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On extension of representations of $so(n+1,1)$ to representations of $so(n+1,2)$

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Abstract

In the present paper, we construct representations of the Lie algebra $so(n+1,2)$ on $C^\infty(S^n)$ by extending a representation of the Lie algebra $so(n+1,1)$ on $C^\infty(S^n)$ which arises from the action of Lorentz group $SO(n+1,1)$ on S^n as conformal transformations.

1 Introduction and statement of the result

Let $\mathbf{R}^{n+1}$ be $(n+1)$-dimensional Euclidean space with cartesian coordinates $x_1,\ldots,x_{n+1}$. For $x=(x_1,\ldots,x_{n+1})\in\mathbf{R}^{n+1}$, the norm of x is defined by $\|x\|=\sqrt{(x_1)^2+\cdots+(x_{n+1})^2}$. Let $S^n=\{x\in\mathbf{R}^{n+1}\mid\|x\|=1\}$ be the unit sphere in $\mathbf{R}^{n+1}$. Let $C^\infty(S^n)$ denote the linear space of complex-valued C^∞ functions on S^n. The special orthogonal group $SO(n+1)$ acts on S^n as an isometry group. This action induces a representation of the Lie algebra $so(n+1)$ on $C^\infty(S^n)$. Let $SO(n+1,1)$ denote the Lorentz group with its Lie algebra $so(n+1,1)$. It is well-known that the action of $SO(n+1)$ on S^n can be extended to an action of $SO(n+1,1)$ on S^n. This action induces an irreducible representation of the Lie algebra $so(n+1,1)$ on $C^\infty(S^n)$. We denote this representation by $\mathrm{Rep}_0(so(n+1,1))$.

Let us now consider the Lie algebra $so(n+1,2)$ of the Lie group $SO(n+1,2)$. Let E_{ij} denote the $(n+3)\times(n+3)$ matrix with the (i,j)-component 1 and the others are 0. The following are basis of $so(n+1,2)$.

$$E_{ij}-E_{ji}\quad(1\le i<j\le n+1),$$
$$E_{j,n+2}+E_{n+2,j}\quad(1\le j\le n+1),$$
$$E_{j,n+3}+E_{n+3,j}\quad(1\le j\le n+1),$$
$$E_{n+2,n+3}-E_{n+3,n+2}.$$

Note that the following hold.

(1) $[E_{ij}-E_{ji}, E_{kl}-E_{lk}] = \delta_{jk}(E_{il}-E_{li})+\delta_{il}(E_{jk}-E_{kj})-\delta_{jl}(E_{ik}-E_{ki})-\delta_{ik}(E_{jl}-E_{lj})$,
(2) $[E_{ij} - E_{ji}, E_{k,n+2} + E_{n+2,k}] = \delta_{jk}(E_{i,n+2} + E_{n+2,i}) - \delta_{ik}(E_{j,n+2} + E_{n+2,j})$,
(3) $[E_{ij} - E_{ji}, E_{k,n+3} + E_{n+3,k}] = \delta_{jk}(E_{i,n+3} + E_{n+3,i}) - \delta_{ik}(E_{j,n+3} + E_{n+3,j})$,
(4) $[E_{ij} - E_{ji}, E_{n+2,n+3} - E_{n+3,n+2}] = 0$,
(5) $[E_{i,n+2} + E_{n+2,i}, E_{j,n+2} + E_{n+2,j}] = E_{ij} - E_{ji}$,
(6) $[E_{i,n+2} + E_{n+2,i}, E_{j,n+3} + E_{n+3,j}] = \delta_{ij}(E_{n+2,n+3} - E_{n+3,n+2})$,
(7) $[E_{j,n+2} + E_{n+2,j}, E_{n+2,n+3} - E_{n+3,n+2}] = E_{j,n+3} + E_{n+3,j}$,
(8) $[E_{i,n+3} + E_{n+3,i}, E_{j,n+3} + E_{n+3,j}] = E_{ij} - E_{ji}$,
(9) $[E_{j,n+3} + E_{n+3,j}, E_{n+2,n+3} - E_{n+3,n+2}] = -(E_{j,n+2} + E_{n+2,j})$.

