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Representation of Ammann-Beenker tilings by an automaton

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ABSTRACT. The Ammann-Beenker tilings are quasiperiodic tilings of the plane, which is constructed by using the Ammann's matching rules. We show that the Ammann-Beenker tilings can be composed by an automaton with 4 states, and note some results concerning composition sequences from the viewpoint of symbolic dynamics.

1. Introduction

In 1984 quasi-crystals with icosahedral symmetry were discovered by Shechtman et al.([9]). Before that, it had been believed that the structure of a crystal was periodic, like a wallpaper pattern. Periodicty is another name for translational symmetry. Icosahedral symmetry is incompatible with translational symmetry and therefore quasi-crystals are not periodic. The most famous 2-dimensional mathematical model for a quasi-crystal would be Penrose tilings of the plane ([6],[7]). The tiles of the Penrose's tilings are two types of rhombs with double and single arrows on the edges, as shown in Figure 1.

In the Penrose maching rules, the common edges of two adjacent tiles must have the same type (single or double) and the same direction of the arrows. The up-down generation introduced by J. Conway is one of the methods to construct such tilings or quasiperiodic tilings. In [3], de Bruijn actually uses the up-down generation to construct Penrose tilings, and represents the method by an automaton.

In this note we interested in Ammann-Beenker tilings (see [1],[2],[5] for example). Decorated Ammann-Beenker tiles are two rhombs with acute angles 45° and a square, and lengths of their edges are the same. Ammann-Beenker

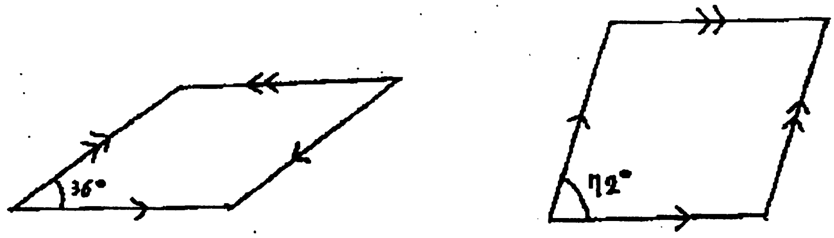

Figure 1: Penrose tiles with double and single arrows

tilings must have the property that the patterns match along the edges, which means that the quarter-disks should be combined to make half-disks, and the other patterns should be combined to give big arrows (see the figure 2).

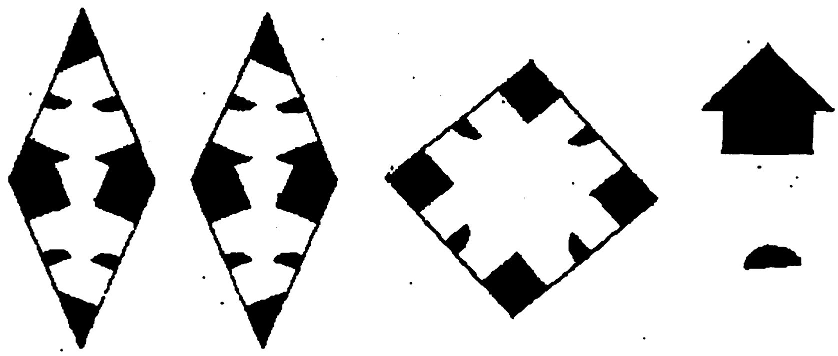

Figure 2: decorated Ammann-Beenker tiles and matching patterns

Ammann-Beenker tilings are obtained by the cut-and-project method corresponding to non-crystallographic $I_2(8)$-type folding of B_4-type Coxeter group. So they are also called B_4-type quasi-periodic tilings.

One of the purposes of this paper is to represent by an automaton a method to generate Ammann-Beenker tilings:

THEOREM. *Ammann-Beenker tilings can be represented by an automaton with 4 states and 24 transition maps (Figure 10).*

This note is arranged as follows. In sections 2 we prove the theorem. In section 3 we give some remarks on infinite paths in the automaton from the viewpoint of symbolic dynamics.

