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Weighted composition operators between weighted Bergman spaces in the unit ball of $\mathbb{C}^n$

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Abstract

Let φ be a holomorphic self-map of the unit ball B in $\mathbb{C}^n$ and ψ a holomorphic function in B. Let $A^p(\nu_\alpha)$ denote the weighted Bergman space in B. In this paper, we characterize the boundedness and the compactness of the weighted composition operator $W_{\varphi,\psi} : f \mapsto \psi(f \circ \varphi)$ from $A^p(\nu_\alpha)$ into $A^q(\nu_\beta)$ $(0 < p \leq q < \infty, -1 \leq \alpha, \beta < \infty)$, in terms of the Carleson-type measures. We also consider the boundedness and the compactness of $W_{\varphi,\psi} : A^p(\nu_\alpha) \to H^\infty(B)$, the space of the bounded holomorphic functions in B.

1 Introduction

Throughout this paper, let n be a fixed integer. Let B and S denote the unit ball and the unit sphere of the complex n-dimensional Euclidean space $\mathbb{C}^n$, respectively. Let ν and σ denote the normalized Lebesgue measures on B and S, respectively. For each $\alpha \in (-1, \infty)$, we set $c_\alpha = \Gamma(n+\alpha+1)/\{\Gamma(n+1)\Gamma(\alpha+1)\}$ and $d\nu_\alpha(z) = c_\alpha(1-|z|^2)^\alpha d\nu(z)$ $(z \in B)$. Note that $\nu_\alpha(B) = 1$. Let $H(B)$ denote the space of all holomorphic functions in B. For each $p \in (0, \infty)$ and $\alpha \in (-1, \infty)$, the *weighted Bergman space* $A^p(\nu_\alpha)$ and the *Hardy space* $H^p(B)$ are defined by

$$A^p(\nu_\alpha) = \left\{ f \in H(B) \,:\, \|f\|^p_{A^p(\nu_\alpha)} \equiv \int_B |f|^p d\nu_\alpha < \infty \right\},$$

$$H^p(B) = \left\{ f \in H(B) \,:\, \|f\|^p_{H^p} \equiv \sup_{0<r<1} \int_S |f_r|^p d\sigma < \infty \right\},$$

where $f_r(z) = f(rz)$ for $r \in (0,1)$, $z \in \mathbb{C}^n$ with $rz \in B$. For convenience' sake, the spaces $H^p(B)$ are denoted by the symbols $A^p(\nu_{-1})$ $(0 < p < \infty)$. Note that $\lim_{\alpha\downarrow -1} \|f\|_{A^p(\nu_\alpha)} = \|f\|_{H^p}$ for $p \in (0,\infty)$ and $f \in H(B)$. (See [1, §0.3 and p.25].)

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If φ is a holomorphic self-map of B and $\psi \in H(B)$, then φ and ψ define the linear operator $W_{\varphi,\psi}$ on $H(B)$ by means of the equation $W_{\varphi,\psi}f = \psi \cdot (f \circ \varphi)$. This operator $W_{\varphi,\psi}$ is called the *weighted composition operator* induced by φ and ψ.

In the case of the dimension $n = 1$, G. Mirzakarimi and K. Seddighi [8] have studied the weighted composition operators on $A^p(\nu_\alpha)$ $(0 < p < \infty,\ -1 < \alpha < \infty)$. Recently, $W_{\varphi,\psi}$ on H^p $(1 \le p < \infty)$ has been studied by M. D. Contreras and A. G. Hernández-Díaz [2]. Subsequently, they also characterized when $W_{\varphi,\psi} : H^p \to H^q$ $(1 \le p \le q < \infty)$ is bounded or compact, in terms of the Carleson measure [3]:

Theorem ([3]). *Let $1 \le p \le q < \infty$. Suppose that φ is a holomorphic self-map of the unit disk $\mathbb{D}$ and $\psi \in H^q$. Define the measure $\mu_{\varphi,\psi,q}$ on $\overline{\mathbb{D}}$ by*

$$\mu_{\varphi,\psi,q}(E) = \int_{\varphi^{*-1}(E)} |\psi^*|^q \, d\sigma,$$

for all Borel sets E of $\overline{\mathbb{D}}$. Here $\varphi^* : \partial\mathbb{D} \to \overline{\mathbb{D}}$ is the radial limit map of φ.

(a) $W_{\varphi,\psi} : H^p \to H^q$ is bounded if and only if $\mu_{\varphi,\psi,q}$ is a $\frac{q}{p}$-Carleson measure on $\overline{\mathbb{D}}$.

(b) $W_{\varphi,\psi} : H^p \to H^q$ is compact if and only if $\mu_{\varphi,\psi,q}$ is a comact $\frac{q}{p}$-Carleson measure on $\overline{\mathbb{D}}$.

In this paper, we consider the boundedness and the compactness of $W_{\varphi,\psi} : A^p(\nu_\alpha) \to A^q(\nu_\beta)$ $(0 < p \le q < \infty, -1 < \alpha, \beta < \infty)$ in the higher dimensional case $n \ge 1$. We also prove the similar results on the operators $W_{\varphi,\psi} : H^p(B) \to H^q(B)$ $(0 < p \le q < \infty)$ and $W_{\varphi,\psi} : H^p(B) \to A^q(\nu_\alpha)$ $(0 < p \le q < \infty, -1 < \alpha < \infty)$.

In Section 3, we characterize the boundedness of $W_{\varphi,\psi}$. In Section 4, we discuss the compactness of $W_{\varphi,\psi}$. Moreover, in Section 5, we consider the boundedness and the compactness of $W_{\varphi,\psi} : A^p(\nu_\alpha) \to H^\infty(B)$ $(0 < p < \infty, -1 \le \alpha < \infty)$, where $H^\infty(B)$ is the space of all bounded holomorphic functions on B, endowed with the norm $\|f\|_\infty \equiv \sup_{z \in B} |f(z)|$.

2 Preliminaries

In order to prove our main result, we will need some notations and lemmas.

Lemma 2.1. *Let $0 < p < \infty$ and $-1 \le \alpha < \infty$. Suppose $f \in H(B)$ and $z \in B$. Then*

$$|f(z)| \le \left(\frac{1}{1 - |z|^2}\right)^{\frac{n+1+\alpha}{p}} \|f\|_{A^p(\nu_\alpha)},$$

where $\|f\|_{A^p(\nu_{-1})} = \|f\|_{H^p(B)}$.

The proof of this lemma is essentially the same as [5, Lemma 3.2].

Let φ_z $(z \in B)$ be the biholomorphic involution of B described in [10, p.25]. For $z \in B$ and $0 < r < 1$, we set $E(z,r) = \varphi_z(rB)$. According to [10, p.29, §2.2.7], $E(z,r)$ consists of all $w \in B$ that satisfy

$$\frac{|P_z w - c|^2}{(r\rho)^2} + \frac{|w - P_z w|^2}{r^2 \rho} < 1,$$

where $P_z w = \frac{\langle w,z\rangle}{\langle z,z\rangle} z$, $c = \frac{(1-r^2)z}{1-(r|z|)^2}$ and $\rho = \frac{1-|z|^2}{1-(r|z|)^2}$.

For $\zeta \in S$ and $\delta > 0$, we introduce the *Carleson set* $\mathcal{S}(\zeta, \delta)$ in $\overline{B}$:

$$\mathcal{S}(\zeta,\delta) = \left\{ z \in \overline{B} : |1 - \langle z, \zeta \rangle| < \delta \right\}.$$

Furthermore, we put $B(\zeta,\delta) = \mathcal{S}(\zeta,\delta) \cap B$ and $S(\zeta,\delta) = \mathcal{S}(\zeta,\delta) \cap S$.

