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K-THEORY OF CONTINUOUS FIELDS OF QUANTUM TORI

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ABSTRACT. In this paper we study K-theory of continuous fields of quantum tori. For this purpose we review and compute K-theory of C^*-algebras of continuous functions on the tori and that of the quantum (or noncommutative) tori by obtaining the formulas for counting generators of their K-groups.

0. INTRODUCTION

Our first motivation for this study is the following:

Problem. *Let $\Gamma(X, \{\mathfrak{A}_t\}_{t\in X})$ be the C^*-algebra of a continuous filed on a locally compact Hausdorff space X with the fibers $\mathfrak{A}_t$. Then how does one compute its K-groups in terms of the base space X and the fibers $\mathfrak{A}_t$?*

See Fell [F] and Dixmier [Dx] for the theory of continuous fields of C^*-algebras. As a step toward solving the problem, we focus our attention to the case where $X = \mathbb{T}^n$ the tori and $\mathfrak{A}_t = \mathfrak{A}_\Theta$ the quantum (or noncommutative) tori. Fortunately, the K-groups of the C^*-algebras $C(\mathbb{T}^n)$ of continuous functions on $\mathbb{T}^n$ as well as $\mathfrak{A}_\Theta$ are well known. Also, the Bott generator for the K_0-group of $C(\mathbb{T}^2)$ and the Rieffel projections for the K_0-group of the quantum 2-tori are well known. However, the generators of the K-groups of $C(\mathbb{T}^n)$ and the quantum n-tori for $n \geq 3$ seem to be little well known in the literature. Therefore, in Section 1 we review and study the K-groups of $C(\mathbb{T}^n)$ by obtaining the formulas for counting generators given by the generalized Bott projections. In Section 2 we review and study the K-groups of $\mathfrak{A}_\Theta$ by obtaining the formulas for counting generators given by the generalized Rieffel projections. Using these explicit formulas for counting generators of the K-groups, in Section 3 we obtain a partial answer to the Problem.

Notation. Let $C(X)$ be the C^*-algebra of all continuous complex-valued functions on a compact Hausdorff space X. Let $K_*(\mathfrak{A})$ for $* = 0, 1$ be the K-groups of a C^*-algebra $\mathfrak{A}$. See [Bl], [RLL], [Wo] for details about the K-theory of C^*-algebras.

1. THE C^*-ALGEBRAS OF CONTINUOUS FUNCTIONS ON THE TORI

In this section we first briefly recall the K-theory of the C^*-algebras of continuous complex-valued functions on the tori.

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Proposition 1.1. *Let $C(\mathbb{T}^n)$ be the C^*-algebra of continuous functions on the n-torus $\mathbb{T}^n$. Then*

$$K_0(C(\mathbb{T}^n)) \cong \mathbb{Z}^{2^{n-1}}, \quad \text{and} \quad K_1(C(\mathbb{T}^n)) \cong \mathbb{Z}^{2^{n-1}}.$$

Proof. This fact is well known. For convenience, we follow the proof as given in [Wo]. Note that $C(\mathbb{T}^n) \cong C(\mathbb{T}, C(\mathbb{T}^{n-1}))$ the C^*-algebra of continuous $C(\mathbb{T}^{n-1})$-valued functions on $\mathbb{T}$. Then the following short exact sequence:

$$0 \to C_0(\mathbb{R}, C(\mathbb{T}^{n-1})) \to C(\mathbb{T}, C(\mathbb{T}^{n-1})) \to C(\mathbb{T}^{n-1}) \to 0$$

is obtained and actually splitting, where the closed ideal $C_0(\mathbb{R}, C(\mathbb{T}^{n-1}))$ is the C^*-algebra of all continuous $C(\mathbb{T}^{n-1})$-valued functions on $\mathbb{R}$ vanishing at infinity. Therefore, the following short exact sequences for $* = 0, 1$:

$$0 \to K_*(C_0(\mathbb{R}, C(\mathbb{T}^{n-1}))) \to K_*(C(\mathbb{T}^n)) \to K_*(C(\mathbb{T}^{n-1})) \to 0$$

are gained. Thus, for $* = 0, 1$,

$$\begin{aligned} K_*(C(\mathbb{T}^n)) &\cong K_*(C_0(\mathbb{R}, C(\mathbb{T}^{n-1}))) \oplus K_*(C(\mathbb{T}^{n-1})) \\ &\cong K_{*+1}(C(\mathbb{T}^{n-1})) \oplus K_*(C(\mathbb{T}^{n-1})) \end{aligned}$$

by using the Bott periodicity ($n \geq 2$), where $C_0(\mathbb{R}, C(\mathbb{T}^{n-1})) \cong C_0(\mathbb{R}) \otimes C(\mathbb{T}^{n-1})$ and $* + 1$ means $* + 1 \pmod 1$. On the other hand, we have $K_0(C(\mathbb{T})) \cong \mathbb{Z}$ and $K_1(C(\mathbb{T})) \cong \mathbb{Z}$. By induction the proof is complete. □

Remark. This K-theoretic proof is quite convenient and clear. But a trouble would be to know generators in the K-groups from those isomorphisms in the statement.

From that reason we give an interpretation of Proposition 1.1 by counting the generalized Bott generators, which might be known to specialists but would not be found in the literature, as follows:

Proposition 1.2. *Let $C(\mathbb{T}^n)$ be the C^*-algebra of continuous functions on the n-torus $\mathbb{T}^n$. Then for $n \geq 1$,*

$$\begin{aligned} K_0(C(\mathbb{T}^{2n})) &\cong \mathbb{Z}^{\binom{2n}{0}} \oplus \mathbb{Z}^{\binom{2n}{2}} \oplus \cdots \oplus \mathbb{Z}^{\binom{2n}{2n}} \\ \textit{with} \quad 2^{2n-1} &= \sum_{k=0}^{n} \binom{2n}{2k}, \quad \textit{and} \\ K_0(C(\mathbb{T}^{2n+1})) &\cong \mathbb{Z}^{\binom{2n+1}{0}} \oplus \mathbb{Z}^{\binom{2n+1}{2}} \oplus \cdots \oplus \mathbb{Z}^{\binom{2n+1}{2n}} \\ \textit{with} \quad 2^{2n} &= \sum_{k=0}^{n} \binom{2n+1}{2k}, \end{aligned}$$

where the combinations $\binom{2n}{2k}$, $\binom{2n+1}{2k}$ *correspond to choosing the generalized Bott generators* $[Q_k]$ *of* $K_0(C(\mathbb{T}^{2k}))$ *defined in the proof below. Also, for* $n \geq 1$,