2. Proof of Theorem

We divide the square in Figure 2 into two triangles as Figure 3, and consider tilings composed of four kinds of decorated tiles in Figure 3. These tiles are sequentially named (A), (B), (C) and (D) from the left. Note that the original Ammann-Beenker tilings are restorable from tilings that use four kinds of decorated tiles in Figure 3.

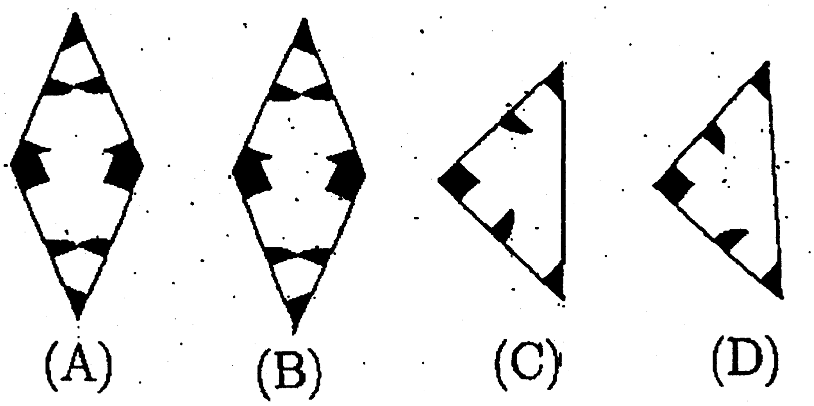

Figure 3: four kinds of decorated tiles

To use the up-down generation method, we utilize subdivided rhomb and triangle with similar and small subtiles such as in Figure 4.

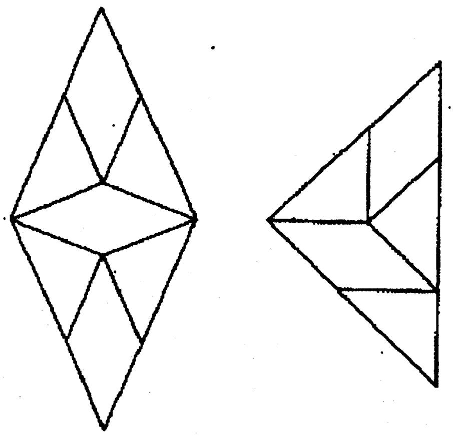

Figure 4: similar patches

We must decorate the subtiles. By checking all the cases, we have just four kinds of decorated patches as in Figure 5 (see [10] for details). These patches is sequentially named (a), (b), (c) and (d) from the left.

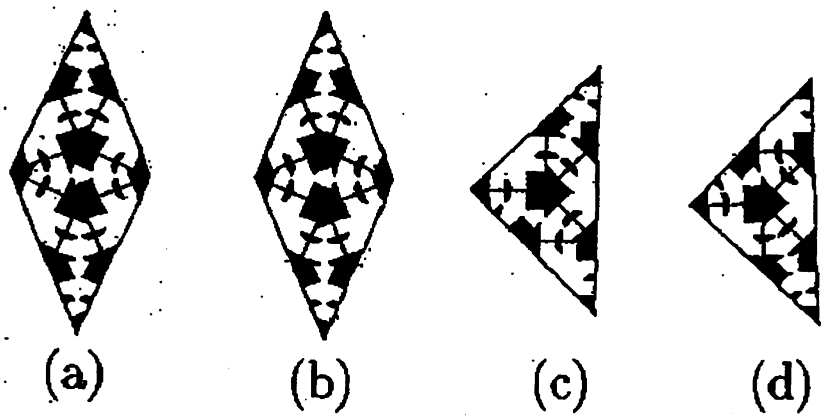

Figure 5: decorated similar patches

Next, we need to determine a correspondence between the decorated patches and the decorated tiles. By checking all the cases again, the only possibility of correspondence in Figure 6 is left. Correspondences other than Figure 6 have no consistency with decorations (see [10] for details).