Lemma 2.2. *For any $z \in B$ and $0 < r < 1$, there exist $\zeta \in S$ and $\delta > 0$ with $E(z,r) \subset B(\zeta,\delta)$. Furthermore, $\delta \sim 1 - |z|^2$.*

Proof. By a simple computation, this lemma is easily verified. See [12, Lemma 1.3]. □

Lemma 2.3. *Let $0 < p \leq q < \infty$ and $-1 < \alpha < \infty$. Suppose that μ is a positive Borel measure on B with*

$$\mu(B(\zeta,\delta)) \leq C\delta^{\frac{q(n+1+\alpha)}{p}} \quad (\zeta \in S, \delta > 0), \tag{2.1}$$

for some constant $C > 0$. Then there is a positive constant K such that

$$\left[\int_B |f|^q d\mu \right]^{\frac{1}{q}} \leq K \|f\|_{A^p(\nu_\alpha)} \quad (f \in A^p(\nu_\alpha)). \tag{2.2}$$

Proof. Let $z \in B$ and $\frac{1}{2} < r < 1$ be fixed. By Lemma 2.2, there exist $\zeta \in S$ and $\delta > 0$ such that

$$E(z,r) \subset B(\zeta,\delta), \quad \delta \sim 1 - |z|^2. \tag{2.3}$$

By (2.1) and (2.3), we obtain

$$\mu(E(z,r)) \leq C'(1 - |z|^2)^{\frac{q(n+1+\alpha)}{p}}, \tag{2.4}$$

for some constant $C' > 0$. Now, take $f \in A^p(\nu_\alpha)$. Since $|f|^q$ is a nonnegative $\mathcal{M}$-subharmonic function in B, it follows from [11, p.33, (4.3) and (4.4)] that for $z \in B$

$$|f(z)|^q \leq 3^n \int_{E(z,\frac{1}{2})} |f(w)|^q (1 - |w|^2)^{-n-1} d\nu(w). \tag{2.5}$$

By Lemma 2.1, (2.5) and Fubini's theorem, we have

$$\int_B |f(z)|^q d\mu(z)$$

$$\leq 3^n \int_B d\mu(z) \int_{E(z,\frac{1}{2})} |f(w)|^q (1-|w|^2)^{-n-1} d\nu(w)$$

$$\leq 3^n \|f\|_{A^p(\nu_\alpha)}^{q-p} \int_B |f(w)|^p (1-|w|^2)^{\alpha - \frac{q(n+1+\alpha)}{p}} d\nu(w) \int_B \chi_{E(z,\frac{1}{2})}(w)\, d\mu(z). \tag{2.6}$$

Since $\chi_{E(z,\frac{1}{2})}(w) \leq \chi_{E(w,r)}(z)$ for each $(z,w) \in B \times B$, (2.4) and (2.6) give

$$\int_B |f(z)|^q d\mu(z) \leq 3^n \|f\|_{A^p(\nu_\alpha)}^{q-p} \int_B |f(w)|^p (1-|w|^2)^{\alpha - \frac{q(n+1+\alpha)}{p}} \mu(E(w,r))\, d\nu(w)$$

$$\leq 3^n C' \|f\|_{A^p(\nu_\alpha)}^{q-p} \int_B |f(w)|^p (1-|w|^2)^{\alpha} d\nu(w) = \frac{3^n C'}{c_\alpha} \|f\|_{A^p(\nu_\alpha)}^{q}.$$

This proves (2.2). □

Remark. By a careful computation, we see that the constant K of (2.2) is taken to be the product of $C^{1/q}$ and a positive constant depending on α and the dimension n.

The proof of the following lemma is essentially the same as that of S. C. Power's theorem in [9].

Lemma 2.4. *Let $0 < p \leq q < \infty$. Suppose that μ is a positive Borel measure on B and there exists a constant $C > 0$ such that*

$$\mu(B(\zeta,\delta)) \leq C\delta^{\frac{qn}{p}} \quad (\zeta \in S, \delta > 0). \tag{2.7}$$

Then there exists a constant $K > 0$ such that

$$\left[\int_B |f|^q d\mu\right]^{\frac{1}{q}} \leq K\|f\|_{H^p} \quad (f \in H^p(B)). \tag{2.8}$$

Proof. Fix $f \in H^p(B)$ and $t > 0$. By the same argument as the proof of Theorem in [9, pp.14 - 15], it follows from (2.7) that there exists a constant $C' > 0$ such that

$$\mu(\{z \in B : |f(z)| \geq t\}) \leq C'\Big[\sigma(\{\zeta \in S : Mf(\zeta) \geq t\})\Big]^{\frac{q}{p}}, \tag{2.9}$$

where Mf is the *admissible maximal function* of f which is defined by

$$Mf(\zeta) = \sup\Big\{|f(z)| : z \in \mathbf{C}^n, |1 - \langle z,\zeta\rangle| < 1 - |z|^2\Big\},$$

for $\zeta \in S$. By (2.9), we have

$$\int_B |f|^q d\mu = q\int_0^\infty \mu\{|f| > t\} t^{q-1} dt \leq C' q \int_0^\infty \sigma\{Mf \geq t\}^{\frac{q}{p}} t^{q-1} dt. \tag{2.10}$$

Since $f \in H^p(B)$ $(0 < p < \infty)$, it follows from [10, Theorem 5.6.5] that

$$\sigma\{Mf \geq t\}^{\frac{q}{p}-1} t^{q-p} \leq \left[\int_S \{Mf(\zeta)\}^p d\sigma(\zeta)\right]^{\frac{q}{p}-1} \leq \left[C_p \|f\|_{H^p}^p\right]^{\frac{q}{p}-1}, \tag{2.11}$$

for some positive constant C_p depending on p and n. By (2.10) and (2.11), we have

$$\begin{aligned}
\int_B |f|^q d\mu &\le C'q\int_0^\infty \sigma\{Mf \ge t\}^{\frac{q}{p}} t^{q-1}dt \\
&\le C'\frac{q}{p}\left[C_p\|f\|_{H^p}^p\right]^{\frac{q}{p}-1} p\int_0^\infty \sigma\{Mf \ge t\}t^{p-1}dt \\
&\le C'\frac{q}{p}\left[C_p\|f\|_{H^p}^p\right]^{\frac{q}{p}-1} 2^{p-1}p\int_0^\infty \sigma\left\{Mf > \frac{t}{2}\right\}\left(\frac{t}{2}\right)^{p-1} dt \\
&= C'\frac{q}{p}\left[C_p\|f\|_{H^p}^p\right]^{\frac{q}{p}-1} 2^p \int_S \{Mf(\zeta)\}^p d\sigma(\zeta) \le C'\frac{q}{p}C_p{}^{\frac{q}{p}}2^p\|f\|_{H^p}^q.
\end{aligned}$$

This completes the proof. □

Lemma 2.5. *Let $0 < p \le q < \infty$. Suppose that μ is a positive Borel measure on S such that*

$$\mu(S(\zeta,\delta)) \le C\delta^{\frac{qn}{p}} \quad (\zeta \in S, \delta > 0), \tag{2.12}$$

for some constant $C > 0$.