$$K_1(C(\mathbb{T}^{2n})) \cong \mathbb{Z}^{\binom{2n}{1}} \oplus \mathbb{Z}^{\binom{2n}{3}} \oplus \cdots \oplus \mathbb{Z}^{\binom{2n}{2n-1}}$$

$$\text{with} \quad 2^{2n-1} = \sum_{k=0}^{n-1} \binom{2n}{2k+1}, \quad \text{and}$$

$$K_1(C(\mathbb{T}^{2n+1})) \cong \mathbb{Z}^{\binom{2n+1}{1}} \oplus \mathbb{Z}^{\binom{2n+1}{3}} \oplus \cdots \oplus \mathbb{Z}^{\binom{2n+1}{2n+1}}$$

$$\text{with} \quad 2^{2n} = \sum_{k=0}^{n} \binom{2n+1}{2k+1},$$

where the combinations $\binom{2n}{2k+1}$, $\binom{2n+1}{2k+1}$ *correspond to choosing both unitary generators of tensor factors* $C(\mathbb{T})$ *in* $C(\mathbb{T}^{2n}) \cong \otimes^{2n} C(\mathbb{T})$ *(or* $C(\mathbb{T}^{2n+1}) \cong \otimes^{2n+1} C(\mathbb{T})$*) and the generalized Bott generators of* $K_0(C(\mathbb{T}^{2k}))$*, that is, the classes* $[u_l]$, $[V_k]$ *defined below.*

Proof. Following the description of the Bott generator for $K_0(C(\mathbb{T}^2))$ in the reference [AP], we define

$$K(z) = \begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix} \in M_2(\mathbb{C})$$

for $z \in \mathbb{T}$, and set $S = K(1)$. Furthermore define unitaries

$$Y(t, z) = \exp(i\pi t K(z)/2) \exp(i\pi t S/2) \in M_2(\mathbb{C})$$

for $t \in [0, 1]$. Then

$$Y(1, z) = -\begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}$$

since $\exp(i\pi V/2) = iV$ for a self-adjoint unitary V in general. Define the function P on $\mathbb{T}^2$ by

$$\begin{aligned} P(e^{2\pi i t}, z) &= Y(t, z)^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Y(t, z) \\ &\equiv \mathrm{Ad}(U(w, z))(1 \oplus 0) \in M_2(\mathbb{C}), \end{aligned}$$

where $U(w, z) = Y(t, z)^*$ for $w = e^{2\pi i t} \in \mathbb{T}$, and $\oplus$ means the diagonal sum. The class $[P]$ of P is the Bott generator for $K_0(C(\mathbb{T}^2))$.

Generalizing the above method, we define the function Q_k on $\mathbb{T}^{2k}$ by

$$\begin{aligned} &Q_k(z_{i_1}, z_{i_2}, \cdots, z_{i_{2k}}) = \\ &\mathrm{Ad}(U_1(z_{i_1}, z_{i_2}))\mathrm{Ad}(U_2(z_{i_3}, z_{i_4})) \cdots \mathrm{Ad}(U_k(z_{i_{2k-1}}, z_{i_{2k}}))(1 \oplus 0) \in M_2(\mathbb{C}) \end{aligned}$$

for $1 \leq i_1 < i_2 < \cdots < i_{2k-1} < i_{2k} \leq 2n$ (or $2n+1$) corresponding to $\binom{2n}{2k}$ (or $\binom{2n+1}{2k}$), where $U_j(\cdot, \cdot) = U(\cdot, \cdot)$ $(j \geq 1)$ means the unitary-valued function defined above. Since Q_k are projection-valued functions, their classes $[Q_k]$ become generators for $K_0(C(\mathbb{T}^{2n}))$ (or $K_0(C(\mathbb{T}^{2n+1}))$. If the finite sequences $(i_1 < i_2 < \cdots < i_{2k-1} < i_{2k})$ and $(j_1 < j_2 < \cdots < j_{2l-1} < i_{2l})$ for either $k = l$ or $k \neq l$ are different, then their corresponding functions Q_k, Q_l are inequivalent in matrix algebras over $C(\mathbb{T}^{2n})$ (or $C(\mathbb{T}^{2n+1})$). Moreover, all the functions Q_k are mutually inequivalent. In fact, since Q_k, Q_l are regarded as functions on $\mathbb{T}^{2n}$ (or $\mathbb{T}^{2n+1}$), there exists a direct product factor $\mathbb{T}$ of $\mathbb{T}^{2n}$ (or $\mathbb{T}^{2n+1}$) such that Q_k is constant on the factor $\mathbb{T}$ but Q_l is non constant (or vice versa). Thus, if Q_k, Q_l are (stably) unitarily equivalent, then we have the contradiction. Since all the functions Q_k have different components of variables on which their restrictions are constant, they are mutually inequivalent. By construction, Q_k corresponds to en element of $\binom{2n}{2k}$ (or $\binom{2n+1}{2k}$), and the unit function on $\mathbb{T}^{2n}$ (or $\mathbb{T}^{2n+1}$) corresponds to $\binom{2n}{0}$ (or $\binom{2n+1}{0}$).

Furthermore, recall the following binary expansion:

$$(1+x)^m = \binom{m}{0} + \binom{m}{1}x + \cdots + \binom{m}{m}x^m.$$

Set $m = 2n$ (or $2n+1$). Taking $x = 1$ and $x = -1$, adding both evaluated terms, and factoring by 2 imply the (counting) formulas in the first part of the statement. Subtracting the terms we obtain the (counting) formulas in the latter part.

As for the K_1-group case, recall that $K_1(C(\mathbb{T}))$ is generated by the class of the generating unitary of $C(\mathbb{T})$, that is, the function $z(t) = t$ for $t \in \mathbb{T}$. Also, $K_1(C(\mathbb{T}^2)) \cong \mathbb{Z}^2$ is generated by the classes of two generating unitaries $z_1 \otimes 1$, $1 \otimes z_2$ of $C(\mathbb{T}^2) \cong C(\mathbb{T}) \otimes C(\mathbb{T})$, where $z_j(t) = z(t)$ for $z \in \mathbb{T}$ and $j = 1, 2$. Furthermore, it is known (cf. [Wl, Lemma 3.3]) that $K_1(C(\mathbb{T}^3)) \cong \mathbb{Z}^4$ is generated by the classes of three generating unitaries $u_1 = z_1$, $u_2 = z_2$, $u_3 = z_3$ of $C(\mathbb{T}^3) \cong \otimes^3 C(\mathbb{T})$ and the class of the following function:

$$I_2 + (u_3 - 1) \otimes P_{12} \in M_2(C(\mathbb{T}^3))$$

where I_2 is the 2×2 identity matrix and P_{12} is the Bott projection corresponding to the first two variables of $\mathbb{T}^3$, where $u_3 - 1 \in C(\mathbb{T})$.