$\{E_{ij} - E_{ji} \mid 1 \le i < j \le n+1\}$ generate a Lie subalgebra isomorphic to $so(n+1)$. We identify this subalgebra with $so(n+1)$. $\{E_{ij} - E_{ji} \mid 1 \le i < j \le n+1\} \cup \{E_{j,n+2} + E_{n+2,j} \mid 1 \le j \le n+1\}$ generate a Lie subalgebra isomorphic to $so(n+1, 1)$. We identify this subalgebra with $so(n+1, 1)$. The Lie algebra $so(n+1, 2)$ is generated by $so(n+1, 1)$ and $E_{n+2,n+3} - E_{n+3,n+2}$, since, by (7)

$$E_{j,n+3} + E_{n+3,j} = [E_{j,n+2} + E_{n+2,j}, E_{n+2,n+3} - E_{n+3,n+2}].$$

The vector field $\partial/\partial x_j$ on $\mathbf{R}^{n+1}$ is denoted by X_j. The function $(x_1, \ldots, x_{n+1}) \mapsto x_j$ on $\mathbf{R}^{n+1}$ is denoted by x_j. The restriction to S^n of vector fields and functions on $\mathbf{R}^{n+1}$ are written by the same letter. Let ξ_j be a vector field defined by

$$\xi_j = \sum_{k=1}^{n+1}(\delta_{jk} - x_j x_k)X_k \quad (1 \le j \le n+1).$$

We note that $(x_i X_j - x_j X_i)(\|x\|^2) = 0$ and $\xi_j(\|x\|^2) = 0$. The representation $\mathrm{Rep}_0(so(n+1, 1))$ is given by

$$E_{ij} - E_{ji} \mapsto x_i X_j - x_j X_i \quad (1 \le i < j \le n+1),$$
$$E_{j,n+2} + E_{n+2,j} \mapsto \xi_j \quad (1 \le j \le n+1),$$

where $x_i X_j - x_j X_i$ and ξ_j represents tangent vector fields to S^n.

Let $\Phi_j : C^\infty(S^n) \to C^\infty(S^n)$ be an operator defined by

$$\Phi_j = \xi_j + \mu x_j = \sum_{k=1}^{n+1}(\delta_{jk} - x_j x_k)X_k + \mu x_j \quad (1 \le j \le n+1),$$

where μ is a complex number. Then it is easily proved that,for each μ, the correspondence given by

$$E_{ij} - E_{ji} \mapsto x_i X_j - x_j X_i \quad (1 \le i < j \le n+1),$$
$$E_{j,n+2} + E_{n+2,j} \mapsto \Phi_j \quad (1 \le j \le n+1)$$

is a representation of $so(n+1, 1)$ on $C^\infty(S^n)$ (See Section 4). We denote this representation by $\mathrm{Rep}_\mu(so(n+1, 1))$.

The problem we consider in this paper is the possiblity to extend the representation $\mathrm{Rep}_\mu(so(n+1,1))$ of $so(n+1,1)$ on $C^\infty(S^n)$ to a representation of $so(n+1,2)$ on $C^\infty(S^n)$. The result we have obtained is the following:

Main Theorem *The representation* $\mathrm{Rep}_\mu(so(n+1,1))$ *of* $so(n+1,1)$ *on* $C^\infty(S^n)$ *can be extended to a representation of* $so(n+1,2)$ *on* $C^\infty(S^n)$ *if and only if* $\mu = -\dfrac{n\pm 1}{2}$.
Furthermore, in this case, the representation is given by

$$\begin{aligned} &E_{ij} - E_{ji} \mapsto x_i X_j - x_j X_i \quad (1 \le i < j \le n+1), \\ &E_{j,n+2} + E_{n+2,j} \mapsto \Phi_j \quad (1 \le j \le n+1), \\ &E_{j,n+3} + E_{n+3,j} \mapsto [\Phi_j, \Lambda] \quad (1 \le j \le n+1), \\ &E_{n+2,n+3} - E_{n+3,n+2} \mapsto \Lambda, \end{aligned}$$

where Λ *is an operator defined by*

$$\Lambda : C^\infty(S^n) \to C^\infty(S^n), \quad \Lambda = \pm\sqrt{-1}\sqrt{\Delta + \frac{(n-1)^2}{4}}.$$

Remark 1 $\Delta : C^\infty(S^n) \to C^\infty(S^n)$ denotes the Laplace-Beltrami operator on S^n. The eigenvalues of Δ are $m(m+n-1)$ $(m = 0, 1, 2, \ldots)$. $\sqrt{\Delta + \frac{(n-1)^2}{4}} : C^\infty(S^n) \to C^\infty(S^n)$ denotes the operator on S^n with the same eigenfunctions as Δ, and with the corresponding eigenvalues $\dfrac{2m+n-1}{2}$ $(m = 0, 1, 2, \ldots)$.
Remark 2 Representation of the Lie group $SO(n+1,2)$ and the Lie algebra $so(n+1,2)$ has been studied by several authors in connection with geometric quantization of the Kepler problem. (See References.)