(a) *If $p = q$, then there exist a $g \in L^\infty(S)$ and a constant $C' > 0$ (C' is the product of C and a constant depending only on n) such that $d\mu = gd\sigma$ and $\|g\|_{L^\infty} \le C'$.*

(b) *If $p < q$, then $\mu \equiv 0$ for all Borel sets of S.*

Proof. (cf. [6, p.238, Lemma 1.3].) Since $\sigma(S(\zeta,\delta)) \sim \delta^n$ (see [10, p.67, Proposition 5.1.4]), there exist positive constants C_1 and C_2 depending only on n, such that

$$C_1\delta^n \le \sigma(S(\zeta,\delta)) \le C_2\delta^n \quad (\zeta \in S, \delta > 0). \tag{2.13}$$

By (2.12) and (2.13), we see that for all $\zeta \in S$ and $\delta > 0$

$$\frac{\mu(S(\zeta,\delta))}{\sigma(S(\zeta,\delta))} \le \frac{C}{C_1}\delta^{n(\frac{q}{p}-1)}. \tag{2.14}$$

Note that $S(\zeta,\delta) = S$ when $\delta > 2$. Thus $\mu(S(\zeta,\delta))/\sigma(S(\zeta,\delta)) = \mu(S) = \mu(S(\zeta,3)) \le 3^{\frac{qn}{p}}C$ if $\delta > 2$. If $0 < \delta \le 2$, then we have $\mu(S(\zeta,\delta))/\sigma(S(\zeta,\delta)) \le \frac{2^{n(\frac{q}{p}-1)}C}{C_1}$, by (2.14). Hence we see that the maximal function $M\mu$ of the positive measure μ satisfies $M\mu(\zeta) < \infty$ for all $\zeta \in S$. By [10, Theorem 5.2.7 and Theorem 5.3.1], we obtain $d\mu = gd\sigma$ for some $g \in L^1(\sigma)$ and

$$g(\zeta) = \lim_{\delta\downarrow 0}\frac{1}{\sigma(S(\zeta,\delta))}\int_{S(\zeta,\delta)} g\, d\sigma \le M\mu(\zeta) \equiv \sup_{\delta>0}\frac{\mu(S(\zeta,\delta))}{\sigma(S(\zeta,\delta))}, \tag{2.15}$$

for a.e. $\zeta \in S$.

In the case $p = q$, it follows from (2.14) and (2.15) that $g \in L^\infty(S)$ and $\|g\|_{L^\infty} \le \frac{C}{C_1}$. This proves (a).

In the case $p < q$, by (2.14), we have

$$0 \le \frac{1}{\sigma(S(\zeta,\delta))} \int_{S(\zeta,\delta)} g\, d\sigma = \frac{\mu(S(\zeta,\delta))}{\sigma(S(\zeta,\delta))} \le \frac{C}{C_1} \delta^{n(\frac{q}{p}-1)},$$

for all $\zeta \in S$ and $\delta > 0$. As $\delta \downarrow 0$, we have $g = 0$ a.e. on S. Thus $\mu \equiv 0$. This completes the proof. □

Lemma 2.6. *Let $0 < p \le q < \infty$. Suppose that μ is a positive Borel measure on $\overline{B}$ such that*

$$\mu(S(\zeta,\delta)) \le C\delta^{\frac{qn}{p}} \quad (\zeta \in S, \delta > 0), \tag{2.16}$$

for some constant $C > 0$. Then there exists a constant $K > 0$ such that

$$\left[\int_{\overline{B}} |f^*|^q d\mu\right]^{\frac{1}{q}} \le K\|f\|_{H^p}, \tag{2.17}$$

for all $f \in H^p(B)$. Here the notation f^* denotes the function defined on $\overline{B}$ by $f^* = f$ in B and $f^* = \lim_{r\uparrow 1} f_r$ a.e. $[\sigma]$ on S.

Proof. (cf. [6, p.239].) Put $\mu_1 \equiv \mu\mid_B$ and $\mu_2 \equiv \mu\mid_S$. By (2.16) we have

$$\mu_1(B(\zeta,\delta)) \le C\delta^{\frac{qn}{p}}, \tag{2.18}$$

$$\mu_2(S(\zeta,\delta)) \le C\delta^{\frac{qn}{p}}, \tag{2.19}$$

for all $\zeta \in S$ and $\delta > 0$. By (2.18) and Lemma 2.4, there exists a constant $K' > 0$ such that

$$\left[\int_B |f|^q d\mu_1\right]^{\frac{1}{q}} \le K'\|f\|_{H^p} \quad (f \in H^p(B)). \tag{2.20}$$

Moreover, it follows from (2.19) and Lemma 2.5 that $d\mu_2 = g d\sigma$ for some $g \in L^\infty(S)$ when $p = q$ and $\mu_2 \equiv 0$ when $p < q$. Thus using (2.20), we have

$$\begin{aligned}\int_{\overline{B}} |f^*|^q d\mu &= \int_B |f|^q d\mu_1 + \int_S |f^*|^q d\mu_2 \\ &\le \begin{cases} K'^q\|f\|_{H^p}^q + \|g\|_{L^\infty} \int_S |f^*|^p d\sigma, & \text{if } p = q, \\ K'^q \|f\|_{H^p}^q, & \text{if } p < q. \end{cases}\end{aligned}$$

This proves (2.17). □

Remark. In Lemma 2.6 (or Lemma 2.4), we see that the constant K of (2.17) (or (2.8)) can be chosen to be the product of $C^{1/q}$ and a positive constant depending only on p, q and the dimension n.

3 Boundedness of $W_{\varphi,\psi} : A^p(\nu_\alpha) \longrightarrow A^q(\nu_\beta)$

Let $0 < q < \infty$ and $-1 < \beta < \infty$. For a holomorphic map $\varphi : B \to B$ and $\psi \in A^q(\nu_\beta)$, we define a finite positive Borel measure $\mu_{\varphi,\psi,q,\beta}$ on B by

$$\mu_{\varphi,\psi,q,\beta}(E) = \int_{\varphi^{-1}(E)} |\psi|^q d\nu_\beta,$$

for all Borel sets E of B. Moreover, for $\psi \in H^q(B)$, we also define a finite positive Borel measure $\mu_{\varphi,\psi,q}$ on $\overline{B}$ by

$$\mu_{\varphi,\psi,q}(E) = \int_{\varphi^{*-1}(E)} |\psi^*|^q d\sigma \quad \text{(for all Borel sets } E \text{ of } \overline{B}),$$

where φ^* denotes the *radial limit map* of the mapping φ considered as a map of $S \to \overline{B}$.

The following lemma is a change of variables formula from measure theory.

Lemma 3.1. *Let $0 < q < \infty$ and $-1 \leq \beta < \infty$. Suppose that φ is a holomorphic self-map of B and $\psi \in A^q(\nu_\beta)$.*

(a) If $\beta > -1$, then for each nonnegative measurable function g in B

$$\int_B g \, d\mu_{\varphi,\psi,q,\beta} = \int_B |\psi|^q (g \circ \varphi) \, d\nu_\beta. \tag{3.1}$$

(b) If $\beta = -1$, then for each nonnegative measurable function g on $\overline{B}$

$$\int_{\overline{B}} g \, d\mu_{\varphi,\psi,q} = \int_S |\psi^*|^q (g \circ \varphi^*) \, d\sigma. \tag{3.2}$$

Proof. By adopting the way to prove [7, p.16, Theorem 1.19], we can verify this lemma. □

We introduce a γ-Carleson measure on B or $\overline{B}$. For $\gamma \geq n$, we say a finite positive Borel measure μ on B (resp. on $\overline{B}$) is a γ-*Carleson measure* if there exists a constant $C > 0$ such that $\mu(B(\zeta,\delta)) \leq C\delta^\gamma$ (resp. $\mu(\mathcal{S}(\zeta,\delta)) \leq C\delta^\gamma$) for all $\zeta \in S$ and $\delta > 0$.

