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SOME STARLIKENESS CONDITIONS FOR ANALYTIC FUNCTIONS

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Abstract

The object of this paper is to improve some sufficient conditions for p-valently starlikeness and for starlikeness of order α. Some new starlikeness conditions are also presented.

1. Introduction

Let $\mathcal{H}$ denote the class of functions that are analytic in the open unit disk $\mathcal{U}$ and let

$$\mathcal{A} := \{f \in \mathcal{H} : f(0) = f'(0) - 1 = 0\}.$$

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Denote by $\mathcal{A}(p)$ the subclass of $\mathcal{H}$ consisting of functions

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \qquad (p \in \mathbb{N} := \{1, 2, 3, \cdots\}).$$

If

$$f(z) \in \mathcal{A}(p) \qquad (0 \leq \alpha < p)$$

and

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in \mathcal{U}),$$

then $f(z)$ is said to be *p-valently starlike of order* α. We represent the class of such functions by $\mathcal{S}_p^*(\alpha)$. Obviously, $\mathcal{S}_1^*(\alpha) = \mathcal{S}^*(\alpha)$ is the familiar class of *starlike functions of order* α.

Let $f(z) \in \mathcal{H}$ and $g(z) \in \mathcal{H}$. We say that $f(z)$ is subordinate to $g(z)$, written $f(z) \prec g(z)$ or $f \prec g$, if there exists a Schwarz function $w(z) \in \mathcal{H}$, $w(0) = 0$, and $|w(z)| < 1$ in $\mathcal{U}$, such that $f(z) = g(w(z))$. It is known that, if $g(z)$ is univalent in $\mathcal{U}$, then $f \prec g$ is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\mathcal{U}) \subset g(\mathcal{U}).$$

In this note, we improve some sufficient conditions for p-valently starlikeness and for starlikeness of order α, and also obtain some new starlikeness conditions in terms of

$$\frac{zf'(z)}{f(z)} \quad \text{and} \quad \frac{zf''(z)}{f'(z)}.$$

2. Preliminaries

We need the following lemmas to prove our results.

Lemma 1 (Miller *et al.* [5]). *Let $g(z) \in \mathcal{H}$ be a convex function in $\mathcal{U}$ (i.e., $g(z)$ is univalent and $g(\mathcal{U})$ is a convex domain). If $\gamma \neq 0$, $\Re(\gamma) \geq 0$, and $f(z) \in \mathcal{H}$, then*

$$f \prec g \Rightarrow \frac{1}{z^\gamma}\int_0^z f(t)\,t^{\gamma-1}\,dt \prec \frac{1}{z^\gamma}\int_0^z g(t)\,t^{\gamma-1}\,dt. \tag{1}$$

If $f(z) \in \mathcal{H}$ with $f(0) = 0$, then (1) *holds true in the case $\gamma = 0$.*

Lemma 2 (Miller and Mocanu [4]). *Let $s > 0$, $A, B \in [-1,1]$, and $A \neq B$. If $p(z) \in \mathcal{H}$ satisfies*

$$p(z) + \frac{zp'(z)}{sp(z)} \prec \frac{1+Az}{1+Bz},$$

then

$$p(z) \prec q(z) \prec \frac{1+Az}{1+Bz},$$

where $q(z)$ given by

$$q(z) = \begin{cases} \dfrac{z^s(1+Bz)^{s(A-B)/B}}{s\int\limits_0^z t^{s-1}(1+Bt)^{s(A-B)/B}\,dt} & (B \neq 0) \\[2ex] \dfrac{z^s\, e^{sAz}}{s\int\limits_0^z t^{s-1}\, e^{sAt}\,dt} & (B = 0) \end{cases}$$

is the best dominant.

Lemma 3 (Jack [2]). *Let $w(z) \in \mathcal{H}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at z_0, then*

$$z_0 w'(z_0) = m w(z_0) \qquad (m \geq 1).$$

3. Two Sufficient Conditions for p-Valently Starlikeness

Nunokawa [7] proved that each of the following two conditions:

$$\left|1 + \frac{zf''(z)}{f'(z)}\right| < \frac{1}{p}\left|\frac{zf'(z)}{f(z)}\right| \log(4e^{p-1}) \qquad (f \in \mathcal{A}(p);\ z \in \mathcal{U}) \tag{2}$$

and

$$\left|1 + \frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right| < \left|\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right| \log 4 \qquad (f \in \mathcal{A}(p);\ z \in \mathcal{U}) \tag{3}$$

implies that $f(z)$ is p-valently starlike in $\mathcal{U}$, that is, $f(z) \in \mathcal{S}_p^*(0) = \mathcal{S}_p^*$.
Nunokawa [7] also proposed the problem of finding the best constants in the above two starlikeness conditions. Subsequently, Nunokawa [10, pp. 206-211] showed that the same conclusion holds true if the constants in (2) and (3) are replaced by $(p + \frac{1}{2})/p$ and $\frac{3}{2}$, respectively. In this section, we shall improve the above results as follows.