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2 Harmonic functions $H_{i_1\cdots i_m}$ on $\mathbf{R}^{n+1}$

For each positive integer m, and $i_1, \ldots, i_m \in \{1, \ldots, n+1\}$, $H_{i_1\cdots i_m}$ denotes the function on $\mathbf{R}^{n+1} - \{0\}$ defined by

$$H_{i_1\cdots i_m} = \frac{(-1)^m}{(n-1)(n+1)\cdots(2m+n-3)} \cdot \frac{\partial^m(\|x\|^{1-n})}{\partial x_{i_1}\cdots\partial x_{i_m}}.$$

If $m = 0$, we put $H_{i_1\cdots i_m} = \|x\|^{1-n}$. Note that $H_{i_1\cdots i_m}$ is invariant under each permutation of $i_1, \ldots, i_m$, and that $\sum_{k=1}^{n+1} H_{i_1\cdots i_m kk} = 0$ for each non-negative integer m.

Lemma 1 *For each non-negative integer m, we have*

(1) $X_j H_{i_1\cdots i_m} = -(2m+n-1)H_{i_1\cdots i_m j}$,

(2) $x_j H_{i_1\cdots i_m} = \|x\|^2 H_{i_1\cdots i_m j} + \dfrac{1}{2m+n-1}\displaystyle\sum_{a=1}^{m}\delta_{i_a j}H_{i_1\cdots\widehat{i_a}\cdots i_m}$

$$- \frac{2}{(2m+n-1)(2m+n-3)}\sum_{1\le a<b\le m}\delta_{i_a i_b}H_{i_1\cdots\widehat{i_a}\cdots\widehat{i_b}\cdots i_m j},$$

(3) $\displaystyle\sum_{k=1}^{n+1} x_k X_k H_{i_1\cdots i_m} = -(m+n-1)H_{i_1\cdots i_m}$,

(4) $(x_i X_j - x_j X_i)H_{i_1\cdots i_m} = \displaystyle\sum_{a=1}^{m}(\delta_{i_a j}H_{i_1\cdots\widehat{i_a}\cdots i_m i} - \delta_{i_a i}H_{i_1\cdots\widehat{i_a}\cdots i_m j})$,

(5) $\xi_j H_{i_1\cdots i_m} = \{(m+n-1)\|x\|^2 - (2m+n-1)\}H_{i_1\cdots i_m j} + \dfrac{m+n-1}{2m+n-1}\displaystyle\sum_{a=1}^{m}\delta_{i_a j}H_{i_1\cdots\widehat{i_a}\cdots i_m}$

$$- \frac{2(m+n-1)}{(2m+n-1)(2m+n-3)}\sum_{1\le a<b\le m}\delta_{i_a i_b}H_{i_1\cdots\widehat{i_a}\cdots\widehat{i_b}\cdots i_m j}.$$

Proof We prove (2) by induction on m. If $m = 0$, (2) holds. Now, assume that (2) holds for some m. Differentiating (2) with respect to $x_{i_{m+1}}$, we have

$$-(2m+n-1)x_j H_{i_1\cdots i_m i_{m+1}} + \delta_{i_{m+1} j}H_{i_1\cdots i_m}$$
$$= 2x_{i_{m+1}}H_{i_1\cdots i_m j} - (2m+n+1)\|x\|^2 H_{i_1\cdots i_m i_{m+1} j}$$
$$- \frac{2m+n-3}{2m+n-1}\sum_{a=1}^{m}\delta_{i_a j}H_{i_1\cdots\widehat{i_a}\cdots i_m i_{m+1}} + \frac{2}{2m+n-1}\sum_{1\le a<b\le m}\delta_{i_a i_b}H_{i_1\cdots\widehat{i_a}\cdots\widehat{i_b}\cdots i_m i_{m+1} j}.$$

Transposing $\delta_{i_{m+1} j}H_{i_1\cdots i_m}$ to the right-hand side, and then dividing by $-(2m+n-1)$, we have

$$x_j H_{i_1\cdots i_m i_{m+1}} = -\frac{2}{2m+n-1}x_{i_{m+1}}H_{i_1\cdots i_m j} + \frac{2m+n+1}{2m+n-1}\|x\|^2 H_{i_1\cdots i_m i_{m+1} j}$$
$$+ \frac{1}{2m+n-1}\delta_{i_{m+1} j}H_{i_1\cdots i_m} + \frac{2m+n-3}{(2m+n-1)^2}\sum_{a=1}^{m}\delta_{i_a j}H_{i_1\cdots\widehat{i_a}\cdots i_m i_{m+1}}$$
$$- \frac{2}{(2m+n-1)^2}\sum_{1\le a<b\le m}\delta_{i_a i_b}H_{i_1\cdots\widehat{i_a}\cdots\widehat{i_b}\cdots i_m i_{m+1} j}. \tag{1-1}$$