Now, we give a characterization of the boundedness of $W_{\varphi,\psi} : A^p(\nu_\alpha) \longrightarrow A^q(\nu_\beta)$ in terms of a $\frac{q(n+1+\alpha)}{p}$-Carleson measure.

Theorem 3.1. *Let $0 < p \leq q < \infty$ and $-1 < \alpha, \beta < \infty$. Suppose that φ is a holomorphic self-map of B and $\psi \in A^q(\nu_\beta)$. Then the following conditions are equivalent:*

(a) $W_{\varphi,\psi} : A^p(\nu_\alpha) \longrightarrow A^q(\nu_\beta)$ is bounded.

(b) $\mu_{\varphi,\psi,q,\beta}$ is a $\frac{q(n+1+\alpha)}{p}$-Carleson measure on B.

(c) φ and ψ satisfy

$$M \equiv \sup_{a\in B} \int_B |\psi(z)|^q \left\{ \frac{1-|a|^2}{|1-\langle\varphi(z),a\rangle|^2} \right\}^{\frac{q(n+1+\alpha)}{p}} d\nu_\beta(z) < \infty. \tag{3.3}$$

Remark. When $p = q$, $\alpha = \beta > -1$ and $n = 1$, the above result appears in [8, Theorem 4.3].

Proof. (c) $\Rightarrow$ (b). For $\zeta \in S$ and $\delta \in (0,1)$, put $a = (1-\delta)\zeta \in B$. Define the function f_a on $\overline{B}$ by

$$f_a(z) = \left\{ \frac{1-|a|^2}{(1-\langle z,a\rangle)^2} \right\}^{\frac{n+1+\alpha}{p}} \quad (z \in \overline{B}).$$

We can easily see that f_a is in the *ball algebra* $A(B) \subset A^p(\nu_\alpha)$ and $|f_a(z)|^q \geq (4\delta)^{-\frac{q(n+1+\alpha)}{p}}$ for all $z \in B(\zeta,\delta)$. By (3.1) and (3.3), we have

$$\begin{aligned}\mu_{\varphi,\psi,q,\beta}(B(\zeta,\delta))(4\delta)^{-\frac{q(n+1+\alpha)}{p}} &\leq \int_{B(\zeta,\delta)} |f_a|^q d\mu_{\varphi,\psi,q,\beta} \\ &\leq \int_B |\psi|^q |f_a \circ \varphi|^q d\nu_\beta \leq M.\end{aligned}$$

That is, $\mu_{\varphi,\psi,q,\beta}(B(\zeta,\delta)) \leq 4^{\frac{q(n+1+\alpha)}{p}} M \delta^{\frac{q(n+1+\alpha)}{p}}$ if $\delta \in (0,1)$. If $\delta \geq 1$, we see that $\mu_{\varphi,\psi,q,\beta}(B(\zeta,\delta)) \leq \|\psi\|^q_{A^q(\nu_\beta)} \delta^{\frac{q(n+1+\alpha)}{p}}$. These prove that $\mu_{\varphi,\psi,q,\beta}$ is a $\frac{q(n+1+\alpha)}{p}$-Carleson measure on B.

(b) $\Rightarrow$ (a). By the assumption (b) and Lemma 2.3, there exists a constant $K > 0$ such that

$$\left[\int_B |f|^q d\mu_{\varphi,\psi,q,\beta} \right]^{\frac{1}{q}} \leq K\|f\|_{A^p(\nu_\alpha)}, \tag{3.4}$$

for all $f \in A^p(\nu_\alpha)$. On the other hand, by Lemma 3.1, we have

$$\int_B |f|^q d\mu_{\varphi,\psi,q,\beta} = \int_B |\psi|^q |f \circ \varphi|^q d\nu_\beta \equiv \|W_{\varphi,\psi} f\|^q_{A^q(\nu_\beta)}. \tag{3.5}$$

Hence (3.4) and (3.5) show that $W_{\varphi,\psi}$ is a bounded operator from $A^p(\nu_\alpha)$ to $A^q(\nu_\beta)$.

(a) $\Rightarrow$ (c). Let $a \in B$ be fixed. Take the function f_a as in the proof of (c) $\Rightarrow$ (b). Then $f_a \in A(B) \subset A^p(\nu_\alpha)$. Moreover, by [10, Proposition 1.4.10], we see that there exists a constant $C > 0$ such that

$$\int_B \frac{(1-|a|^2)^{n+1+\alpha}(1-|z|^2)^\alpha}{|1-\langle z,a\rangle|^{2(n+1+\alpha)}} d\nu(z) \leq C(1-|a|^2)^{n+1+\alpha}(1-|a|^2)^{-(n+1+\alpha)} = C.$$

Thus, we have $\sup_{a\in B} \|f_a\|_{A^p(\nu_\alpha)} \leq \{c_\alpha C\}^{\frac{1}{p}}$. Since $W_{\varphi,\psi} : A^p(\nu_\alpha) \to A^q(\nu_\beta)$ is a bounded operator, there exists a constant $K > 0$ such that $\|W_{\varphi,\psi} f_a\|^q_{A^q(\nu_\beta)} \leq$

$\{K\|f_a\|_{A^p(\nu_\alpha)}\}^q \le K^q\{c_\alpha C\}^{\frac{q}{p}}$ for all $a \in B$. It follows from the form of the function f_a that

$$\begin{aligned}&\int_B |\psi(z)|^q \left\{\frac{1-|a|^2}{|1-\langle\varphi(z),a\rangle|^2}\right\}^{\frac{q(n+1+\alpha)}{p}} d\nu_\beta(z)\\ &= \|W_{\varphi,\psi}f_a\|^q_{A^q(\nu_\beta)} \le K^q\{c_\alpha C\}^{\frac{q}{p}} < \infty,\end{aligned}$$

for all $a \in B$. This gives (3.3). □

Theorem 3.2. *Let $0 < p \le q < \infty$. Suppose that φ is a holomorphic self-map of B and $\psi \in H^q(B)$. Then the following conditions are equivalent:*

(a) $W_{\varphi,\psi} : H^p(B) \to H^q(B)$ is bounded.

(b) $\mu_{\varphi,\psi,q}$ is a $\frac{qn}{p}$-Carleson measure on $\overline{B}$.

(c) φ and ψ satisfy

$$\sup_{a\in B}\int_S |\psi^*(\zeta)|^q \left\{\frac{1-|a|^2}{|1-\langle\varphi^*(\zeta),a\rangle|^2}\right\}^{\frac{qn}{p}} d\sigma(\zeta) < \infty. \tag{3.6}$$

Proof. The proofs of (a) $\Rightarrow$ (c) and (c) $\Rightarrow$ (b) are entirely similar to those of Theorem 3.1 except that we choose the test function

$$f_a(z) = \left\{\frac{1-|a|^2}{(1-\langle z,a\rangle)^2}\right\}^{\frac{n}{p}} \quad (z \in \overline{B}).$$

(b) $\Rightarrow$ (a). By Lemma 2.6, there exists a constant $K > 0$ such that

$$\int_{\overline{B}} |f^*|^q d\mu_{\varphi,\psi,q} \le \{K\|f\|_{H^p}\}^q, \tag{3.7}$$

for all $f \in H^p(B)$. Let f be in $H^p(B)$. Since $A(B)$ is dense in $H^p(B)$, there is a sequence $\{f_j\}$ in $A(B)$ such that $\lim_{j\to\infty}\|f_j - f\|_{H^p} = 0$. Noting that $f_j \in A(B)$ implies that $(f_j \circ \varphi)^* = f_j \circ \varphi^*$ a.e. on S. Thus, by Lemma 3.1 and (3.7), we have

$$\|W_{\varphi,\psi}f_j\|^q_{H^q} = \int_S |\psi^*|^q |f_j \circ \varphi^*|^q d\sigma = \int_{\overline{B}} |f_j|^q d\mu_{\varphi,\psi,q} \le \{K\|f_j\|_{H^p}\}^q, \tag{3.8}$$

for all $j \in \mathbb{N}$. Since $\lim_{j\to\infty}\|f_j - f\|_{H^p} = 0$, it follows from (3.8) that $\{W_{\varphi,\psi}f_j\}$ is a Cauchy sequence in $H^q(B)$. The completeness of $H^q(B)$ gives $W_{\varphi,\psi}f \in H^q(B)$ and $\|W_{\varphi,\psi}f\|_{H^q} \le K\|f\|_{H^p}$. This proves that $W_{\varphi,\psi} : H^p(B) \to H^q(B)$ is bounded. □

The proof of the following theorem is entirely similar to that of Theorem 3.2, except for using Lemma 2.4 instead of Lemma 2.6. In fact, the proof is much easier than that of Theorem 3.2 because the boundary functions are not involved.