Generalizing the method above, we define the functions on $\mathbb{T}^{2k+1}$:

$$V_k = I_2 + (u_l - 1) \otimes Q_k \in M_2(C(\mathbb{T}^{2k+1}))$$

where Q_k are the generalized Bott projections corresponding to $(1 \leq i_1 < i_2 < \cdots < i_{2k} \leq s)$, and u_l is a generating unitary of a tensor factor $C(\mathbb{T})$ in $C(\mathbb{T}^s) \cong \otimes^s C(\mathbb{T})$ ($s = 2n$ or $2n+1$) where $l \neq i_j$ for $1 \leq j \leq 2k$. Thus, the pairs (u_l, Q_k) correspond to the combination $\binom{s}{2k+1}$. Then the classes $[u_l], [V_k]$ of u_l, V_k for $1 \leq l \leq s$ and $\binom{s}{2k+1}$ $(3 \leq 2k+1 \leq s)$ are mutually distinct in $K_1(C(\mathbb{T}^s))$. In fact, by definition the classes $[u_l], [V_k]$ are homotopy classes of u_l, V_k in the unitary groups of matrix algebras over $C(\mathbb{T})$ respectively. Since homotopy classes of u_l and Q_k are mutually different, the claim follows. Moreover, the classes $[u_l], [V_k]$ exhaust generators of $K_1(C(\mathbb{T}^s))$ by the counting formulas above. $\square$

2. THE QUANTUM TORI

In this section we first briefly recall the K-theory of the quantum tori, which are also called the noncommutative tori.

Proposition 2.1. *Let $\mathfrak{A}_{\Theta_n}$ be the quantum n-torus generated by unitaries $\{U_j\}_{j=1}^n$ subject to the commutation relations: $U_kU_j = e^{2\pi i\theta_{jk}}U_jU_k$ for $1 \le k, j \le n$ and $\Theta_n = (\theta_{jk})$ an $n \times n$ skew adjoint matrix. Then*

$$K_0(\mathfrak{A}_{\Theta_n}) \cong \mathbb{Z}^{2^{n-1}}, \quad \textit{and} \quad K_1(\mathfrak{A}_{\Theta_n}) \cong \mathbb{Z}^{2^{n-1}}.$$

Proof. Note that $\mathfrak{A}_{\Theta_n}$ is decomposed into a successive crossed product by $\mathbb{Z}$ as follows:

$$\mathfrak{A}_{\Theta_n} \cong (\cdots(C(\mathbb{T}) \rtimes_{\alpha_2} \mathbb{Z}) \rtimes_{\alpha_3} \mathbb{Z} \cdots) \rtimes_{\alpha_n} \mathbb{Z},$$

where α_j $(2 \le j \le n)$ are actions of $\mathbb{Z}$ defined by $\alpha_j(1) = \mathrm{Ad}(U_j)$ for $1 \in \mathbb{Z}$ on $\mathfrak{A}_{\Theta_{j-1}}$ the quantum $(j-1)$-torus generated by $\{U_k\}_{k=1}^{j-1}$. Thus, use the Pimsner-Voiculesce exact sequence for K-theory of crossed products of C^*-algebras by $\mathbb{Z}$ inductively. $\square$

Remark. A trouble would be to know generators in K-groups from the isomorphisms above. Thus, we would like to give an interpretation of Proposition 2.1 by counting the generalized Rieffel projections in the following.

Let $\mathfrak{A}_{\Theta_n}$ denote the quantum n-torus defined above. We say that $\mathfrak{A}_{\Theta_n}$ is irrational if any canonical quantum l-tori $\mathfrak{A}_{\Theta_l}$ $(2 \le l \le n-1)$ in $\mathfrak{A}_{\Theta_n}$ generated by some $U_{j_1}, U_{j_2}, \cdots, U_{j_l}$ with $1 \le j_1 < j_2 < \cdots < j_l \le n$ is non-rational, where $\mathfrak{A}_{\Theta_l}$ is rational if corresponding components θ_{jk} in Θ_l are all rational. Note that there are $\binom{n}{l}$ quantum l-tori of the form $\mathfrak{A}_{\Theta_l}$ in $\mathfrak{A}_{\Theta_n}$ for $2 \le l \le n-1$, where $\binom{n}{l}$ means the combination.

Proposition 2.2. *Let $\mathfrak{A}_{\Theta_n}$ be the quantum n-torus generated by unitaries $\{U_j\}_{j=1}^n$ subject to the relations: $U_kU_j = e^{2\pi i\theta_{jk}}U_jU_k$ $(1 \le k, j \le n)$. Then for $n \ge 1$,*

$$\begin{aligned}
K_0(\mathfrak{A}_{\Theta_{2n}}) &\cong \mathbb{Z}^{\binom{2n}{0}} \oplus \mathbb{Z}^{\binom{2n}{2}} \oplus \cdots \oplus \mathbb{Z}^{\binom{2n}{2n}} \\
&\cong \mathbb{Z} + \sum_{i<j} \mathbb{Z}\theta_{ij} + \sum_{i<j<k<l} \mathbb{Z}\theta_{ijkl} + \cdots + \mathbb{Z}\theta_{12\cdots 2n-1,2n} \\
&\textit{with} \quad 2^{2n-1} = \sum_{k=0}^{n} \binom{2n}{2k}, \quad \textit{and} \\
K_0(\mathfrak{A}_{\Theta_{2n+1}}) &\cong \mathbb{Z}^{\binom{2n+1}{0}} \oplus \mathbb{Z}^{\binom{2n+1}{2}} \oplus \cdots \oplus \mathbb{Z}^{\binom{2n+1}{2n}} \\
&\cong \mathbb{Z} + \sum_{i<j} \mathbb{Z}\theta_{ij} + \sum_{i<j<k<l} \mathbb{Z}\theta_{ijkl} + \cdots + \sum^{2n+1} \mathbb{Z}\theta_{i_1 i_2 \cdots i_{2n-1} i_{2n}} \\
&\textit{with} \quad 2^{2n} = \sum_{k=0}^{n} \binom{2n+1}{2k},
\end{aligned}$$