Theorem 1. *Let $f(z) \in \mathcal{A}(p)$ with $f(z) \neq 0$ for $0 < |z| < 1$. If $f(z)$ satisfies*

$$\left|1 + \frac{zf''(z)}{f'(z)}\right| < \left|\frac{zf'(z)}{f(z)}\right| k(p) \qquad (z \in \mathcal{U}), \tag{4}$$

where $k(p) \in (1,2)$ is the unique root of the equation:

$$1 + p(1-k^2)\ln\left(1 + \frac{1}{k}\right) = 0, \tag{5}$$

then $f(z) \in S_p^*$.

Proof. Let $g(z) = p\,f(z)/(zf'(z))$. Then, from the assumption, $g(z) \in \mathcal{H}$ with $g(0) = 1$ and $g(z) \neq 0$ in $\mathcal{U}$, and

$$|p - zg'(z)| < p\,k(p) \qquad (z \in \mathcal{U}). \tag{6}$$

Let $w(z)$ be defined by

$$zg'(z) = (1-k^2)\,\frac{p\,w(z)}{k + w(z)} \qquad (z \in \mathcal{U}), \tag{7}$$

where $k = k(p) > 1$.

It follows from (6) that

$$\left|\frac{1 + k\,w(z)}{k + w(z)}\right| < 1 \qquad (z \in \mathcal{U}),$$

which is equivalent to $|w(z)| < 1$ in $\mathcal{U}$.

From (7), we have

$$zg'(z) \prec (1-k^2)\,\frac{p\,z}{k+z}, \tag{8}$$

and by Lemma 1, we easily get

$$g(z) \prec 1 + p(1-k^2)\ln\left(1 + \frac{z}{k}\right). \tag{9}$$

Since

$$\Re\left\{1 + p(1-k^2)\ln\left(1 + \frac{z}{k}\right)\right\} > 1 + p(1-k^2)\ln\left(1 + \frac{1}{k}\right) = 0 \qquad (z \in \mathcal{U}),$$

we deduce from (9) that

$$\Re\{g(z)\} > 0 \qquad (z \in \mathcal{U}),$$

which shows that $f(z) \in \mathcal{S}_p^*$. This evidently completes the proof of Theorem 1.

It may be of interest to note that, if we let

$$\phi(k) = 1 + p(1-k^2)\ \ln\left(1+\frac{1}{k}\right),$$

then

$$\phi'(k) < 0 \quad (k \in (1,\infty)), \quad \phi(2) < 0, \quad \text{and} \quad \phi\left(1+\frac{7}{10p}\right) > 0 \quad (p \in \mathbb{N}).$$

Thus the constant in (4) satisfies the inequality:

$$k(p) > \frac{1}{p}\left(p+\frac{7}{10}\right).$$

In the cases when $p = 1$ and $p = 2$, we have

$$k(1) = 1.809\ldots \quad \text{and} \quad k(2) = 1.3857\ldots.$$

Theorem 2. *Let $f(z) \in \mathcal{A}(p)$ with $f^{(p-1)}(z) \neq 0$ in $0 < |z| < 1$. If f satisfies*

$$\left|1 + \frac{z\,f^{(p+1)}(z)}{f^{(p)}(z)}\right| < \left|\frac{z\,f^{(p)}(z)}{f^{(p-1)}(z)}\right|\ k(1) \qquad (z \in \mathcal{U}), \tag{10}$$

then $f(z) \in \mathcal{S}_p^*(p-1)$.

Proof. Let $g(z) = f^{(p-1)}(z)/p!$. Then $g(z) \in \mathcal{A}(1) = \mathcal{A}$, $g(z) \neq 0$ in $0 < |z| < 1$, and

$$\left|1 + \frac{z\,g''(z)}{g'(z)}\right| < \left|\frac{zg'(z)}{g(z)}\right|\ k(1) \qquad (z \in \mathcal{U}).$$

Applying Theorem 1, we have

$$\Re\left\{\frac{z\,f^{(p)}(z)}{f^{(p-1)}(z)}\right\} > 0 \qquad (z \in \mathcal{U}). \tag{11}$$