Theorem 3.3. *Let* $0 < p \leq q < \infty$ *and* $\alpha \in (-1, \infty)$. *Suppose that* φ *is a holomorphic self-map of* B *and* $\psi \in A^q(\nu_\alpha)$. *Then the following conditions are equivalent:*

(a) $W_{\varphi,\psi} : H^p(B) \to A^q(\nu_\alpha)$ *is bounded.*

(b) $\mu_{\varphi,\psi,q,\alpha}$ *is a* $\frac{qn}{p}$*-Carleson measure on* B.

(c) φ *and* ψ *satisfy*

$$\sup_{a\in B} \int_B |\psi(z)|^q \left\{ \frac{1-|a|^2}{|1-\langle \varphi(z), a\rangle|^2} \right\}^{\frac{qn}{p}} d\nu_\alpha(z) < \infty.$$

4 Compactness of $W_{\varphi,\psi} : A^p(\nu_\alpha) \to A^q(\nu_\beta)$

The proofs of the results in Sections 4 and 5 depend on the following characterization of the compactness of $W_{\varphi,\psi} : A^p(\nu_\alpha) \to A^q(\nu_\beta)$ $(0 < p \leq q \leq \infty, -1 \leq \alpha, \beta < \infty)$ expressed in terms of sequential convergence. Here $A^\infty(\nu_\alpha) \equiv H^\infty(B)$. By using Lemma 2.1 and the fact that bounded subsets of $A^p(\nu_\alpha)$ are normal families, we can prove the next proposition in the same way as we prove [4, Proposition 3.11].

Proposition 4.1. *Let* $0 < p \leq q \leq \infty$ *and* $-1 \leq \alpha, \beta < \infty$. *Let* φ *be a holomorphic self-map of* B *and* $\psi \in A^q(\nu_\beta)$. *Suppose that* $W_{\varphi,\psi}(A^p(\nu_\alpha)) \subset A^q(\nu_\beta)$. *Then* $W_{\varphi,\psi} : A^p(\nu_\alpha) \to A^q(\nu_\beta)$ *is compact if and only if for every bounded sequence* $\{f_j\}$ *in* $A^p(\nu_\alpha)$ *which converges to* 0 *uniformly on compact subsets of* B, $\{W_{\varphi,\psi} f_j\}$ *converges to* 0 *in* $A^q(\nu_\beta)$.

In order to state our results, we introduce a compact γ-Carleson measure on B or $\overline{B}$. For $\gamma \geq n$, a finite positive Borel measure μ on B (resp. on $\overline{B}$) is called a *compact* γ*-Carleson measure* if

$$\lim_{\delta\downarrow 0} \sup_{\zeta\in S} \frac{\mu(B(\zeta,\delta))}{\delta^\gamma} = 0 \quad \left(\text{resp. } \lim_{\delta\downarrow 0} \sup_{\zeta\in S} \frac{\mu(S(\zeta,\delta))}{\delta^\gamma} = 0\right).$$

In this section, we characterize the compactness of the operators $W_{\varphi,\psi} : A^p(\nu_\alpha) \to A^q(\nu_\beta)$, $H^p(B) \to H^q(B)$ and $H^p(B) \to A^q(\nu_\alpha)$ $(0 < p \leq q < \infty)$.

Theorem 4.1. *Let* $0 < p \leq q < \infty$ *and* $-1 < \alpha, \beta < \infty$. *Suppose that* φ *is a holomorphic self-map of* B *and* $\psi \in A^q(\nu_\beta)$. *Then the following conditions are equivalent:*

(a) $W_{\varphi,\psi} : A^p(\nu_\alpha) \to A^q(\nu_\beta)$ *is compact.*

(b) $\mu_{\varphi,\psi,q,\beta}$ *is a compact* $\frac{q(n+1+\alpha)}{p}$*-Carleson measure on* B.

(c) φ and ψ satisfy

$$\lim_{|a|\uparrow 1}\int_B |\psi(z)|^q \left\{\frac{1-|a|^2}{|1-\langle\varphi(z),a\rangle|^2}\right\}^{\frac{q(n+1+\alpha)}{p}} d\nu_\beta(z) = 0. \tag{4.1}$$

Proof. (a) $\Rightarrow$ (c). Take a sequence $\{a_j\}$ in B with $\lim_{j\to\infty}|a_j| = 1$. We define functions f_j on $\overline{B}$ by

$$f_j(z) = \left\{\frac{1-|a_j|^2}{(1-\langle z,a_j\rangle)^2}\right\}^{\frac{n+1+\alpha}{p}} \quad (z\in\overline{B}, j\in\mathbb{N}). \tag{4.2}$$

As in the proof of Theorem 3.1, we see that $f_j \in A(B)$ and $\|f_j\|^p_{A^p(\nu_\alpha)} \le c_\alpha C < \infty$ for some constant $C > 0$. Moreover, we can easily see that $\{f_j\}$ converges to 0 uniformly on compact subsets of B. By Proposition 4.1, we have $\{W_{\varphi,\psi}f_j\}$ converges to 0 in $A^q(\nu_\beta)$. Hence,

$$\lim_{j\to\infty}\int_B |\psi(z)|^q \left\{\frac{1-|a_j|^2}{|1-\langle\varphi(z),a_j\rangle|^2}\right\}^{\frac{q(n+1+\alpha)}{p}} d\nu_\beta(z) = \lim_{j\to\infty}\|W_{\varphi,\psi}f_j\|^q_{A^q(\nu_\beta)} = 0.$$

This implies (4.1).

(c) $\Rightarrow$ (b). Fix $\varepsilon > 0$. By (4.1) and Lemma 3.1, we can choose $r_0 \in (\frac{1}{2}, 1)$ such that

$$\int_B \left\{\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right\}^{\frac{q(n+1+\alpha)}{p}} d\mu_{\varphi,\psi,q,\beta}(z) < \frac{\varepsilon}{4^{\frac{q(n+1+\alpha)}{p}}}, \tag{4.3}$$

for all $a \in B$ with $r_0 < |a| < 1$. Put $\delta_0 = 1 - r_0$. For $\zeta \in S$ and $\delta \in (0,\delta_0)$, we put $a = (1-\delta)\zeta$. Then $a \in B$ and $r_0 < |a| < 1$. Moreover, we see that for each $z \in B(\zeta,\delta)$

$$\left\{\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right\}^{\frac{q(n+1+\alpha)}{p}} \ge \left\{\frac{\delta}{(2\delta)^2}\right\}^{\frac{q(n+1+\alpha)}{p}} = (4\delta)^{-\frac{q(n+1+\alpha)}{p}}. \tag{4.4}$$

By (4.3) and (4.4), we have

$$\frac{\mu_{\varphi,\psi,q,\beta}(B(\zeta,\delta))}{\delta^{\frac{q(n+1+\alpha)}{p}}} \le 4^{\frac{q(n+1+\alpha)}{p}} \int_{B(\zeta,\delta)} \left\{\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right\}^{\frac{q(n+1+\alpha)}{p}} d\mu_{\varphi,\psi,q,\beta}(z) < \varepsilon,$$

for all $\delta \in (0,\delta_0)$ and $\zeta \in S$. This implies that $\mu_{\varphi,\psi,q,\beta}$ is a compact $\frac{q(n+1+\alpha)}{p}$-Carleson measure on B.