where $\theta_{i_1 i_2 \cdots i_{2k}} = \sum_{j=1}^{k} \sum_{l=1}^{k} \theta_{i_j i_{k+l}}$ *with* $1 \le i_1 < i_2 < \cdots < i_{2k} \le 2n$ *(or* $2n+1$*), and the combinations* $\binom{2n}{2k}$, $\binom{2n+1}{2k}$ *correspond to choosing the generalized Rieffel projections* E_k *of* $K_0(\mathfrak{A}_{\Theta_{2k}})$ *defined in the proof below. Furthermore suppose that* $\mathfrak{A}_{\Theta_n}$ *is irrational as defined above. Then, for* $n \ge 1$,

$$K_1(\mathfrak{A}_{\Theta_{2n}}) \cong \mathbb{Z}^{\binom{2n}{1}} \oplus \mathbb{Z}^{\binom{2n}{3}} \oplus \cdots \oplus \mathbb{Z}^{\binom{2n}{2n-1}}$$

$$\textit{with} \quad 2^{2n-1} = \sum_{k=0}^{n-1} \binom{2n}{2k+1}, \quad \textit{and}$$

$$K_1(\mathfrak{A}_{\Theta_{2n+1}}) \cong \mathbb{Z}^{\binom{2n+1}{1}} \oplus \mathbb{Z}^{\binom{2n+1}{3}} \oplus \cdots \oplus \mathbb{Z}^{\binom{2n+1}{2n+1}},$$

$$\textit{with} \quad 2^{2n} = \sum_{k=0}^{n} \binom{2n+1}{2k+1},$$

where the combinations $\binom{2n}{2k+1}$, $\binom{2n+1}{2k+1}$ *correspond to choosing both unitary generators* U_l *of* $\mathfrak{A}_{\Theta_{2n}}$ *(or* $\mathfrak{A}_{\Theta_{2n+1}}$*) and the generalized Rieffel projections* E_k *of* $K_0(\mathfrak{A}_{\Theta_{2k}})$*, that is, unitaries* U_l*,* $V_{k,l}$ *defined below.*

Proof. We first review the case $n = 2$, that is, $\mathfrak{A}_{\Theta_2} = \mathfrak{A}_\theta$ with $\theta = \theta_{12}$ the rotation algebras. It is well known (cf. [Wo]) that

$$K_0(\mathfrak{A}_\theta) \cong \mathbb{Z} + \mathbb{Z}\theta, \quad K_1(\mathfrak{A}_\theta) \cong \mathbb{Z} + \mathbb{Z},$$

where the group $\mathbb{Z} + \mathbb{Z}\theta$ also means an ordered subgroup of the real line $\mathbb{R}$, the generators of $K_0(\mathfrak{A}_\theta)$ are the classes of the unit and the Rieffel projection of $\mathfrak{A}_\theta$, and the generators of $K_1(\mathfrak{A}_\theta)$ are the classes of two generating unitaries of $\mathfrak{A}_\theta$.

We next consider the case $n = 3$. Recall that $\mathfrak{A}_{\Theta_3}$ is regarded as the twisted crossed product $C^*(\mathbb{Z}^3, \sigma)$ of $\mathbb{Z}^3$ with σ a cocycle. Define $C^*(\mathbb{Z}^2, \sigma_{ij})$ for $1 \le i < j \le 3$ to be the twisted crossed products of the canonical subgroups $\mathbb{Z}^2$ of $\mathbb{Z}^3$ corresponding to pairs U_i, U_j of the generators in $\mathfrak{A}_{\Theta_3}$. Then $C^*(\mathbb{Z}^2, \sigma_{ij}) \cong \mathfrak{A}_{\theta_{ij}}$. Thus, we have

$$K_0(\mathfrak{A}_{\Theta_3}) \cong \mathbb{Z} + \mathbb{Z}\theta_{12} + \mathbb{Z}\theta_{23} + \mathbb{Z}\theta_{13},$$

where the generators of $K_0(\mathfrak{A}_{\Theta_3})$ are the classes of the unit of $\mathfrak{A}_{\Theta_3}$ and the Rieffel projections e_{ij} of $\mathfrak{A}_{\theta_{ij}}$ with the traces θ_{ij} $(1 \le i < j \le 3)$. Recall that the Rieffel projections e_{ij} of $\mathfrak{A}_{\theta_{ij}}$ are given by

$$U_j^* \bar{g}(U_i) + f(U_i) + g(U_i)U_j$$

where $f, g \in C(\mathbb{T})$ satisfying some compatible relations (see [Rf], [Wo]).

Moreover, we have

$$K_1(\mathfrak{A}_{\Theta_3}) \cong \mathbb{Z}^3 + \mathbb{Z},$$

where three generators of $K_1(\mathfrak{A}_{\Theta_3})$ correspond to the generating unitaries U_j of $\mathfrak{A}_{\Theta_3}$ $(1 \le j \le 3)$, and the forth generator is given by the class of the following unitary:

$$1 - e_{12} + e_{12}U_3 w_3^* e_{12},$$

where e_{12} is the Rieffel projection of $\mathfrak{A}_{\theta_{12}}$ corresponding to θ_{12}, and w_3 is a unitary of $\mathfrak{A}_{\theta_{12}}$ such that $U_3^* e_{12} U_3 = w_3^* e_{12} w_3$, which is deduced from Rieffel's cancellation theorem (see [Rf2, Corollary 7.8 and Proposition 8.2] and [Wl, Lemma at p.495]) since $\mathfrak{A}_{\theta_{12}}$ is an irrational rotation algebra.

Generalizing the method above, we define the generalized Rieffel projections in $\mathfrak{A}_{\Theta_{2k}}$ by

$$E_k \equiv W_k^* \bar{g}(V_k) + f(V_k) + g(V_k) W_k,$$
$$\text{where} \quad V_k = U_{i_1} U_{i_2} \cdots U_{i_k}, \quad \text{and} \quad W_k = U_{i_{k+1}} U_{i_{k+2}} \cdots U_{i_{2k}}$$

for $1 \leq i_1 < i_2 < \cdots < i_{2k} \leq 2n$ (or $2n+1$). Note that

$$\begin{aligned} W_k U_k &= \Pi_{j=1}^k (\Pi_{l=1}^k e^{2\pi i \theta_{i_j i_{k+l}}}) U_k W_k \\ &= \exp(2\pi i \sum_{j=1}^k \sum_{l=1}^k \theta_{i_j i_{k+l}}) U_k W_k. \end{aligned}$$

Set $\theta_{i_1 i_2 \cdots i_{2k}} = \sum_{j=1}^k \sum_{l=1}^k \theta_{i_j i_{k+l}}$. Then the class of the unit of $\mathfrak{A}_{\Theta_{2n}}$ (or $\mathfrak{A}_{\Theta_{2n+1}}$) and the classes of E_k for $\binom{2n}{2k}$ (or $\binom{2n+1}{2k}$) $(1 \leq k \leq n)$ generate $K_0(\mathfrak{A}_{\Theta_s})$ $(s = 2n$ or $2n+1)$.