Let

$$q_j(z) = \frac{z\,f^{(p-j)}(z)}{(j+1)\,f^{(p-j-1)}(z)} \qquad (j = 0,1,\cdots,p-1).$$

Then $q_j(0) = 1$ and

$$q_j(z) + \frac{z\,q_j'(z)}{(j+1)\,q_j(z)} = \frac{1+j\,q_{j-1}(z)}{j+1} \qquad (j = 1,2,\cdots,p-1). \tag{12}$$

It follows from (11) and a result of Nunokawa [10, pp. 206-211] that

$$\Re\{q_j(z)\} > 0 \qquad (z \in \mathcal{U};\ j = 0, 1, \cdots, p-1)$$

i.e.,

$$\frac{(j+1)!}{p!} f^{(p-j-1)}(z) \in \mathcal{S}^*_{j+1} \qquad (j = 0, 1, \cdots, p-1).$$

However, if we apply Lemma 2, we find that (11) or $q_0(z) \prec (1+z)/(1-z)$ implies that

$$q_j(z) + \frac{zq_j'(z)}{(j+1)q_j(z)} \prec \frac{1}{j+1}\left(j + \frac{1+z}{1-z}\right) \qquad (j = 1, \cdots, p-1), \tag{13}$$

which yields

$$\Re\{(j+1)\, q_j(z)\} > j \qquad (j = 1, \cdots, p-1;\ z \in \mathcal{U})$$

or, equivalently,

$$\frac{(j+1)!}{p!} f^{(p-j-1)}(z) \in \mathcal{S}^*_{j+1}(j) \qquad (j = 1, \cdots, p-1;\ z \in \mathcal{U}).$$

Hence the required result follows. This completes the proof of Theorem 2.

4. Sufficient Conditions for Starlikeness

Let $f(z) \in \mathcal{A}$, $\beta + \gamma \geq 0$, and $\beta \geq 0$. Obradović and Owa [10, pp. 220-233] considered a problem of Ruscheweyh and some related topics. They showed that, if

$$\left|\frac{zf''(z)}{f'(z)}\right|^{\beta} \left|\frac{zf'(z)}{f(z)} - 1\right|^{\gamma} < \frac{1}{2^{\beta}\, 3^{\gamma}} \qquad (z \in \mathcal{U}), \tag{14}$$

then $f(z)$ is starlike in $\mathcal{U}$, that is, $f(z) \in \mathcal{S}^*(0) = \mathcal{S}^*$. In this section, we shall extend this result and some related results.

Theorem 3. *Let $f(z) \in \mathcal{A}$ with $f(z) \neq 0$ in $0 < |z| < 1$, $\frac{1}{2} \leq \alpha < 1$, $\beta + \gamma \geq 0$, and $\beta \geq 0$. If $f(z)$ satisfies the inequality:*

$$\left|\frac{zf''(z)}{f'(z)}\right|^{\beta} \left|\frac{zf'(z)}{f(z)} - 1\right|^{\gamma} < 2^{\beta}(1-\alpha)^{\beta+\gamma} \qquad (z \in \mathcal{U}), \tag{15}$$

then

$$f(z) \in S^*(\alpha) \quad \text{and} \quad \frac{zf'(z)}{f(z)} \prec \frac{\alpha}{\alpha - (1-\alpha)\,z}.$$

Proof. Let us put

$$\frac{zf'(z)}{f(z)} = \frac{\alpha}{\alpha - (1-\alpha)\,w(z)}. \tag{16}$$

Then $w(z) \in \mathcal{H}$ with $w(0) = 0$ and, by an easy calculation, we have

$$\frac{|zw'(z) + w(z)|^{\beta}\,|w(z)|^{\gamma}}{|\alpha - (1-\alpha)\,w(z)|^{\beta+\gamma}} < 2^{\beta} \qquad (z \in \mathcal{U}). \tag{17}$$

To prove Theorem 3, it suffices to show that $|w(z)| < 1$ in $\mathcal{U}$. If this is not the case, then there is a point z_0 with $|z_0| = \rho < 1$, such that

$$\max_{|z| \leq \rho} |w(z)| = |w(z_0)| = 1.$$

By Lemma 3, there exists a real number $m \geq 1$ such that

$$z_0 w'(z_0) = m w(z_0).$$

Thus we get

$$\frac{|z_0 w'(z_0) + w(z_0)|^{\beta}\,|w(z_0)|^{\gamma}}{|\alpha - (1-\alpha)\,w(z_0)|^{\beta+\gamma}} = \frac{(m+1)^{\beta}}{|\alpha - (1-\alpha)\,w(z_0)|^{\beta+\gamma}} \geq 2^{\beta}, \tag{18}$$

which contradicts (17). Therefore, we have $|w(z)| < 1$ in $\mathcal{U}$, and it follows from (16) that

$$\frac{zf'(z)}{f(z)} \prec \frac{\alpha}{\alpha - (1-\alpha)\,z} \quad \text{and} \quad \Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in \mathcal{U})$$

This completes the proof of Theorem 3.