(b) $\Rightarrow$ (a). We assume that $\mu_{\varphi,\psi,q,\beta}$ is a compact $\frac{q(n+1+\alpha)}{p}$-Carleson measure on B. For $\zeta \in S$ and $\delta > 0$, set $D(\zeta,\delta) = \{z \in B : 1-\delta < |z|, z/|z| \in S(\zeta,\delta)\}$. Then we easily see that

$$B(\zeta,\delta/2) \subset D(\zeta,\delta) \subset B(\zeta,2\delta) \quad \text{for } \zeta \in S \text{ and } \delta > 0. \tag{4.5}$$

By the assumption on $\mu_{\varphi,\psi,q,\beta}$ and (4.5), we obtain $\mu_{\varphi,\psi,q,\beta}(D(\zeta,\delta))/\delta^{\frac{q(n+1+\alpha)}{p}} \to 0$ as $\delta \downarrow 0$ uniformly in $\zeta \in S$. Hence, as in the proof of [6, Theorem 1.1(ii)], we can prove that there exists a constant $C > 0$ depending only on p, q, α and n such that

$$\widetilde{\mu}(B(\zeta,\delta)) \leq C\varepsilon\delta^{\frac{q(n+1+\alpha)}{p}} \quad (\zeta \in S, \delta > 0), \tag{4.6}$$

where $\varepsilon > 0$ is fixed and $\widetilde{\mu} \equiv \mu_{\varphi,\psi,q,\beta}|_{B\setminus(1-\delta_0)\overline{B}}$ for some $\delta_0 \in (0,1)$.

Suppose that $\{f_j\} \subset A^p(\nu_\alpha)$ satisfies $M \equiv \sup_{j\in\mathbb{N}} \|f_j\|_{A^p(\nu_\alpha)} < \infty$ and converges to 0 uniformly on compact subsets of B. By Lemma 3.1, we have for each $j \in \mathbb{N}$

$$\begin{aligned}\|W_{\varphi,\psi}f_j\|^q_{A^q(\nu_\beta)} &= \int_B |\psi(z)(f_j\circ\varphi)(z)|^q d\nu_\beta(z) = \int_B |f_j(z)|^q d\mu_{\varphi,\psi,q,\beta}(z) \\ &= \int_B |f_j(z)|^q d\widetilde{\mu}(z) + \int_{(1-\delta_0)\overline{B}} |f_j(z)|^q d\mu_{\varphi,\psi,q,\beta}(z).\end{aligned} \tag{4.7}$$

By (4.6) and Lemma 2.3, there exists a positive constant K depending only on p, q, α and n such that

$$\int_B |f_j|^q d\widetilde{\mu} \leq KC\varepsilon\|f_j\|^q_{A^p(\nu_\alpha)} \leq KM^qC\varepsilon, \tag{4.8}$$

for each $j \in \mathbb{N}$. Since $\{f_j\}$ converges to 0 uniformly on $(1-\delta_0)\overline{B}$, it follows that

$$\lim_{j\to\infty} \int_{(1-\delta_0)\overline{B}} |f_j(z)|^q d\mu_{\varphi,\psi,q,\beta}(z) = 0. \tag{4.9}$$

Hence (4.7)–(4.9) show that $\{W_{\varphi,\psi}f_j\}$ converges to 0 in $A^q(\nu_\beta)$. By Proposition 4.1, we see that $W_{\varphi,\psi} : A^p(\nu_\alpha) \to A^q(\nu_\beta)$ is compact. □

In order to prove Theorem 4.2, we show two lemmas. These are the extensions of [6, Corollary 1.4 and Lemma 1.6] to the weighted composition operator $W_{\varphi,\psi}$.

Lemma 4.1. *Let $0 < p \leq q < \infty$. Suppose that $\varphi : B \to B$ is a holomorphic map and $\psi \in H^q(B)\setminus\{0\}$ such that $W_{\varphi,\psi} : H^p(B) \to H^q(B)$ is bounded. Then φ^* cannot carry a set of positive σ-measure in S into a set of σ-measure 0 in S.*

Proof. Suppose $E, F \subset S$ and $\varphi^*(E) \subset F$ with $\sigma(E) > 0$ and $\sigma(F) = 0$. Put $\lambda \equiv \mu_{\varphi,\psi,q}|_S$. Since $W_{\varphi,\psi} : H^p(B) \to H^q(B)$ is bounded, by Theorem 3.2, we have

$$\lambda(S(\zeta,\delta)) \leq C\delta^{\frac{qn}{p}} \quad (\zeta \in S, \delta > 0),$$

for some positive constant C. By Lemma 2.5, we see that $\lambda \equiv 0$ (if $p < q$) or λ is absolutely continuous with respect to σ (if $p = q$). Thus we have

$$0 \geq \lambda(\varphi^*(E)) \equiv \int_{\varphi^{*-1}(\varphi^*(E))} |\psi^*|^q d\sigma \geq \int_E |\psi^*|^q d\sigma.$$

That is, $\psi^* = 0$ a.e. on E. Hence [10, Theorem 5.5.9] gives that $\psi \equiv 0$ in B. This contradicts $\psi \not\equiv 0$. □

Lemma 4.2. *Let $0 < p \le q < \infty$ and $f \in H^p(B)$. Suppose that $\varphi : B \to B$ is a holomorphic map and $\psi \in H^q(B) \setminus \{0\}$ such that $W_{\varphi,\psi} : H^p(B) \to H^q(B)$ is bounded. Then $\psi^*(f \circ \varphi)^* = \psi^*(f^* \circ \varphi^*)$ a.e. $[\sigma]$ on S. Here the notation f^* is used as in Lemma 2.6.*

Proof. (cf. [6, Lemma 1.6].) Since φ^* cannot carry a set of positive measure in S into a set of measure 0 in S (by Lemma 4.1) and since the radial limit of φ, f and ψ exist on a set of full measure in S, we have $\lim_{r\uparrow 1} \psi^*(f_r \circ \varphi^*) = \psi^*(f^* \circ \varphi^*)$ a.e. $[\sigma]$ on S.

On the other hand, since $f_r \in A(B)$ and $f_r \to f$ as $r \uparrow 1$ in $H^p(B)$, the boundedness of $W_{\varphi,\psi}$ shows that

$$\begin{aligned} 0 &\le \int_S |\psi^*(\zeta)(f\circ\varphi)^*(\zeta) - \psi^*(\zeta)(f^*\circ\varphi^*)(\zeta)|^q d\sigma(\zeta) \\ &= \int_S \lim_{r\uparrow 1} |\psi^*(\zeta)(f\circ\varphi)^*(\zeta) - \psi^*(\zeta)(f_r\circ\varphi^*)(\zeta)|^q d\sigma(\zeta) \\ &\le \liminf_{r\uparrow 1} \int_S |\psi^*(\zeta)(f\circ\varphi)^*(\zeta) - \psi^*(\zeta)(f_r\circ\varphi)^*(\zeta)|^q d\sigma(\zeta) \\ &= \liminf_{r\uparrow 1} \|W_{\varphi,\psi} f - W_{\varphi,\psi} f_r\|_{H^q}^q = 0. \end{aligned}$$

This implies that $\psi^*(f \circ \varphi)^* = \psi^*(f^* \circ \varphi^*)$ a.e. $[\sigma]$ on S. □

Theorem 4.2. *Let $0 < p \le q < \infty$. Suppose that φ is a holomorphic self-map of B and $\psi \in H^q(B)$. Then the following conditions are equivalent:*

(a) $W_{\varphi,\psi} : H^p(B) \to H^q(B)$ is compact.