Furthermore, we define the following unitaries in $\mathfrak{A}_{\Theta_{2k}}$:

$$V_{k,l} \equiv 1 - E_k + E_k U_l w_l^* E_k$$

where E_k is the generalized Rieffel projection of $\mathfrak{A}_{\Theta_{2k}}$ corresponding to the finite sequence $(i_1, i_2, \cdots, i_{2k})$, and U_l is a generating unitary of $\mathfrak{A}_{\Theta_{2n}}$ (or $\mathfrak{A}_{\Theta_{2n+1}}$) such that $l \neq i_j$ for $1 \leq j \leq 2k$, and w_l is a unitary of $\mathfrak{A}_{\Theta_{2k}}$ such that $U_l^* E_k U_l = w_l^* E_k w_l$ by Rieffel's cancellation theorem ([Rf2, Corollary 7.8 and Proposition 8.2]) since $\mathfrak{A}_{\Theta_{2k}}$ is a non-rational noncommutative torus. Then the classes of U_j $(1 \leq j \leq 2n$ (or $2n+1$)) and the classes of $V_{k,l}$ for $\binom{2n}{2k+1}$ (or $\binom{2n+1}{2k+1}$) $(1 \leq k \leq n-1$ (or n)) generate $K_1(\mathfrak{A}_{\Theta_s})$ $(s = 2n$ or $2n+1)$. $\square$

Remark. As an example of rational quantum tori $\mathfrak{A}_{\Theta_n}$, it is known that the generators of $K_1(\mathfrak{A}_\theta \otimes C(\mathbb{T}))$ for $\mathfrak{A}_\theta = \mathfrak{A}_{\Theta_2}$ with θ irrational correspond to two unitary generators of $\mathfrak{A}_\theta$, the generating unitary W of $C(\mathbb{T})$, and the unitary defined by

$$(1-e) \otimes 1 + e \otimes W,$$

where e means the Rieffel projection of $\mathfrak{A}_\theta$ with the trace θ (see [Wl, 2.1 for $A_\theta^{5,1}$]). Note also that rational quantum tori and their K-theory are reduced to the case of matrix algebras over $C(\mathbb{T}^k)$ for some k (cf. [EL]).

3. Continuous fields of the quantum tori

As an application of the results obtained in the previous sections we have

Theorem 3.1. *Let $\Gamma(\mathbb{T}^m, \{\mathfrak{A}_{\Theta_n^t}\}_{t\in\mathbb{T}^m})$ be the C^*-algebra of a continuous field on $\mathbb{T}^m$ with the fibers $\mathfrak{A}_{\Theta_n^t}$ the quantum n-tori generated by unitaries U_j $(1 \leq j \leq n)$ such that their commutation relations at each $t = (t_1, \cdots, t_m) \in \mathbb{T}^m$ are given by $U_k U_j = t_1 \cdots t_m U_j U_k$ $(1 \leq j < k \leq n)$. Then the fibers $\mathfrak{A}_{\Theta_n^t}$ are irrational for irrational points $t \in \mathbb{T}^m = [0,1]^m$ (mod 1). Suppose that the constant operator fields: $\mathbb{T}^m \ni t \mapsto U_j$ $(1 \leq j \leq n)$ belong to $\Gamma(\mathbb{T}^m, \{\mathfrak{A}_{\Theta_n^t}\}_{t\in\mathbb{T}^m})$. Then for $* = 0, 1$,*

$$\begin{aligned} K_*(\Gamma(\mathbb{T}^m, \{\mathfrak{A}_{\Theta_n^t}\}_{t\in\mathbb{T}^m})) &\cong K_*(C(\mathbb{T}^m)) \oplus (K_*(C^*(C(\mathbb{T}^m), U_1))/K_*(C(\mathbb{T}^m))) \\ &\oplus (K_*(C^*(C(\mathbb{T}^m), U_2))/K_*(C(\mathbb{T}^m))) \oplus \cdots \\ &\cdots \oplus (K_*(C^*(C(\mathbb{T}^m), U_n))/K_*(C(\mathbb{T}^m))) \\ &\cong \mathbb{Z}^{2^{m-1}} \oplus (\oplus_{j=1}^n (\mathbb{Z}^{2^m}/\mathbb{Z}^{2^{m-1}})) \cong \mathbb{Z}^{(n+1)2^{m-1}}, \end{aligned}$$

where $C^*(C(\mathbb{T}^m), U_j)$ $(1 \leq j \leq n)$ mean the C^*-algebras generated by $C(\mathbb{T}^m)$ and U_j, and in fact $C^*(C(\mathbb{T}^m), U_j) \cong C(\mathbb{T}^{m+1})$, and $K_*(C^*(C(\mathbb{T}^m), U_j))/K_*(C(\mathbb{T}^m))$ $(* = 0, 1)$ mean the quotients by the subgroups corresponding to the K-groups of $C(\mathbb{T}^m) \cong \Gamma(\mathbb{T}^m, \{\mathbb{C}1_t\}_{t\in\mathbb{T}^m})$ in $\Gamma(\mathbb{T}^m, \{\mathfrak{A}_{\Theta_n^t}\}_{t\in\mathbb{T}^m})$ with 1_t the unit of $\mathfrak{A}_{\Theta_n^t}$ for $t \in \mathbb{T}^m$.

Proof. First note that $\Gamma(\mathbb{T}^m, \{\mathfrak{A}_{\Theta_n^t}\}_{t\in\mathbb{T}^m})$ is regarded as the C^*-algebra generated by those unitaries U_j $(1 \leq j \leq n)$ and commuting unitaries Z_k $(1 \leq k \leq m)$ such that $C(\mathbb{T}^m) \cong C^*(Z_1, \cdots, Z_m)$ the C^*-algebra generated by $\{Z_k\}_{k=1}^m$. Thus, the first direct summands of those K-groups in the statement correspond to $K_*(C^*(Z_1, \cdots, Z_m))$ $(* = 0, 1)$.