Rewriting j to i_{m+1}, and i_{m+1} to j, we have

$$x_{i_{m+1}}H_{i_1\cdots i_m j} = -\frac{2}{2m+n-1}x_j H_{i_1\cdots i_m i_{m+1}} + \frac{2m+n+1}{2m+n-1}\|x\|^2 H_{i_1\cdots i_m i_{m+1} j}$$
$$+ \frac{1}{2m+n-1}\delta_{i_{m+1} j}H_{i_1\cdots i_m} + \frac{2m+n-3}{(2m+n-1)^2}\sum_{a=1}^{m}\delta_{i_a i_{m+1}}H_{i_1\cdots\widehat{i_a}\cdots i_m j}$$
$$- \frac{2}{(2m+n-1)^2}\sum_{1\le a<b\le m}\delta_{i_a i_b}H_{i_1\cdots\widehat{i_a}\cdots\widehat{i_b}\cdots i_m i_{m+1} j}. \tag{1-2}$$

Eliminating $x_{i_{m+1}}H_{i_1\cdots i_m j}$ from (1-1) and (1-2), we have

$$x_jH_{i_1\cdots i_{m+1}} = \|x\|^2H_{i_1\cdots i_{m+1}j} + \frac{1}{2m+n+1}\sum_{a=1}^{m+1}\delta_{i_a j}H_{i_1\cdots\widehat{i_a}\cdots i_{m+1}}$$
$$-\frac{2}{(2m+n+1)(2m+n-1)}\sum_{1\le a<b\le m+1}\delta_{i_a i_b}H_{i_1\cdots\widehat{i_a}\cdots\widehat{i_b}\cdots i_{m+1}j}.$$

This completes the proof of (2). The rest are easily obtained. q.e.d.

3 Spherical harmonics $h_{i_1\cdots i_m}$

We denote by $h_{i_1\cdots i_m}$ the restriction of $H_{i_1\cdots i_m}$ onto S^n. It is well-known that $h_{i_1\cdots i_m}$ are eigenfunctions of the Laplace-Beltrami operator Δ on S^n corresponding to the eigenvalue $m(m+n-1)$. $\mathcal{H}_m(S^n)$ denotes the linear subspace of $C^\infty(S^n)$ spanned by $h_{i_1\cdots i_m}$ $(i_1,\ldots,i_m\in\{1,2,\ldots,n+1\})$.
The following lemma is obtained easily from Lemma 1.

Lemma 2 *For each non-negative integer m, we have*

(1) $$x_jh_{i_1\cdots i_m} = h_{i_1\cdots i_m j} + \frac{1}{2m+n-1}\sum_{a=1}^{m}\delta_{i_a j}h_{i_1\cdots\widehat{i_a}\cdots i_m}$$
$$-\frac{2}{(2m+n-1)(2m+n-3)}\sum_{1\le a<b\le m}\delta_{i_a i_b}h_{i_1\cdots\widehat{i_a}\cdots\widehat{i_b}\cdots i_m j},$$

(2) $$(x_iX_j - x_jX_i)h_{i_1\cdots i_m} = \sum_{a=1}^{m}(\delta_{i_a j}h_{i_1\cdots\widehat{i_a}\cdots i_m i} - \delta_{i_a i}h_{i_1\cdots\widehat{i_a}\cdots i_m j}),$$

(3) $$\xi_jh_{i_1\cdots i_m} = -mh_{i_1\cdots i_m j} + \frac{m+n-1}{2m+n-1}\sum_{a=1}^{m}\delta_{i_a j}h_{i_1\cdots\widehat{i_a}\cdots i_m}$$
$$-\frac{2(m+n-1)}{(2m+n-1)(2m+n-3)}\sum_{1\le a<b\le m}\delta_{i_a i_b}h_{i_1\cdots\widehat{i_a}\cdots\widehat{i_b}\cdots i_m j}.$$

4 Representation $\mathbf{Rep}_\mu(so(n+1,1))$ of $so(n+1,1)$

Let $\Phi_j : C^\infty(S^n)\to C^\infty(S^n)$ be the operator defined in Section 1.

Lemma 3 *We have*

(1) $[x_iX_j - x_jX_i, x_kX_l - x_lX_k]$
$= \delta_{jk}(x_iX_l - x_lX_i) + \delta_{il}(x_jX_k - x_kX_j) - \delta_{jl}(x_iX_k - x_kX_i) - \delta_{ik}(x_jX_l - x_lX_j),$

(2) $[x_iX_j - x_jX_i, \Phi_k] = \delta_{jk}\Phi_i - \delta_{ik}\Phi_j,$

(3) $[\Phi_i, \Phi_j] = x_i X_j - x_j X_i$.

