Conversely, suppose that $S = U|T|^p$ is quasihyponormal. Then

$$S^*(S^*S - SS^*)S = |T|^pU^*(|T|^pU^*U|T|^p - U|T|^p|T|^pU^*)U|T|^p \geq 0.$$

This implies that

$$T^*(|T|^{2p} - |T^*|^{2p})T \geq 0$$

and hence T is p-quasihyponormal.

The following corollary is corresponding to [9, Theorem 3].

Corollary 5. *Every p-quasihyponormal and cohyponormal operator is normal.*

Proof. Let $T = U|T|$ be the polar decomposition of p-quasihyponormal and let $S = U|T|^p$. Then, by Theorem 4, S is quasihyponormal and cohyponormal. By [9], S is normal. Since $|T^*|^{2p} = |T|^{2p}$, we have T is normal.

McCarthy[10] proposed the following inequalities as an operator variant of the Hölder inequality.

Hölder-McCarthy Inequality. *Let A be a positive operator on $\mathcal{H}$. Then the following inequalities hold:*

(1) $(A^rx, x) \leq ||x||^{2(1-r)}(Ax, x)^r$ *for* $x \in \mathcal{H}$ *if* $0 < r \leq 1$.
(2) $(A^rx, x) \geq ||x||^{2(1-r)}(Ax, x)^r$ *for* $x \in \mathcal{H}$ *if* $r \geq 1$.

Based on this inequalities, we have

Theorem 6. *Every p-quasihyponormal operator is paranormal.*

Proof. For a unit vector x in $\mathcal{H}$, by Hölder-McCarthy Inequality,

$$\begin{aligned}(T^*(TT^*)^pTx, x) &= ((T^*T)^{p+1}x, x)\\ &\geq ((T^*T)x, x)^{p+1}\\ &= ||Tx||^{2(p+1)}\end{aligned}$$

and

$$\begin{aligned}(T^*(T^*T)^pTx, x) &\leq ||Tx||^{2(1-p)}(T^{*2}T^2x, x)^p\\ &= ||Tx||^{2(1-p)}||T^2x||^{2p}.\end{aligned}$$

Since T is p-quasihyponormal, $||Tx||^{4p} \leq ||T^2x||^{2p}$ for all unit vector x in $\mathcal{H}$. Hence T is paranormal.

According to the above theorem, every p-quasihyponormal operator is normaloid . Applying results of [6], we shall give another proof of Theorem 6. It is easy to see that Theorem 4 implies that every p-quasihyponormal operator is p-paranormal and so paranormal(see [6, Lemma 3 and Theorem 4]).

Next, we give an example to show that the class of p-quasihyponormal operators is properly contained in the class of paranormal operators.

Example 7. Let $\mathcal{K}$ be the direct sum of countably many copies of a Hilbert space $\mathcal{H}$. For given positive operators A, B definded on $\mathcal{H}$ and $n \in \mathbf{N}$, define the operator $T_{A,B,n}$ on $\mathcal{K}$ as follows

$$T_{A,B,n}(x_1, x_2, \cdots) = (0, Ax_1, Ax_2, \cdots, Ax_n, Bx_{n+1}, Bx_{n+2}, \cdots).$$

The operator $T_{A,B,n}$ is paranormal if and only if $AB^2A + 2\lambda A^2 + \lambda^2 \geq 0$ for all $\lambda \in \mathbf{R}$ and is p-quasihyponormal if and only if $AB^{2p}A \geq A^{2(1+p)}$.

Let $\mathcal{H}$ be a two-dimensional Hilbert space and let

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 3 \\ 3 & 10 \end{pmatrix}$$

and let $p = \frac{1}{2}$. Then

$$AB^2A + 2\lambda A^2 + \lambda^2 = \begin{pmatrix} 40 + 8\lambda + \lambda^2 & 0 \\ 0 & \lambda^2 \end{pmatrix}$$

which is positive for all $\lambda \in \mathbf{R}$. But

$$AB^{2p}A - A^{2(1+p)} = \begin{pmatrix} -4 & 0 \\ 0 & 0 \end{pmatrix}$$

is not positive. Therefore the operator $T_{A,B,n}$ is not semi-quasihyponormal.

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