(b) $\mu_{\varphi,\psi,q}$ is a compact $\frac{qn}{p}$-Carleson measure on $\overline{B}$.

(c) φ and ψ satisfy

$$\lim_{|a|\uparrow 1} \int_S |\psi^*(\zeta)|^q \left\{ \frac{1 - |a|^2}{|1 - \langle \varphi^*(\zeta), a\rangle|^2} \right\}^{\frac{qn}{p}} d\sigma(\zeta) = 0. \tag{4.10}$$

Proof. If $\psi \equiv 0$, then $W_{\varphi,\psi}$ is compact. Thus, we consider the case $\psi \not\equiv 0$.

(a) ⇒ (c). For any sequence $\{a_j\}$ in B with $\lim_{j\to\infty} |a_j| = 1$, put

$$f_j(z) = \left\{ \frac{1 - |a_j|^2}{(1 - \langle z, a_j\rangle)^2} \right\}^{\frac{n}{p}} \quad (z \in \overline{B}, j \in \mathbb{N}).$$

We can easily see that $\{f_j\}(\subset A(B))$ is a bounded sequence in $H^p(B)$ which converges to 0 uniformly on compact subsets of B. By Proposition 4.1, we have $\lim_{j\to\infty} \|W_{\varphi,\psi} f_j\|_{H^q} = 0$. Since $(f_j \circ \varphi)^* = f_j \circ \varphi^*$ a.e. on S, we obtain

$$\int_S |\psi^*(\zeta)|^q \left\{ \frac{1 - |a_j|^2}{|1 - \langle \varphi^*(\zeta), a_j\rangle|^2} \right\}^{\frac{qn}{p}} d\sigma(\zeta) = \|W_{\varphi,\psi} f_j\|_{H^q}^q \to 0 \quad \text{as } j \to \infty.$$

This proves (4.10).

The proof of (c) $\Rightarrow$ (b) is entirely similar to that of Theorem 4.1.

(b) $\Rightarrow$ (a). Fix $\varepsilon > 0$. As in the proof of (b) $\Rightarrow$ (a) in Theorem 4.1, there exists a constant $C > 0$ dpending only on p, q and n such that

$$\widetilde{\mu}(\mathcal{S}(\zeta,\delta)) \le C\varepsilon\delta^{\frac{qn}{p}} \quad (\zeta \in S, \delta > 0), \tag{4.11}$$

where $\widetilde{\mu} \equiv \mu_{\varphi,\psi,q}|_{\overline{B}\setminus(1-\delta_0)\overline{B}}$ for some $\delta_0 \in (0,1)$.

Now, suppose that $\{f_j\}$ is a bounded sequence in $H^p(B)$ which converges to 0 uniformly on compact subsets of B. By the assumption (b) and Theorem 3.2, $W_{\varphi,\psi} : H^p(B) \to H^q(B)$ is bounded. Hence, it follows from Lemma 4.2 that $\psi^*(f_j \circ \varphi)^* = \psi^*(f_j^* \circ \varphi^*)$ a.e. on S ($j \in \mathbb{N}$). By Lemma 3.1, we obtain for each $j \in \mathbb{N}$

$$\begin{aligned} \|W_{\varphi,\psi} f_j\|_{H^q}^q &= \int_S |\psi^*(f_j \circ \varphi)^*|^q d\sigma = \int_S |\psi^*|^q |f_j^* \circ \varphi^*|^q d\sigma \\ &= \int_{\overline{B}} |f_j^*|^q d\mu_{\varphi,\psi,q} \\ &= \int_{\overline{B}} |f_j^*|^q d\widetilde{\mu} + \int_{(1-\delta_0)\overline{B}} |f_j|^q d\mu_{\varphi,\psi,q}. \end{aligned} \tag{4.12}$$

On the other hand, by (4.11) and Lemma 2.6, there exists a constant $K > 0$ depending only on p, q and n such that

$$\int_{\overline{B}} |f_j^*|^q d\widetilde{\mu} \le KC\varepsilon \|f_j\|_{H^p}^q \le KM^q C\varepsilon, \tag{4.13}$$

where $M \equiv \sup_{j\in\mathbb{N}} \|f_j\|_{H^p} < \infty$. Since $\{f_j\}$ converges to 0 uniformly on $(1-\delta_0)\overline{B}$, we have

$$\lim_{j\to\infty} \int_{(1-\delta_0)\overline{B}} |f_j|^q d\mu_{\varphi,\psi,q} = 0. \tag{4.14}$$

Hence (4.12)–(4.14) show that $\lim_{j\to\infty} \|W_{\varphi,\psi} f_j\|_{H^q} = 0$. By Proposition 4.1, $W_{\varphi,\psi}$ is compact from $H^p(B)$ to $H^q(B)$. □

Theorem 4.3. *Let $0 < p \le q < \infty$ and $\alpha \in (-1,\infty)$. Suppose that φ is a holomorphic self-map of B and $\psi \in A^q(\nu_\alpha)$. Then the following conditions are equivalent:*

(a) $W_{\varphi,\psi} : H^p(B) \to A^q(\nu_\alpha)$ is compact.

(b) $\mu_{\varphi,\psi,q,\alpha}$ is a compact $\frac{qn}{p}$-Carleson measure on B.

(c) φ and ψ satisfy

$$\lim_{|a|\uparrow 1} \int_B |\psi(z)|^q \left\{ \frac{1-|a|^2}{|1-\langle\varphi(z),a\rangle|^2} \right\}^{\frac{qn}{p}} d\nu_\alpha(z) = 0.$$

Proof. By replacing Lemma 2.6 with Lemma 2.4 in the proof of Theorem 4.2, we can prove this theorem. In fact, the proof of this theorem become much easier than that of Theorem 4.2 because the boundary functions are not involved. □

5 $W_{\varphi,\psi}$ from $A^p(\nu_\alpha)$ to $H^\infty(B)$

In this section, we study the boundedness and the compactness of $W_{\varphi,\psi}$ from $A^p(\nu_\alpha)$ $(0 < p < \infty, -1 \le \alpha < \infty)$ to $H^\infty(B)$. Our results in this section are the extensions of the results by M. D. Contreras and A. G. Hernández-Díaz ([3]).

From now on, till the end of this paper, we fix $\alpha \in [-1, \infty)$ and $p \in (0, \infty)$.

Theorem 5.1. *Suppose that φ is a holomorphic self-map of B and $\psi \in H(B)$. Then the following conditions are equivarent:*

(a) $W_{\varphi,\psi} : A^p(\nu_\alpha) \to H^\infty(B)$ is bounded.