It is easily proved from this lemma that the correspondence given by

$$E_{ij} - E_{ji} \mapsto x_i X_j - x_j X_i \quad (1 \le i < j \le n{+}1),$$
$$E_{j,n+2} + E_{n+2,j} \mapsto \Phi_j \quad (1 \le j \le n{+}1)$$

is a representation of $so(n{+}1,1)$ on $C^\infty(S^n)$ for each μ.
We denote this representation by $\mathrm{Rep}_\mu(so(n{+}1,1))$. The following lemma follows easily from (1) and (3) of Lemma 2.

Lemma 4 *For each non-negative integer m, we have*

$$\Phi_j h_{i_1 \cdots i_m} = -(m-\mu) h_{i_1 \cdots i_m j} + \frac{m+n-1+\mu}{2m+n-1} \sum_{a=1}^{m} \delta_{i_a j} h_{i_1 \cdots \widehat{i_a} \cdots i_m}$$
$$- \frac{2(m+n-1+\mu)}{(2m+n-1)(2m+n-3)} \sum_{1 \le a < b \le m} \delta_{i_a i_b} h_{i_1 \cdots \widehat{i_a} \cdots \widehat{i_b} \cdots i_m j}.$$

5 Representation of $so(n{+}1,2)$

We will consider the problem whether the representation $\mathrm{Rep}_\mu(so(n{+}1,1))$ can be extended to a representation of $so(n{+}1,2)$. Choose a linear operator $\Lambda : C^\infty(S^n) \to C^\infty(S^n)$, and define an operator $\Psi_j : C^\infty(S^n) \to C^\infty(S^n)$ by $\Psi_j = [\Phi_j, \Lambda]$. Throughout this section, we will assume that the correspondence given by

$$E_{ij} - E_{ji} \mapsto x_i X_j - x_j X_i \quad (1 \le i < j \le n{+}1),$$
$$E_{j,n+2} + E_{n+2,j} \mapsto \Phi_j \quad (1 \le j \le n{+}1),$$
$$E_{j,n+3} + E_{n+3,j} \mapsto \Psi_j \quad (1 \le j \le n{+}1),$$
$$E_{n+2,n+3} - E_{n+3,n+2} \mapsto \Lambda$$

is a representation of $so(n{+}1,2)$.

Lemma 5 *There exist complex numbers λ_m ($m \in \{0,1,2,\ldots\}$) such that*

$$\Lambda h_{i_1 \cdots i_m} = \lambda_m h_{i_1 \cdots i_m}$$

for each $i_1, \ldots, i_m \in \{1,2,\ldots,n{+}1\}$.

Proof This follows from $[x_i X_j - x_j X_i, \Lambda] = 0$. q.e.d.

Let us define complex numbers c_m ($m \in \{0,1,2,\ldots\}$) by $c_0 = \lambda_0$, and $c_m = \lambda_m - \lambda_{m-1}$ ($m \ge 1$). The following lemma is obtained easily from the definition of Ψ_j, Lemma 4 and Lemma 5.

Lemma 6 *For each non-negative integer* m, *we have*

$$\Psi_j h_{i_1\cdots i_m} = c_{m+1}(m-\mu)h_{i_1\cdots i_m j} + \frac{c_m(m+n-1+\mu)}{2m+n-1}\sum_{a=1}^{m}\delta_{i_a j}h_{i_1\cdots\widehat{i_a}\cdots i_m}$$
$$-\frac{2c_m(m+n-1+\mu)}{(2m+n-1)(2m+n-3)}\sum_{1\leq a<b\leq m}\delta_{i_a i_b}h_{i_1\cdots\widehat{i_a}\cdots\widehat{i_b}\cdots i_m j}.$$

The following lemma is obtained easily from Lemma 5 and Lemma 6.

Lemma 7 *For each non-negative integer* m, *we have*

$$[\Psi_j, \Lambda]h_{i_1\cdots i_m} = -(c_{m+1})^2(m-\mu)h_{i_1\cdots i_m j} + \frac{(c_m)^2(m+n-1+\mu)}{2m+n-1}\sum_{a=1}^{m}\delta_{i_a j}h_{i_1\cdots\widehat{i_a}\cdots i_m}$$
$$-\frac{2(c_m)^2(m+n-1+\mu)}{(2m+n-1)(2m+n-3)}\sum_{1\leq a<b\leq m}\delta_{i_a i_b}h_{i_1\cdots\widehat{i_a}\cdots\widehat{i_b}\cdots i_m j}.$$

The following lemma is obtained by direct calculation using Lemma 4 and Lemma 6.