Since the unitaries Z_k $(1 \leq k \leq m)$ and U_j commute, we have

$$C^*(C(\mathbb{T}^m), U_j)) \cong C(\mathbb{T}^{m+1})$$

for $1 \leq j \leq n$. Note that we have the following inclusions:

$$C^*(Z_1, \cdots, Z_m) \subset C^*(C(\mathbb{T}^m), U_j)) \subset \Gamma(\mathbb{T}^m, \{\mathfrak{A}_{\Theta_n^t}\}_{t\in\mathbb{T}^m})$$

by the assumption on U_j. Since the K-groups of $C^*(Z_1, \cdots, Z_m) \cong C(\mathbb{T}^m)$ are included in those of $C^*(C(\mathbb{T}^m), U_j)) \cong C(\mathbb{T}^{m+1})$, the following mutually disjoint quotients:

$$K_*(C^*(C(\mathbb{T}^m), U_j))/K_*(C(\mathbb{T}^m))$$

for $* = 0, 1$ and $1 \leq j \leq n$ are embedded in the K-groups of $\Gamma(\mathbb{T}^m, \{\mathfrak{A}_{\Theta_n^t}\}_{t\in\mathbb{T}^m})$. Their mutual disjointness follows from the analysis for Proposition 1.2. In fact, the generalized Bott projections involved with U_j are mutually inequivalent for different j.

Furthermore, we have the following inclusions:

$$\Gamma(\mathbb{T}^m, \{\mathfrak{A}_{\Theta_k^t}\}_{t\in\mathbb{T}^m}) \subset \Gamma(\mathbb{T}^m, \{\mathfrak{A}_{\Theta_n^t}\}_{t\in\mathbb{T}^m})$$

for $2 \leq k \leq n$, where there are $\binom{n}{k}$-inclusions for k fixed by choosing k-generators of $\mathfrak{A}_{\Theta_k^t}$ among n-generators of the fibers $\mathfrak{A}_{\Theta_n^t}$. Thus, the following mutually disjoint quotients:

$$\oplus^{\binom{n}{k}} K_*(\Gamma(\mathbb{T}^m, \{\mathfrak{A}_{\Theta_k^t}\}_{t\in\mathbb{T}^m}))/\oplus^{\binom{n}{k-1}} K_*(\Gamma(\mathbb{T}^m, \{\mathfrak{A}_{\Theta_{k-1}^t}\}_{t\in\mathbb{T}^m}))$$

for $* = 0, 1$ and $1 \leq k \leq n$ are embedded in the K-groups of $\Gamma(\mathbb{T}^m, \{\mathfrak{A}_{\Theta_n^t}\}_{t\in\mathbb{T}^m})$. However, these quotients are all trivial because the fibers are noncommutative tori, and the generators of their K-groups are given by the generalized Rieffel projections as in Proposition 2.2, but they are not continuous at the point $(1, \cdots, 1) \in \mathbb{T}^m$ since they are not definable at the point. Note that there are no non-trivial projections of $\mathfrak{A}_{\Theta_n^{(1,\cdots,1)}} = C(\mathbb{T}^n)$. Therefore, the K-classes of the generalized Rieffel projections of the fibers do not produce those for $\Gamma(\mathbb{T}^m, \{\mathfrak{A}_{\Theta_n^t}\}_{t\in\mathbb{T}^m})$. Hence, in other words, the K-groups of the fibers are not continuous over $\mathbb{T}^m$. Note also that any projection (or unitary) of matrix algebras over $\Gamma(\mathbb{T}^m, \{\mathfrak{A}_{\Theta_n^t}\}_{t\in\mathbb{T}^m})$ does produce projections (or unitaries) of matrix algebras over the fibers. □

Remark. The structure of continuous fields on $\mathbb{T}^m$ is crucial to the K-groups of the C^*-algebras of continuous fields. For example, if the relations $U_kU_j = t_1 \cdots t_m U_j U_k$ are constant, that is, $t_1 \cdots t_m = z_1 \cdots z_m$ for $(t_1, \cdots, t_m) \in \mathbb{T}^m$ and some fixed $z = (z_1, \cdots, z_m) \in \mathbb{T}^m$, then $\Gamma(\mathbb{T}^m, \{\mathfrak{A}_{\Theta_n^t}\}_{t\in\mathbb{T}^m}) \cong C(\mathbb{T}^m) \otimes \mathfrak{A}_{\Theta_n^z}$. In particular, both K-groups in the case where $m = 1$ and $n = 2$ are $\mathbb{Z}^4$.

Now recall that the discrete Heisenberg group H_3^d of rank 3 is defined by the following matrices:

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \in GL_3(\mathbb{Z}).$$

Corollary 3.2. *Let $C^*(H_3^d)$ be the group C^*-algebra of H_3^d. Then*

$$K_0(C^*(H_3^d)) \cong \mathbb{Z}^3, \quad \textit{and} \quad K_1(C^*(H_3^d)) \cong \mathbb{Z}^3.$$

Proof. By definition, the group H_3^d is isomorphic to the semi-direct product $\mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}$ with the action α defined by $\alpha_a(c, b) = (c + ab, b)$ for $a, b, c \in \mathbb{Z}$. Hence, $C^*(H_3^d)$ is isomorphic to the crossed product $C^*(\mathbb{Z}^2) \rtimes_\alpha \mathbb{Z}$ (cf. [Pd] for crossed products of C^*-algebras). By the Fourier transform, $C^*(\mathbb{Z}^2) \rtimes_\alpha \mathbb{Z} \cong C(\mathbb{T}^2) \rtimes_{\hat{\alpha}} \mathbb{Z}$, where the action $\hat{\alpha}$ is defined by $\hat{\alpha}_a(z, w) = (z, z^a w)$ for $z, w \in \mathbb{T}$. Moreover, we have

$$C(\mathbb{T}^2) \rtimes_{\hat{\alpha}} \mathbb{Z} \cong \Gamma(\mathbb{T}, \{\mathfrak{A}_{\theta^z}\}_{z\in\mathbb{T}}),$$

where $\mathfrak{A}_{\theta^z}$ mean the quantum 2-tori for $z = e^{2\pi i \theta_z} \in \mathbb{T}$ with $\theta_z \in [0, 1]$. Let U_1, U_2 be two unitary generators of $\mathfrak{A}_{\theta^z}$ with the commutation relation $U_2U_1 = e^{2\pi i\theta_z} U_1 U_2$. Using Theorem 3.1, we obtain

$$\begin{aligned} K_*(\Gamma(\mathbb{T}, \{\mathfrak{A}_{\theta^z}\}_{z\in\mathbb{T}})) &\cong K_*(C(\mathbb{T})) \oplus K_*(C^*(C(\mathbb{T}), U_1))/K_*(C(\mathbb{T})) \\ &\oplus K_*(C^*(C(\mathbb{T}), U_2))/K_*(C(\mathbb{T})) \\ &\cong \oplus^3 K_*(C(\mathbb{T})) \cong \mathbb{Z}^3 \quad \text{for } * = 0, 1. \end{aligned}$$