We will show that the correspondence in Figure 6 does not contradict. To obtain this fact, it suffices to show that the subdivision can be repeated without contradiction. The subdivision is defined as follows; the subtiles in the patches (a), (b), (c) or (d) are replaced by patches by using the correspondence in Figure 6. These have an arrow pattern and a semicircle pattern without contradiction. Note that the correspondence in Figure 6 is obtained such that those have no contradiction. Then, new local configurations in Figure 7 appear in the subdivision of patches (a), (b), (c) or (d).

These local configurations in Figure 7 can be subdivided further. Then, we can also check that those have no contaradiction, that is, those have correct arrow patterns and semicircle patterns. In addition, new local configurations of Figure 8 appear.

The local configurations of Figure 8 are subdivided once more, but we have again correct arrow patterns and semicircle patterns. This subdivision process finishes at this point, because new local configurations do not come out. Therefore, it follows that the correspondence in Figure 6 is well-defined.

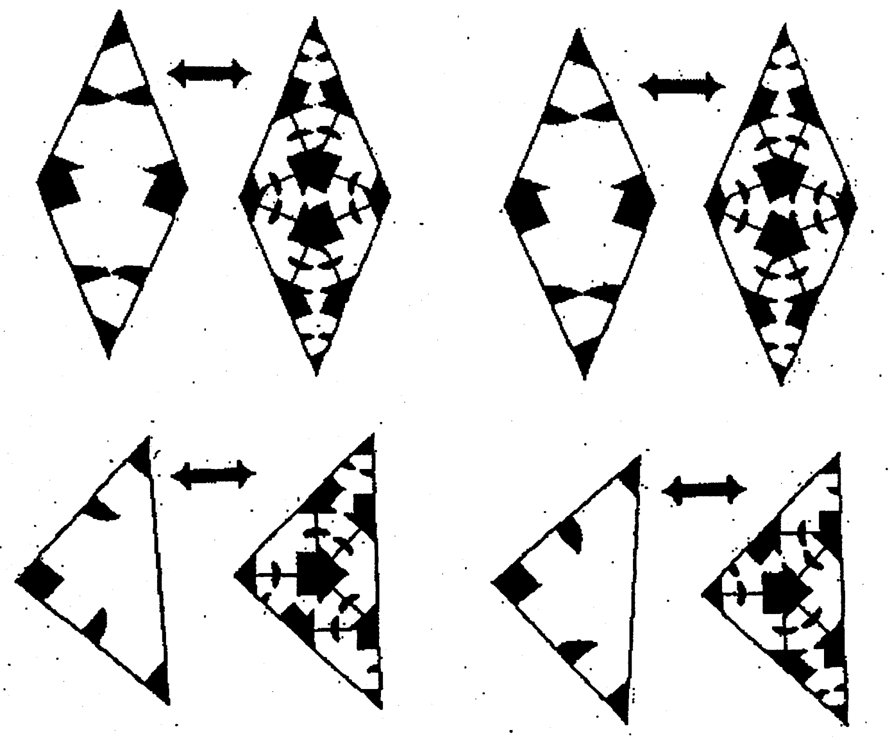

Figure 6: a correspondence between four patches and four tiles with decoration