(b) φ and ψ satisfy

$$\sup_{z\in B} \frac{|\psi(z)|^p}{(1-|\varphi(z)|^2)^{n+1+\alpha}} < \infty.$$

Proof. (b) $\Rightarrow$ (a). Take $f \in A^p(\nu_\alpha)$. By Lemma 2.1, we have $|f(z)| \le (1-|z|^2)^{-\frac{n+1+\alpha}{p}} \|f\|_{A^p(\nu_\alpha)}$ for all $z \in B$. Thus we have for $z \in B$

$$|W_{\varphi,\psi} f(z)| \le \frac{|\psi(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}} \|f\|_{A^p(\nu_\alpha)} \le M^{\frac{1}{p}} \|f\|_{A^p(\nu_\alpha)},$$

where $M \equiv \sup_{z\in B} \frac{|\psi(z)|^p}{(1-|\varphi(z)|^2)^{n+1+\alpha}} < \infty$. This implies $W_{\varphi,\psi} : A^p(\nu_\alpha) \to H^\infty(B)$ is bounded.

(a) $\Rightarrow$ (b). For $z \in B$, we define the function f_z on $\overline{B}$ by

$$f_z(w) = \left\{ \frac{1-|\varphi(z)|^2}{(1-\langle w, \varphi(z)\rangle)^2} \right\}^{\frac{n+1+\alpha}{p}} \qquad (w \in \overline{B}).$$

We can easily see that $f_z \in A(B)$ and $C \equiv \sup_{z\in B} \|f_z\|^p_{A^p(\nu_\alpha)} < \infty$. Since $W_{\varphi,\psi}$: $A^p(\nu_\alpha) \to H^\infty(B)$ is bounded, there exists a constant $K > 0$ such that $\|W_{\varphi,\psi} f_z\|^p_\infty \le K^p \|f_z\|^p_{A^p(\nu_\alpha)} \le CK^p$ for all $z \in B$. Hence, we obtain

$$\frac{|\psi(z)|^p}{(1-|\varphi(z)|^2)^{n+1+\alpha}} = |(W_{\varphi,\psi} f_z)(z)|^p \le CK^p,$$

for all $z \in B$. This completes the proof. □

Theorem 5.2. *Suppose that φ is a holomorphic self-map of B and $\psi \in H^\infty(B)$. Then the following conditions are equivarent:*

(a) $W_{\varphi,\psi} : A^p(\nu_\alpha) \to H^\infty(B)$ is compact.

(b) φ and ψ satisfy either $\sup_{z\in B} |\varphi(z)| < 1$ or

$$\lim_{|\varphi(z)|\uparrow 1} \frac{|\psi(z)|^p}{(1-|\varphi(z)|^2)^{n+1+\alpha}} = 0.$$

Proof. (a) $\Rightarrow$ (b). Assume, to reach a contradiction, that $\sup_{z\in B}|\varphi(z)| = 1$ and

$$\limsup_{|\varphi(z)|\uparrow 1} \frac{|\psi(z)|^p}{(1-|\varphi(z)|^2)^{n+1+\alpha}} \neq 0.$$

Then there exist a sequence $\{z_j\}$ in B and an $\varepsilon_0 > 0$ such that

$$\lim_{j\to\infty} |\varphi(z_j)| = 1, \tag{5.1}$$

$$\frac{|\psi(z_j)|^p}{(1-|\varphi(z_j)|^2)^{n+1+\alpha}} \geq \varepsilon_0, \tag{5.2}$$

for all $j \in \mathbb{N}$. We define functions f_j by

$$f_j(z) = \left\{\frac{1-|\varphi(z_j)|^2}{(1-\langle z, \varphi(z_j)\rangle)^2}\right\}^{\frac{n+1+\alpha}{p}} \qquad (z \in \overline{B}).$$

By (5.1), we see that $\{f_j\}$ is a bounded sequence in $A^p(\nu_\alpha)$ which converges to 0 uniformly on compact subsets of B. Since $W_{\varphi,\psi}$ is compact, it follows from Proposition 4.1 that $\lim_{j\to\infty} \|W_{\varphi,\psi} f_j\|_\infty = 0$.

On the other hand, by (5.2),

$$\begin{aligned}\|W_{\varphi,\psi} f_j\|_\infty &\geq |(W_{\varphi,\psi} f_j)(z_j)| = |\psi(z_j)|\left\{\frac{1-|\varphi(z_j)|^2}{(1-|\varphi(z_j)|^2)^2}\right\}^{\frac{n+1+\alpha}{p}} \\ &= \frac{|\psi(z_j)|}{(1-|\varphi(z_j)|^2)^{\frac{n+1+\alpha}{p}}} \geq \varepsilon_0^{\frac{1}{p}} > 0,\end{aligned}$$

for all $j \in \mathbb{N}$. This contradicts $\lim_{j\to\infty} \|W_{\varphi,\psi} f_j\|_\infty = 0$.

(b) $\Rightarrow$ (a). Suppose that $\{f_j\}$ is a bounded sequence in $A^p(\nu_\alpha)$ which converges to 0 uniformly on compact subsets of B. By Proposition 4.1, it suffices to show that $\|W_{\varphi,\psi} f_j\|_\infty \to 0$ as $j \to \infty$.

First, we assume that $\sup_{z\in B}|\varphi(z)| < 1$. Since $\overline{\varphi(B)}$ is a compact subset of B, we have $\sup_{w\in\overline{\varphi(B)}} |f_j(w)| \to 0$ as $j \to \infty$. Thus,

$$0 \leq \|W_{\varphi,\psi} f_j\|_\infty = \sup_{z\in B} |\psi(z) f_j(\varphi(z))| \leq \|\psi\|_\infty \cdot \sup_{w\in\overline{\varphi(B)}} |f_j(w)| \to 0,$$

as $j \to \infty$. That is, $\lim_{j\to\infty} \|W_{\varphi,\psi} f_j\|_\infty = 0$.

Suppose now that $\lim_{|\varphi(z)|\uparrow 1} \frac{|\psi(z)|^p}{(1-|\varphi(z)|^2)^{n+1+\alpha}} = 0$. Let $\varepsilon > 0$ be given. By the hypothesis, we can choose $r_0 \in (\frac{1}{2}, 1)$ such that

$$|\psi(z)|^p < \varepsilon(1-|\varphi(z)|^2)^{n+1+\alpha}, \tag{5.3}$$

for all $z \in B$ with $|\varphi(z)| > r_0$.

For each $z \in B$ with $|\varphi(z)| > r_0$, it follows from Lemma 2.1 and (5.3) that

$$\begin{aligned} |(W_{\varphi,\psi} f_j)(z)| &= |\psi(z) f_j(\varphi(z))| \\ &\leq \varepsilon^{\frac{1}{p}} (1 - |\varphi(z)|^2)^{\frac{n+1+\alpha}{p}} (1 - |\varphi(z)|^2)^{-\frac{n+1+\alpha}{p}} \|f_j\|_{A^p(\nu_\alpha)} \leq C \varepsilon^{\frac{1}{p}}, \end{aligned} \tag{5.4}$$

for all $j \in \mathbb{N}$. Here $C \equiv \sup_{j\in\mathbb{N}} \|f_j\|_{A^p(\nu_\alpha)}$.

On the other hand, since $r_0 \overline{B}$ is a compact subset of B and $\{f_j\}$ converges to 0 uniformly on compact subsets of B, we have

$$\begin{aligned} 0 \leq \sup_{z\in B, |\varphi(z)| \leq r_0} |(W_{\varphi,\psi} f_j)(z)| &\leq \|\psi\|_\infty \cdot \sup_{w \in r_0 \overline{B}} |f_j(w)| \\ &\to 0 \quad \text{as } j \to \infty. \end{aligned} \tag{5.5}$$

Hence (5.4) and (5.5) show that $\|W_{\varphi,\psi} f_j\|_\infty \to 0$ as $j \to \infty$. This completes the proof. □

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