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GROWTH SEQUENCES FOR FLAT DIFFEOMORPHISMS OF THE INTERVAL

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1. Introduction and statement of results

Let f be a C^1-diffeomorphism of the interval $[0;1]$. We define a *growth sequence* for f by

$$\Gamma_n(f)=\exp\|\log Df^n\|=\max(\|Df^n\|,\|Df^{-n}\|),$$

where f^n is nth iteration of f and $\|Df^n\|=\max_{x\in[0;1]}|Df^n(x)|$.

Let $\mathrm{Fix}(f)$ be the set of fixed points of f. In the case f is of class C^r, for $x\in\mathrm{Fix}(f)$ x is called *r-flat* if $Df(x)=1$ and $D^nf(x)=0$ for $2\le n\le[r]$. f is called *r-flat* if every $x\in\mathrm{Fix}(x)$ is r-flat.

In this paper, we answer the question raised in the paper by L. Polterovich and M. Sodin [2]. We show :

Theorem 1. *Let f be a 2-flat diffeomorphism of the interval. Then,*

$$\lim_{n\to\infty}\frac{\Gamma_n(f)}{n^2}=0.$$

Theorem 2. *There exists an ∞-flat diffeomorphism f of the interval such that for every $\alpha<2$,*

$$\limsup_{n\to\infty}\frac{\Gamma_n(f)}{n^\alpha}=\infty.$$

Independently, A. Borichev shows similar results [1].

2. proof of Theorem 1

The argument in Proof of Theorem 1 is a slight modification of its in [2]. The following is useful.

Lemma 3. (Denjoy) *Let f be a C^2-diffeommorphism of $[0;1]$. If $J\in[0;1]$ is a closed interval such that $\mathrm{Int}(J)\cap f(\mathrm{Int}(J))=\emptyset$ then there exists a positive constant C depending on f such that for every $n\in\mathbb{N}$ and every $x,y\in J$*

$$\frac{1}{C}\le\frac{Df^n(x)}{Df^n(y)}\le C.$$

Lemma 4. (Growth lemma, [2]) *Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers such that for each $n \geq 1$*

$$2a_n - a_{n-1} - a_{n+1} \leq Ce^{-a_n}, \quad C > 0,$$

and $a_0 = 0$. Then either for each $n \geq 0$

$$a_n \leq 2\log\left(n\sqrt{\frac{C}{2}}+1\right), \ or \ \lim_{n\to\infty}\frac{a_n}{n} > 0.$$

By the following lemma estimations can be done in neighbourhoods of Fix(f).

Lemma 5. *Let U be a neighbourhood of* Fix(f). *There exists $N \in \mathbb{N}$ such that if $n \geq N$ and $Df^n(x) \geq 1$, then $x \in U$.*

Proof. Let $x \in [0;1] - U$ and $J(x) = [x; f(x)]$. Then $m = \min_x |J(x)| > 0$ and $M_n = \max_x |f^n(J(x))| \to 0$ as $n \to \infty$.

On the other hand by Lemma 3,

$$Df^n(x) \leq C\frac{|f^n(J(x))|}{|J(x)|} \leq C\frac{M_n}{m}.$$

□

For $n \geq 0$ put

$$a_n = \max_{x\in[0;1]} \log Df^n(x), \ a_n^- = \max_{x\in[0;1]} \log Df^{-n}(x).$$

We consider for a_n . For a_n^- the argument is the same.

Lemma 6. *For any $\varepsilon > 0$, there exist $N \in \mathbb{N}$ such that if $n \geq N$ then the sequence a_n satisfies inequality*

$$2a_n - a_{n-1} - a_{n+1} \leq \varepsilon e^{-a_n}.$$

Proof. Let δ be any positive number and U_δ be a neibourhood of Fix(f) which has the property that if $x \in U_\delta$ then $|Df(x) - 1| \leq \delta$ and $D^2f(x) \leq \delta$. By Lemma 5, there exists $N \in \mathbb{N}$ such that if $n \geq N$ and $a_n = \log Df^n(x_0)$ then $x_0 \in U_\delta$. Put $x_j = f^j(x_0)$. Then, for $n \geq N$ by Lemma 3 we have

$$2a_n - a_{n-1} - a_{n+1} \leq 2\log Df^n(x_0) - \log Df^{n-1}(x_1) - \log Df^{n+1}(x_{-1})$$

$$= \log Df(x_0) - \log Df(x_{-1}) = D\log Df(y_1)(x_0 - x_{-1}) = \frac{D^2f(y_1)}{Df(y_1)}(x_0 - x_{-1})$$

$$\leq \frac{\delta}{1-\delta}|x_0 - x_{-1}| \leq \frac{\delta}{1-\delta}\frac{|x_0 - x_{-1}|}{|x_n - x_{n-1}|} = \frac{\delta}{1-\delta}\frac{1}{Df^n(y_2)}$$

$$\leq \frac{\delta C}{1-\delta}\frac{1}{Df^n(x_0)} \leq \frac{C\delta}{1-\delta}e^{-a_n},$$

where $y_1, y_2 \in [x_{-1}; x_0]$. □

For $\varepsilon > 0$, N is in Lemma 6. Put $b_n = a_{n+N} - a_N$. Then, $b_0 = 0$ and for $n \geq 1$ b_n satisfies

$$2b_n - b_{n-1} - b_{n+1} = 2a_{N+n} - a_{N+n-1} - a_{N+n+1} \leq \varepsilon e^{-a_{N+n}} = \varepsilon e^{-a_N} e^{-b_n}.$$