Lemma 8 *For each non-negative integer* m, *we have*

$$[\Phi_i, \Psi_j]h_{i_1\cdots i_m} = (c_{m+2} - c_{m+1})(m-\mu)(m+1-\mu)h_{i_1\cdots i_m ij}$$
$$+\frac{2c_{m+1}(m-\mu)(m+n+\mu)}{2m+n+1}\delta_{ij}h_{i_1\cdots i_m}$$
$$+\frac{c_{m+1}(m-\mu)(m+n+\mu)(2m+n-3) - c_m(m-1-\mu)(m+n-1+\mu)(2m+n+1)}{(2m+n+1)(2m+n-1)}$$
$$\times\left\{\sum_{a=1}^{m}\delta_{i_a i}h_{i_1\cdots\widehat{i_a}\cdots i_m j} + \sum_{a=1}^{m}\delta_{i_a j}h_{i_1\cdots\widehat{i_a}\cdots i_m i} - \frac{2}{2m+n-3}\sum_{a\neq b}^{m}\delta_{i_a i_b}h_{i_1\cdots\widehat{i_a}\cdots\widehat{i_b}\cdots i_m ij}\right\}$$
$$+\frac{(c_m - c_{m-1})(m+n-1+\mu)(m+n-2+\mu)}{(2m+n-1)(2m+n-3)^2}$$
$$\times\left\{(2m+n-3)\sum_{a\neq b}^{m}\delta_{i_a i}\delta_{i_b j}h_{i_1\cdots\widehat{i_a}\cdots\widehat{i_b}\cdots i_m} - \sum_{a\neq b}^{m}\delta_{ij}\delta_{i_a i_b}h_{i_1\cdots\widehat{i_a}\cdots\widehat{i_b}\cdots i_m}\right.$$
$$-\sum_{a,b,c\neq}^{m}\delta_{i_a i}\delta_{i_b i_c}h_{i_1\cdots\widehat{i_a}\cdots\widehat{i_b}\cdots\widehat{i_c}\cdots i_m j} - \sum_{a,b,c\neq}^{m}\delta_{i_a j}\delta_{i_b i_c}h_{i_1\cdots\widehat{i_a}\cdots\widehat{i_b}\cdots\widehat{i_c}\cdots i_m i}$$
$$\left.+\frac{1}{2m+n-5}\sum_{a,b,c,d\neq}^{m}\delta_{i_a i_b}\delta_{i_c i_d}h_{i_1\cdots\widehat{i_a}\cdots\widehat{i_b}\cdots\widehat{i_c}\cdots\widehat{i_d}\cdots i_m ij}\right\}.$$

Remark $\sum_{a,b,c\neq}^{m}$ implies to sum over all ordered 3-tuples (a, b, c) consisting of mutually different $a, b, c \in \{1, 2, \ldots, m\}$.

Lemma 9 *For each positive integer m, we have*
$$(c_m)^2 = -1.$$

Proof Since $[\Psi_j, \Lambda] = -\Phi_j$, we have $[\Psi_j, \Lambda]h_{i_1\cdots i_m} + \Phi_j h_{i_1\cdots i_m} = 0$ for each non-negative integer m and $j, i_1, \ldots, i_m \in \{1, 2, \ldots, n+1\}$. Then, using Lemma 7 and Lemma 4, we have

$$\{(c_{m+1})^2+1\}(m-\mu)h_{i_1\cdots i_m j} - \frac{\{(c_m)^2+1\}(m+n-1+\mu)}{2m+n-1}\sum_{a=1}^{m}\delta_{i_a j}h_{i_1\cdots\widehat{i_a}\cdots i_m}$$
$$+\frac{2\{(c_m)^2+1\}(m+n-1+\mu)}{(2m+n-1)(2m+n-3)}\sum_{1\le a<b\le m}\delta_{i_a i_b}h_{i_1\cdots\widehat{i_a}\cdots\widehat{i_b}\cdots i_m j} = 0.$$

Putting $m = 0$, we have $\{(c_1)^2+1\}\mu = 0$. Putting $m = 1$, we have $\{(c_2)^2+1\}(1-\mu) = 0$ and $\{(c_1)^2+1\}(n+\mu) = 0$. From $\{(c_1)^2+1\}\mu = 0$ and $\{(c_1)^2+1\}(n+\mu) = 0$, we have $(c_1)^2 = -1$. If $m \ge 2$, putting $i_1 = i_2 = \cdots = i_m = 1$ and $j = 2$, we have $\{(c_{m+1})^2+1\}(m-\mu) = 0$ and $\{(c_m)^2+1\}(m+n-1+\mu) = 0$. From these, we have $(c_m)^2 = -1$ for each positive integer m. q.e.d.