The following theorem can be proved along similar lines and so we omit its proof.

Theorem 4. *Let $f(z) \in \mathcal{A}$ with $f(z) \neq 0$ in $0 < |z| < 1$, $\frac{1}{2} \leq \alpha < 1$, $\beta > 0$, and $\beta + \gamma \geq 0$. If $f(z)$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\right|^{\beta} \left|\frac{zf'(z)}{f(z)}\right|^{\gamma} < 2^{\beta}\,\alpha^{\gamma}(1-\alpha)^{\beta} \qquad (z \in \mathcal{U}), \tag{19}$$

then $f(z) \in S^*(\alpha)$ and

$$\frac{zf'(z)}{f(z)} \prec \frac{\alpha}{\alpha - (1-\alpha)\,z} \qquad (z \in \mathcal{U}).$$

Several cases of Theorem 3 and Theorem 4, for special values of the parameters α, β, and γ, will improve some other interesting known results.

In the rest of this section we restrict our discussion to the case when $\beta = 1$ and $\gamma = 0$. It follows from Theorem 3 or Theorem 4 that, if $\frac{1}{2} \le \alpha < 1$, then

$$\left|\frac{zf''(z)}{f'(z)}\right| < 2(1-\alpha) \Rightarrow f(z) \in \mathcal{S}^*(\alpha) \quad \text{and} \quad \frac{zf'(z)}{f(z)} \prec \frac{\alpha}{\alpha - (1-\alpha)z}. \tag{20}$$

Owa [8] showed that

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{(1-\alpha)(3-\alpha)}{2-\alpha} \Longrightarrow f(z) \in \mathcal{S}^*(\alpha) \qquad (0 \le \alpha < 1).$$

In the case when $0 < \alpha < \frac{2}{3}$, Li Jian Lin [3] improved Owa's result and showed that

$$\left|\frac{zf''(z)}{f'(z)}\right| < M(\alpha) \Longrightarrow f(z) \in \mathcal{S}^*(\alpha) \qquad \left(0 < \alpha < \frac{2}{3}\right),$$

where

$$M(\alpha) := \begin{cases} \frac{3}{2} - \alpha & (0 < \alpha < \frac{1}{2}) \\ (1-\alpha)\left(1 + \frac{1}{2\alpha}\right) & (\frac{1}{2} \le \alpha < \frac{2}{3}). \end{cases}$$

Obviously, in the case when $\frac{1}{2} \le \alpha < 1$, (20) is much better than Owa's assertions. The following theorem will improve this result further.

Theorem 5. *Let $f(z) \in \mathcal{H}$ with $f(z) \neq 0$ in $0 < |z| < 1$.*

(i) *If*

$$0 < \mu \le 1 \quad \textit{and} \quad \left|\frac{f''(z)}{f'(z)}\right| \le \mu \qquad (z \in \mathcal{U}),$$

then

$$f(z) \in \mathcal{S}^*(\rho) \qquad \left(\rho = \frac{\mu}{e^\mu - 1}\right).$$

The value of ρ is the best possible.

(ii) *If*

$$\frac{1}{e-1} \le \alpha < 1 \quad \textit{and} \quad \left|\frac{f''(z)}{f'(z)}\right| \le \mu(\alpha),$$

where $\mu(\alpha) \in (0,1]$ is the unique root of the equation

$$\mu + \alpha(1 - e^\mu) = 0,$$

then $f(z) \in \mathcal{S}^*(\alpha)$. This sufficient condition for starlikeness of order α is sharp.

Proof. (i) If we put

$$p(z) = \frac{zf'(z)}{f(z)} \qquad (z \in \mathcal{U}),$$

then

$$\left|\frac{f''(z)}{f'(z)}\right| \leq \mu$$

reduces to

$$p(z) + \frac{zp'(z)}{p(z)} \prec 1 + \mu z. \tag{21}$$

By using Lemma 2, we have the best dominant

$$p(z) \prec \frac{\mu z e^{\mu z}}{e^{\mu z} - 1}. \tag{22}$$

Ruscheweyh and Singh [9] showed that

$$g(z) := \frac{e^{\mu z} - 1}{\mu} \in \mathcal{S}^*(\rho),$$

that is, that

$$\Re\left\{\frac{\mu z e^{\mu z}}{e^{\mu z} - 1}\right\} > \frac{\mu}{e^{\mu} - 1} \qquad (z \in \mathcal{U}),$$

and this result is sharp. Combining this result with (22), we have the desired assertion of Theorem 5(i).