Remark. This corollary was first proved by [AP, Proposition 1.4] using the Pimser-Voiculesce exact sequence. Thus, our proof above is quite different from their result since our method is based on continuous fields of C^*-algebras.

Furthermore, recall that the generalized discrete Heisenberg group H_{2n+1}^d of rank $2n+1$ is defined by the following matrices:

$$\begin{pmatrix} 1 & a & c \\ 0_n^t & 1_n & b^t \\ 0 & 0_n & 1 \end{pmatrix} \in GL_{n+2}(\mathbb{Z}),$$

where $a = (a_j), b = (b_j), 0_n = (0) \in \mathbb{Z}^n$, $c \in \mathbb{Z}$ and b^t, 0_n^t are the transposes of b, 0_n respectively, and 1_n is the $n \times n$ identity matrix. Then we obtain the following:

Theorem 3.3. *Let $C^*(H_{2n+1}^d)$ be the group C^*-algebra of H_{2n+1}^d. Then*

$$\begin{aligned}
&K_0(C^*(H_{2n+1}^d)) \\
&\cong \begin{cases} \mathbb{Z}^{2n+1} \oplus \mathbb{Z}^{2^2\binom{n}{2}} \oplus \mathbb{Z}^{2^4\binom{n}{4}} \oplus \cdots \oplus \mathbb{Z}^{2^{2m}\binom{2m}{2m}} & \\ \oplus \mathbb{Z}^{2^3\binom{n}{3}} \oplus \mathbb{Z}^{2^5\binom{n}{5}} \oplus \cdots \oplus \mathbb{Z}^{2^{2m-1}\binom{2m}{2m-1}} & n = 2m, \\ \mathbb{Z}^{2n+1} \oplus \mathbb{Z}^{2^2\binom{n}{2}} \oplus \mathbb{Z}^{2^4\binom{n}{4}} \oplus \cdots \oplus \mathbb{Z}^{2^{2m}\binom{2m+1}{2m}} & \\ \oplus \mathbb{Z}^{2^3\binom{n}{3}} \oplus \mathbb{Z}^{2^5\binom{n}{5}} \oplus \cdots \oplus \mathbb{Z}^{2^{2m+1}\binom{2m+1}{2m+1}} & n = 2m+1, \end{cases} \\
&= \mathbb{Z}^{2n+1} \oplus \mathbb{Z}^{2^2\binom{n}{2}} \oplus \mathbb{Z}^{2^3\binom{n}{3}} \oplus \cdots \oplus \mathbb{Z}^{2^n\binom{n}{n}} \\
&= \mathbb{Z}^{1+2\binom{n}{1}+2^2\binom{n}{2}+\cdots+2^n\binom{n}{n}} = \mathbb{Z}^{3^n}, \quad \textit{and} \\
&K_1(C^*(H_{2n+1}^d)) \\
&\cong \begin{cases} \mathbb{Z}^{2n+1} \oplus \mathbb{Z}^{2^3\binom{n}{3}} \oplus \mathbb{Z}^{2^5\binom{n}{5}} \oplus \cdots \oplus \mathbb{Z}^{2^{2m-1}\binom{2m}{2m-1}} & \\ \oplus \mathbb{Z}^{2^2\binom{n}{2}} \oplus \mathbb{Z}^{2^4\binom{n}{4}} \oplus \cdots \oplus \mathbb{Z}^{2^{2m}\binom{2m}{2m}} & n = 2m, \\ \mathbb{Z}^{2n+1} \oplus \mathbb{Z}^{2^3\binom{n}{3}} \oplus \mathbb{Z}^{2^5\binom{n}{5}} \oplus \cdots \oplus \mathbb{Z}^{2^{2m+1}\binom{2m+1}{2m+1}} & \\ \oplus \mathbb{Z}^{2^2\binom{n}{2}} \oplus \mathbb{Z}^{2^4\binom{n}{4}} \oplus \cdots \oplus \mathbb{Z}^{2^{2m}\binom{2m+1}{2m}} & n = 2m+1, \end{cases} \\
&= \mathbb{Z}^{2n+1} \oplus \mathbb{Z}^{2^2\binom{n}{2}} \oplus \mathbb{Z}^{2^3\binom{n}{3}} \oplus \cdots \oplus \mathbb{Z}^{2^n\binom{n}{n}} \\
&= \mathbb{Z}^{1+2\binom{n}{1}+2^2\binom{n}{2}+\cdots+2^n\binom{n}{n}} = \mathbb{Z}^{3^n},
\end{aligned}$$

where $C^*(H_{2n+1}^d) \cong \Gamma(\mathbb{T}, \{\mathfrak{A}_{\Theta^z}\}_{z\in\mathbb{T}})$ with the fibers $\mathfrak{A}_{\Theta^z}$ the quantum $2n$-tori isomorphic to the tensor product $\otimes^n \mathfrak{A}_{\theta^z}$ of the quantum 2-tori $\mathfrak{A}_{\theta^z}$ for $z = e^{2\pi i\theta_z} \in \mathbb{T}$, and the combination $\binom{n}{2k}$ (or $\binom{n}{2k+1}$) corresponds to choosing $2k$-tensor factors $\otimes^{2k}\mathfrak{A}_{\theta^z}$ of $\otimes^n \mathfrak{A}_{\theta^z}$, and the power 2^{2k} (or 2^{2k+1}) corresponds to choosing either the generalized Bott projections associated with commuting unitaries, each of which is chosen from two unitaries of each factor $\mathfrak{A}_{\theta^z}$ of the $2k$-tensor factors (or those projections and generating unitaries, each of which is chosen from two unitaries of one factor $\mathfrak{A}_{\theta^z}$ of the $2k$-tensor factors) in the K_0 (or K_1)-case, and the combination $\binom{n}{2k+1}$ (or $\binom{n}{2k}$) and the power 2^{2k+1} (or 2^{2k}) correspond to the choosings above and the unitary generator of $C(\mathbb{T})$ for $\mathbb{T}$ the base space (respectively).