In the case that $Df(x) = 1$ for every $x \in \mathrm{Fix}(f)$, second case in Lemma 4 does not occur. So

$$\|Df^{n+N}\| = \exp(a_{n+N}) = \exp(b_n + a_N) \leq \left((n+N)\sqrt{\frac{\varepsilon}{2}} - N\sqrt{\frac{\varepsilon}{2}} + \sqrt{a_N}\right)^2.$$

Hence,

$$0 \leq \lim_{n\to\infty} \frac{\|Df^n\|}{n^2} = \lim_{n\to\infty} \frac{\exp(a_{n+N})}{(n+N)^2} \leq \frac{\varepsilon}{2}.$$

Since ε is arbitrarily small positive number, we have

$$\lim_{n\to\infty} \frac{\|Df^n\|}{n^2} = 0.$$

3. Proof of Theorem 2

We work on the neibourhood of 0 in $[0;1]$. We construct a ∞-flat diffeomorphism f near 0 with $\mathrm{Fix}(f) = \{0\}$ as the time-one map of the flow generated by a C^∞-vector field Z near 0 with $D^n Z(0) = 0$ for every $n \in \mathbb{N}$.

Firstly we consider C^∞-vector field ;

$$X(x) = \frac{1}{2}(1 - \cos\frac{1}{x}) \exp\left(-\frac{1}{x}\right), \quad X(0) = 0.$$

Set $a_k = \frac{1}{2\pi k}$, $k \in \mathbb{N}$, then

$$X(a_k) = 0, \; DX(a_k) = 0, \; D^2X(a_k) = \frac{1}{2}(2\pi k)^4 \exp(-2\pi k).$$

Set $\varepsilon_k = D^2X(a_k)$. We can take a positive number of the form $b_k = \frac{1}{2\pi k - M}$ such that for $x \in [a_k; b_k]$

$$\frac{1}{2}\varepsilon_k \leq D^2X(x) \leq 2\varepsilon_k.$$

Notice that M can be chosen independently with k.

Then we have for $x \in [a_k; b_k]$

$$\frac{1}{4}\varepsilon_k (x - a_k)^2 \leq X(x) \leq \varepsilon_k (x - a_k)^2.$$

Set

$$Y_k^-(x) = \frac{1}{4}\varepsilon_k (x - a_k)^2, \; Y_k^+(x) = \varepsilon_k (x - a_k)^2.$$

Let $\phi_t(x), \phi_t^-(x)$ and $\phi_t^+(x)$ be flows generated by X, Y_k^- and Y_k^+ respectively.

Then we have

$$D\phi_t^-(x) = (1 + \frac{1}{4}\varepsilon_k(b_k - a_k)t)^2 \geq \frac{1}{16}\varepsilon_k^2 (b_k - a_k)^2 t^2.$$

We define $s_k, t_k, u_k > 0$ and $c_k \in [a_k; b_k]$ by

$$\frac{1}{16}\varepsilon_k^2(b_k - a_k)^2 \log s_k = 4,\ b_k = \phi_{s_k}^-(c_k),\ b_k = \phi_{t_k}(c_k),\ b_k = \phi_{u_k}^+(c_k).$$

Since $Y_k^-(x) \le X(x) \le Y_k^+(x)$ on $[a_k; b_k]$ we have $u_k \le t_k \le s_k$. Notice that $s_k,\ u_k,\ t_k \to \infty$ as $k \to \infty$.

Then we have

$$D\phi_{t_k}(c_k) = \frac{X(b_k)}{X(c_k)} \ge \frac{Y^-(b_k)}{Y^+(c_k)} = \frac{1}{4}\frac{Y^-(b_k)}{Y^-(c_k)} = \frac{1}{4}D\phi_{s_k}^-(c_k) \ge \frac{s_k^2}{\log s_k} \ge \frac{t_k^2}{\log t_k},$$

and since $\varepsilon_k \to 0, (b_k - a_k)s_k \to \infty$ as $k \to \infty$ for sufficiently large k,

$$c_k - a_k = \phi_{-s_k}^-(b_k) - a_k = \frac{b_k - a_k}{1 + \varepsilon_k(b_k - a_k)s_k}$$

$$\ge \frac{1}{s_k} = \exp\left(\frac{-64}{\varepsilon_k^2(b_k - a_k)^2}\right) \ge \exp(-\exp(4\pi k)).$$

We set

$$X^+(x) = \exp\left(-\exp\left(\frac{3}{x}\right)\right),\ Z(x) = X(x) + X^+(x).$$

Then for sufficiently large k, we have $X(c_k) \ge X^+(c_k)$.

Let $\psi_t(x)$ be a flow generated by Z and w_k the positive number satisfying $\psi_{w_k}(c_k) = b_k$. Obviously $w_k \le t_k$.

We have

$$D\psi_{w_k}(c_k) \ge \frac{1}{2}\frac{X(b_k)}{X(c_k)} = \frac{1}{2}D\phi_{t_k}(c_k) \ge \frac{1}{2}\frac{t_k^2}{\log t_k} \ge \frac{1}{2}\frac{w_k^2}{\log w_k}.$$

Notice that $w_k \to \infty$ as $k \to \infty$.

So $f = \psi_1$ satisfies the desired properties.

References

[1] A. Borichev, *Distortion growth for iterations of diffeomorphisms of the interval,* Geometrical and Functional Analysis, to appear.

[2] L. Polterovich and M. Sodin, *A growth gap for diffeomorphisms of the interval,* Journal d'Analyse Math. 92 (2004), 191-209.

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