(ii) Let

$$\phi(\mu) = \frac{\mu}{e^{\mu} - 1}.$$

Then

$$\phi'(\mu) < 0 \quad (0 < \mu < 1), \quad \phi(0) = 1, \quad \text{and} \quad \phi(1) = \frac{1}{e-1}.$$

If

$$\frac{1}{e-1} \leq \alpha < 1,$$

then (ii) follows from (i) by letting

$$\frac{\mu}{e^{\mu} - 1} = \alpha.$$

This completes the proof of Theorem 5.

Miller and Mocanu [10, pp. 171-178] showed that, if $\mu \leq 4.046\ldots$, then (21) implies (22). Hence, if

$$\mu(\alpha) = \sup\left\{\mu : \frac{e^{\mu z}-1}{\mu} \in S^*(\alpha)\right\},$$

then

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \mu(\alpha) \Longrightarrow f(z) \in S^*(\alpha) \qquad (0 \leq \alpha < 1),$$

and this sufficient condition is sharp. Anisiu and Mocanu [1] proved that $\mu(0) = 2.83\ldots$.

5. A Result Involving the Alexander Integral Operator

The integral operator introduced by J.W. Alexander is defined by

$$f(z) = \int_0^z \frac{g(t)}{t}\,dt \qquad (z \in \mathcal{U};\ g(z) \in \mathcal{A}). \tag{23}$$

Recently, Mocanu [6] proved that, if $g(z)$ satisfies the inequality:

$$|g'(z) - 1| < \frac{8}{2+\sqrt{15}} \qquad (z \in \mathcal{U}),$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 \qquad (z \in \mathcal{U}).$$

In this section, we shall improve this result to the following form.

Theorem 6. *For $f(z)$ and $g(z)$ given by* (23), *let*

$$|g'(z) - 1| < \frac{3}{2} \qquad (z \in \mathcal{U}). \tag{24}$$

Then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 \qquad (z \in \mathcal{U}).$$

Proof. It is easily seen from (24) that

$$g'(z) \prec 1 + \frac{3}{2}z.$$

By Lemma 1, we also have

$$f'(z) \prec 1 + \frac{3}{4}\, z. \tag{25}$$

Applying Lemma 1 once again, we obtain

$$\frac{f(z)}{z} \prec 1 + \frac{3}{8}\, z$$

or, equivalently,

$$\left|\frac{f(z)}{z} - 1\right| < \frac{3}{8} \qquad (z \in \mathcal{U}). \tag{26}$$

Let

$$f'(z) - \frac{f(z)}{z} = \frac{5}{8}\, w(z). \tag{27}$$

Then $w(z) \in \mathcal{H}$ with $w(0) = 0$. If we suppose that there exists a point $z_0 \in \mathcal{U}$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1,$$

then, from Lemma 3, we have

$$z_0 w'(z_0) = m\ w(z_0) \qquad (m \geq 1).$$

Since

$$g'(z) = \frac{f(z)}{z} + \frac{5}{8}\, \{z w'(z) + 2w(z)\}\,,$$

we have

$$|g'(z_0) - 1| = \left|\frac{f(z_0)}{z_0} - 1 + \frac{5}{8}\,(m+2)\, w(z_0)\right| < \frac{3}{2}. \tag{28}$$

The inequality in (28) yields

$$\left|\frac{f(z_0)}{z_0} - 1\right| > \frac{5}{8}\,(m+2) - \frac{3}{2} \geq \frac{3}{8},$$

which contradicts (26). Therefore, we have $|w(z)| < 1$ $(z \in \mathcal{U})$ or, equivalently,

$$\left|f'(z) - \frac{f(z)}{z}\right| < \frac{5}{8} \qquad (z \in \mathcal{U}). \tag{29}$$

Finally, we observe from (26) that

$$\left|\frac{f(z)}{z}\right| > \frac{5}{8} \qquad (z \in \mathcal{U}). \tag{30}$$

The inequality (30), in conjunction with (29), yields

$$\left|f'(z) - \frac{f(z)}{z}\right| < \left|\frac{f(z)}{z}\right| \qquad (z \in \mathcal{U}),$$

which shows that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 \qquad (z \in \mathcal{U}).$$

This evidently completes the proof of Theorem 6.

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