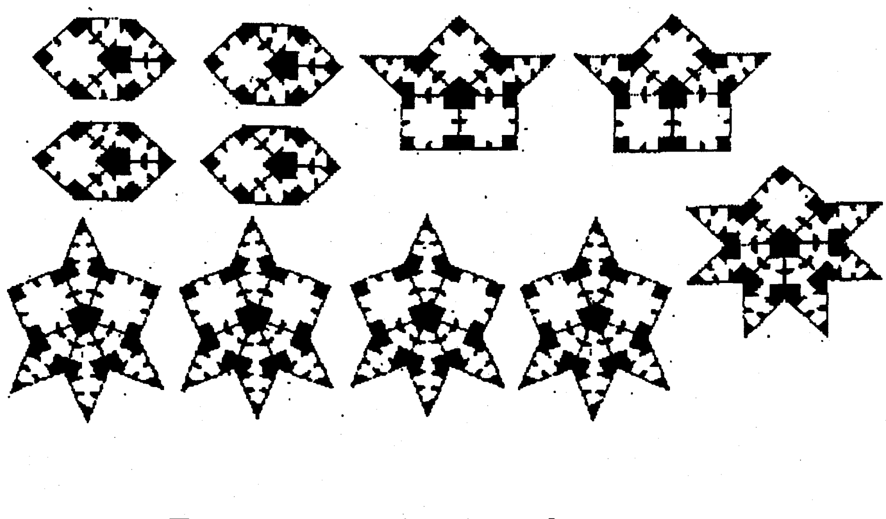

Figure 7: new local configurations

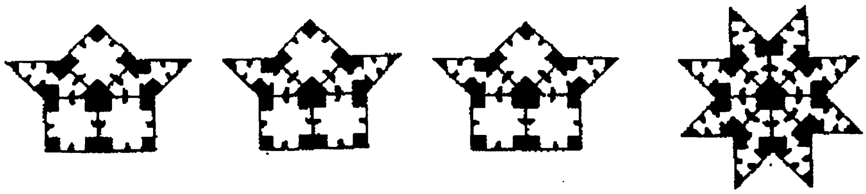

Figure 8: new local configurations of addition

An automaton will be drawn by using the similarity of patches. First, patches (a), (b), (c) or (d) in Figure 5 are identified with tiles (A), (B), (C) or (D) in Figure 6 respectively. Next, if some tile, say (A), is in some pach, say (a), then we can consider an operation to costruct the pach (a) by adding some other tiles to the tile (A). By the identification of the tiles and the paches ((a)=(A)), such an operation is considered to be a transition from (A) to (A), so we have a translation α_a from (A) to (A) as in Figure 9.

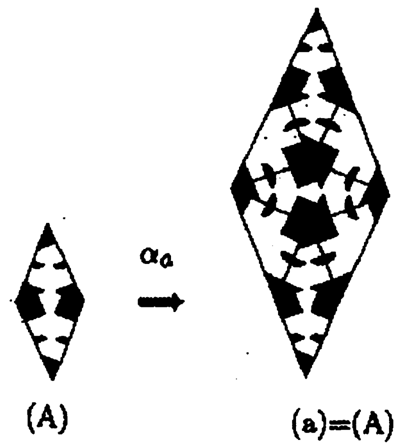

Figure 9: the transition map α_a

For all combinations of patches and tiles, we get the 24 transition map in the same way. Hence, we obtain the automaton of the following Figure 10 which consists of 4 states $\{A, B, C, D\}$ and 24 transition maps, and the proof of Theorem is completed.

Remark. Ammann-Beenker tilings admit only restricted decorated local configurations. All decorated local configurations for Ammann-Beenker tilings appear in Figure 5, 7 and 8.