Lemma 10 *We have*
$$c_0 = \frac{n-1}{2}c, \quad c_m = c\ (m \ge 1) \quad \textit{and} \quad \mu = -\frac{n\pm 1}{2},$$
where $c = \pm\sqrt{-1}$.

Proof Since $[\Phi_i, \Psi_j] = \delta_{ij}\Lambda$, we have $[\Phi_i, \Psi_j]h_{i_1\cdots i_m} - \delta_{ij}\Lambda h_{i_1\cdots i_m} = 0$ for each non-negative integer m and $i, j, i_1, \ldots, i_m \in \{1, 2, \ldots, n+1\}$. Then, using Lemma 8 and Lemma 5, we have

$$(c_{m+2}-c_{m+1})(m-\mu)(m+1-\mu)h_{i_1\cdots i_m ij}$$
$$+\left(\frac{2c_{m+1}(m-\mu)(m+n+\mu)}{2m+n+1} - \lambda_m\right)\delta_{ij}h_{i_1\cdots i_m}$$
$$+\frac{c_{m+1}(m-\mu)(m+n+\mu)(2m+n-3) - c_m(m-1-\mu)(m+n-1+\mu)(2m+n+1)}{(2m+n+1)(2m+n-1)}$$
$$\times\left\{\sum_{a=1}^{m}\delta_{i_a i}h_{i_1\cdots\widehat{i_a}\cdots i_m j} + \sum_{a=1}^{m}\delta_{i_a j}h_{i_1\cdots\widehat{i_a}\cdots i_m i} - \frac{2}{2m+n-3}\sum_{a\ne b}\delta_{i_a i_b}h_{i_1\cdots\widehat{i_a}\cdots\widehat{i_b}\cdots i_m ij}\right\}$$
$$+\frac{(c_m-c_{m-1})(m+n-1+\mu)(m+n-2+\mu)}{(2m+n-1)(2m+n-3)^2}$$
$$\times\left\{(2m+n-3)\sum_{a\ne b}\delta_{i_a i}\delta_{i_b j}h_{i_1\cdots\widehat{i_a}\cdots\widehat{i_b}\cdots i_m} - \sum_{a\ne b}\delta_{ij}\delta_{i_a i_b}h_{i_1\cdots\widehat{i_a}\cdots\widehat{i_b}\cdots i_m}\right.$$

$$- \sum_{a,b,c\neq}^{m} \delta_{i_a i}\delta_{i_b i_c} h_{i_1\cdots\widehat{i_a}\cdots\widehat{i_b}\cdots\widehat{i_c}\cdots i_m j} - \sum_{a,b,c\neq}^{m} \delta_{i_a j}\delta_{i_b i_c} h_{i_1\cdots\widehat{i_a}\cdots\widehat{i_b}\cdots\widehat{i_c}\cdots i_m i}$$

$$+ \frac{1}{2m+n-5} \sum_{a,b,c,d\neq}^{m} \delta_{i_a i_b}\delta_{i_c i_d} h_{i_1\cdots\widehat{i_a}\cdots\widehat{i_b}\cdots\widehat{i_c}\cdots\widehat{i_d}\cdots i_m i j} \Bigg\} = 0.$$

Since $\mathcal{H}_{m+2}(S^n)$, $\mathcal{H}_m(S^n)$ and $\mathcal{H}_{m-2}(S^n)$ are linearly independent, we have

$$(c_{m+2} - c_{m+1})(m-\mu)(m+1-\mu)h_{i_1\cdots i_m ij} = 0,$$

$$\left(\frac{2c_{m+1}(m-\mu)(m+n+\mu)}{2m+n+1} - \lambda_m\right)\delta_{ij}h_{i_1\cdots i_m}$$

$$+ \frac{c_{m+1}(m-\mu)(m+n+\mu)(2m+n-3) - c_m(m-1-\mu)(m+n-1+\mu)(2m+n+1)}{(2m+n+1)(2m+n-1)}$$

$$\times \left\{ \sum_{a=1}^{m} \delta_{i_a i} h_{i_1\cdots\widehat{i_a}\cdots i_m j} + \sum_{a=1}^{m} \delta_{i_a j} h_{i_1\cdots\widehat{i_a}\cdots i_m i} - \frac{2}{2m+n-3}\sum_{a\neq b}^{m} \delta_{i_a i_b} h_{i_1\cdots\widehat{i_a}\cdots\widehat{i_b}\cdots i_m i j} \right\} = 0,$$