Proof. By definition, the group H_{2n+1}^d is isomorphic to the semi-direct product $\mathbb{Z}^{n+1} \rtimes_\alpha \mathbb{Z}^n$ with the action α defined by $\alpha_a(c,b) = (c + \sum_{j=1}^n a_j b_j, b)$ for $a = (a_j), b = (b_j) \in \mathbb{Z}^n$, $c \in \mathbb{Z}$. Hence, $C^*(H_{2n+1}^d) \cong C^*(\mathbb{Z}^{n+1}) \rtimes_\alpha \mathbb{Z}^n$. By the Fourier transform, $C^*(\mathbb{Z}^{n+1}) \rtimes_\alpha \mathbb{Z}^n \cong C(\mathbb{T}^{n+1}) \rtimes_{\hat{\alpha}} \mathbb{Z}^n$, where the action $\hat{\alpha}$ is defined by $\hat{\alpha}_a(z,w) = (z, (z^{a_j} w_j))$ for $z \in \mathbb{Z}$ and $w = (w_j) \in \mathbb{T}^n$. Moreover, we have

$$C(\mathbb{T}^{n+1}) \rtimes_{\hat{\alpha}} \mathbb{Z}^n \cong \Gamma(\mathbb{T}, \{\mathfrak{A}_{\Theta^z}\}_{z\in\mathbb{T}}),$$

where $\mathfrak{A}_{\Theta^z} = C(\mathbb{T}^n) \rtimes_{\hat{\alpha^z}} \mathbb{Z}^n$ mean the quantum $2n$-tori for $z = e^{2\pi i\theta_z} \in \mathbb{T}$ with $\theta_z \in [0,1]$ and $\Theta^z = (\theta_{jk})_{j,k=1}^{2n}$ with $\theta_{j,j+n} = \theta_z$ $(1 \le j \le n)$ and $\theta_{jk} = 0$ otherwise, where $\hat{\alpha}^z$ is the restriction of $\hat{\alpha}$ to $\{z\} \times \mathbb{T}^n$ in $\mathbb{T}^{n+1}$. Note that $\mathfrak{A}_{\Theta^z} \cong \otimes^n \mathfrak{A}_{\theta^z}$ the tensor product of n copies of the quantum 2-torus $\mathfrak{A}_{\theta^z}$. Let $\{U_j\}_{j=1}^{2n}$ be $2n$ unitary generators of $\mathfrak{A}_{\Theta^z}$ with the commutation relation $U_{j+n}U_j = e^{2\pi i\theta_z} U_j U_{j+n}$ for $1 \le j \le n$. By using Theorem 3.1 (in part), $K_*(\Gamma(\mathbb{T}, \{\mathfrak{A}_{\theta^z}\}_{z\in\mathbb{T}}))$ contains

$$\begin{aligned}
&K_*(C(\mathbb{T})) \oplus K_*(C^*(C(\mathbb{T}), U_1))/K_*(C(\mathbb{T})) \\
&\oplus K_*(C^*(C(\mathbb{T}), U_2))/K_*(C(\mathbb{T})) \oplus \cdots \\
&\cdots \oplus K_*(C^*(C(\mathbb{T}), U_{2n}))/K_*(C(\mathbb{T})) \\
&\cong \oplus^{2n+1} K_*(C(\mathbb{T})) \cong \mathbb{Z}^{2n+1}
\end{aligned}$$

for $* = 0, 1$. Since the fibers $\mathfrak{A}_{\Theta^z}$ are non-irrational, we need to further consider commuting unitaries among the generators of $\mathfrak{A}_{\Theta^z}$ and $C(\mathbb{T})$ for $\mathbb{T}$ the base space and the generalized Bott generators associated with them. However, this can be done as stated above and done in Proposition 1.2. Furthermore, we use the following binary expansion:

$$(1+2x)^n = 1 + \binom{n}{1} 2x + \binom{n}{2} (2x)^2 + \cdots + \binom{n}{n} (2x)^n.$$

Take $x = 1$. Then

$$3^n = 1 + \binom{n}{1} 2 + \binom{n}{2} 2^2 + \cdots + \binom{n}{n} 2^n. \qquad \square$$

Remark. This theorem could be obtained by using the Pimsner-Voiculesce's six-term exact sequence for K-groups of crossed products of C^*-algebras by $\mathbb{Z}$ (cf. [Bl], [Wo]) repeatedly. However, chasing the maps on the K-groups involved in the six-term exact sequences of K-groups associated with the successive crossed products by $\mathbb{Z}$ such as $C(\mathbb{T}^{n+1}) \rtimes_{\hat{\alpha}} \mathbb{Z}^n \cong (\cdots (C(\mathbb{T}^{n+1}) \rtimes \mathbb{Z}) \rtimes \mathbb{Z} \cdots) \rtimes \mathbb{Z}$ is somewhat complicated in general. Thus, our computing method above seems to be better in those cases.

Remark. Moreover, it is shown by [LP] that the (twisted) group C^*-algebras of (certain) two-step nilpotent discrete groups including the generalized discrete Heisenberg groups of Theorem 3.3 are decomposed into continuous fields of C^*-algebras on the duals of their centers with fibers isomorphic to matrix algebras over the quantum tori. Since K-groups are stable under taking tensor products with matrix algebras over $\mathbb{C}$, K-groups of (most of) those (twisted) group C^*-algebras in that case can be computed by Theorem 3.1. See also [PR] for K-theory of twisted group C^*-algebras, for which it is shown to be the same as that of (untwisted) group C^*-algebras in many cases. Note that their formulations are quite different from our direct ones.

Remark. As the final remark our method (of Theorems 3.1 and 3.3) seems to be giving a general principle for solving the Problem in the introduction, that is, if we once know projections and unitaries of the fibers generating their K-groups, and their continuity (or discontinuity) over the base spaces, then one can (almost) determine the classes of the K-groups of the C^*-algebras $\Gamma(X, \{\mathfrak{A}_t\}_{t\in X})$ of continuous fields. However, in general, it is hard to know projections and unitaries of the fibers (or general C^*-algebras), and also hard to know their continuity (or discontinuity).

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