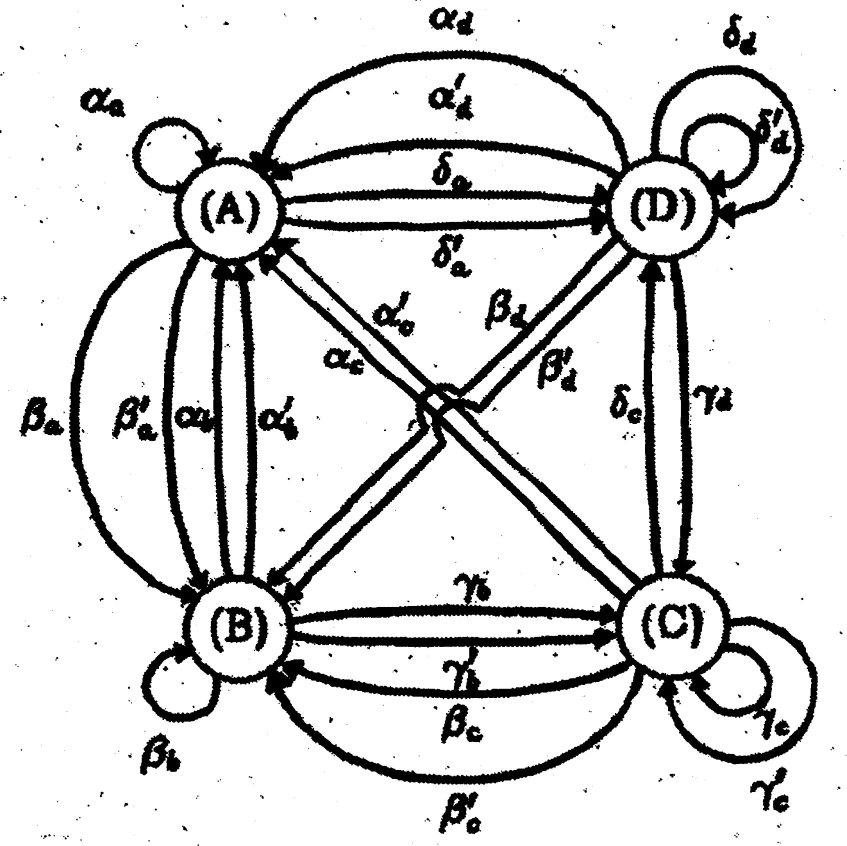

Figure 10: representation of Ammann-Beenker tilings by an automaton

3.Some remarks on infinite paths in the automaton

We identify an infinite path on the automaton of Figure 10 with an infinite sequence $(p_0p_1p_2\ldots)$ of the symbols of 24 transition maps, and it is called a composition sequence. Let $\mathcal{P}$ be the space of all composition sequences with the product topology and the shift map $\sigma((p_0p_1p_2\ldots)) = (p_1p_2p_3\ldots)$. Then, $\mathcal{P}$ is called a one-sided symbolic system. By up-down generation, every composition sequence is associated to a Ammann-Beenker tiling. Let $\mathcal{A}$ be the set of all Ammann-Beenker tilings obtained by the up-down generation. $\mathcal{A}$ is the topological space with the tiling metric (see for example [8]). We define $F : \mathcal{P} \to \mathcal{A}$ by $F(p) = T_p$ if a composition sequence $p = (p_0p_1p_2\ldots)$ corresponds to a tiling T_p, and define the composition map $C : \mathcal{A} \to \mathcal{A}$ by $C(T_p) = T_{\sigma(p)}$ for a composition sequence $p = (p_0p_1p_2\ldots)$. Since subdivision of a decorated patch is unique, we obtain that two composition sequences agree after a finite number of terms if and only if they corresponds to the same tiling. Hence, the definition of the composition map is well-defined. The linear map associated to the composition map is represented by the following matrix:

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 1 & 2 & 2 \\ 0 & 2 & 2 & 1 \\ 2 & 0 & 1 & 2 \end{bmatrix}.$$

Eigenvalues are $-1, 1, 3-2\sqrt{2}$ and $3+2\sqrt{2}$, and the left eigenvector belonging to the eigenvalue $3 + 2\sqrt{2}$ is $(\sqrt{2}, \sqrt{2}, 2, 2)$. Then the ratio of frequencies of tiles A, B, C, D is the ratio $\sqrt{2} : \sqrt{2} : 2 : 2$ of components of the eigenvector $(\sqrt{2}, \sqrt{2}, 2, 2)$. So, we see that the Ammann-Beenker tilings are not periodic.

By the definition of the composition map, F is a factor map, that is surjective continuous map such that $C \circ F = F \circ \sigma$.

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