and

$$\frac{(c_m - c_{m-1})(m+n-1+\mu)(m+n-2+\mu)}{(2m+n-1)(2m+n-3)^2}$$

$$\times \Bigg\{ (2m+n-3)\sum_{a\neq b}^{m} \delta_{i_a i}\delta_{i_b j} h_{i_1\cdots\widehat{i_a}\cdots\widehat{i_b}\cdots i_m} - \sum_{a\neq b}^{m} \delta_{ij}\delta_{i_a i_b} h_{i_1\cdots\widehat{i_a}\cdots\widehat{i_b}\cdots i_m}$$

$$- \sum_{a,b,c\neq}^{m} \delta_{i_a i}\delta_{i_b i_c} h_{i_1\cdots\widehat{i_a}\cdots\widehat{i_b}\cdots\widehat{i_c}\cdots i_m j} - \sum_{a,b,c\neq}^{m} \delta_{i_a j}\delta_{i_b i_c} h_{i_1\cdots\widehat{i_a}\cdots\widehat{i_b}\cdots\widehat{i_c}\cdots i_m i}$$

$$+ \frac{1}{2m+n-5} \sum_{a,b,c,d\neq}^{m} \delta_{i_a i_b}\delta_{i_c i_d} h_{i_1\cdots\widehat{i_a}\cdots\widehat{i_b}\cdots\widehat{i_c}\cdots\widehat{i_d}\cdots i_m i j} \Bigg\} = 0.$$

From the first equation, we have

$$(c_{m+2} - c_{m+1})(m-\mu)(m+1-\mu) = 0 \quad (m \geq 0). \tag{10-1}$$

Putting $i = j = i_1 = \cdots = i_m = 1$ in the third equation, we have

$$(c_m - c_{m-1})(m+n-1+\mu)(m+n-2+\mu) = 0 \quad (m \geq 2). \tag{10-2}$$

From (10-1) and (10-2), we have $c_1 = c_2 = c_3 = \cdots$. Define c by $c = c_1$. Then, from Lemma 9, we have $c^2 = -1$. Putting $m = 0$ in the second equation, we have

$$\frac{2c_1(-\mu)(n+\mu)}{n+1} - \lambda_0 = 0. \tag{10-3}$$

Furthermore, putting $m = 1$, $i = j = 1$, and $i_1 = 2$ in the second equation, we have

$$\frac{2c_2(1-\mu)(1+n+\mu)}{n+3} - \lambda_1 = 0. \tag{10-4}$$

Since $\lambda_0 = c_0$, $\lambda_1 = c_1 + c_0 = c + c_0$, and $c_1 = c_2 = c$, we have, from (10-3) and (10-4),

$$c_0 = \frac{n-1}{2}c.$$

Substituting this into (10-3), we have

$$(2\mu + n+1)(2\mu + n-1)c = 0.$$

Since $c \neq 0$, we have

$$\mu = -\frac{n \pm 1}{2}.$$

q.e.d.

Lemma 11 *We have*

$$\Lambda = \pm\sqrt{-1}\sqrt{\Delta + \frac{(n-1)^2}{4}}.$$

Proof $\Delta + \frac{(n-1)^2}{4}$ is an operator with $\left(\frac{2m+n-1}{2}\right)^2$ as its eigenvalues and $\mathcal{H}_m(S^n)$ as the corresponding eigenspaces ($m \in \{0, 1, 2, \ldots\}$). $\sqrt{\Delta + \frac{(n-1)^2}{4}}$ represents a linear operator with $\frac{2m+n-1}{2}$ as its eigenvalues and $\mathcal{H}_m(S^n)$ as the corresponding eigenspaces.

From Lemma 10, we have $\lambda_m = c\frac{2m+n-1}{2}$ $(c = \pm\sqrt{-1})$. Lemma 5 shows that Λ is an operator with λ_m as its eigenvalues and $\mathcal{H}_m(S^n)$ as the corresponding eigenspaces. Hence, Λ coincides with $c\sqrt{\Delta + \frac{(n-1)^2}{4}}$.

q.e.d.

6 Conclusion

We have proved in Section 5 that if the correspondence given by

$$E_{ij} - E_{ji} \mapsto x_i X_j - x_j X_i \quad (1 \leq i < j \leq n+1),$$
$$E_{j,n+2} + E_{n+2,j} \mapsto \Phi_j \quad (1 \leq j \leq n+1),$$
$$E_{j,n+3} + E_{n+3,j} \mapsto [\Phi_j, \Lambda] \quad (1 \leq j \leq n+1),$$
$$E_{n+2,n+3} - E_{n+3,n+2} \mapsto \Lambda$$

is a representation of $so(n+1, 2)$, then

$$\mu = -\frac{n \pm 1}{2} \text{ and } \Lambda = \pm\sqrt{-1}\sqrt{\Delta + \frac{(n-1)^2}{4}}.$$

The converse can be easily proved. Hence, we obtain Main Theorem.

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