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# $C^*$-ALGEBRAS WITH FREDHOLM OPERATORS WITH INDEX NONZERO ARE NOT APPROXIMATELY SUBHOMOGENEOUS

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ABSTRACT. We show that $C^*$-algebras with Fredholm operators having index nonzero are not approximately subhomogeneous. Using this result we obtain that the unitizations of the $C^*$-algebras of some groups containing all simply connected, solvable Lie groups of non type R are not approximately subhomogeneous

**Introduction.** Approximately homogeneous $C^*$-algebras (AH-algebras), i.e, inductive limits of finite direct sums of homogeneous $C^*$-algebras, have been classified by the K-theory of $C^*$-algebras. It has been one of the interesting problems to determine the structure of concrete examples of AH-algebras. Elliott and Evans [EE] first showed that the irrational rotation algebras are inductive limits of two direct sums of matrix algebras over the $C^*$-algebra of all continuous functions on the torus, and its generalization to certain higher-dimensional noncommutative tori was considered by Elliott and Q. Lin [EL].

On the other hand, it is known that Toeplitz algebra(s) is an extension of the algebra of all continuous functions on the torus (or other spaces) by the $C^*$-algebra of all compact operators (or the commutator ideal) (AH-algebras are not closed under extensions in general), and is not finite since the algebra is generated by the unilateral shift with Fredholm index $-1$ (cf.[Mp, 3.5]). Also, the Fredholm index is an obstruction to whether essentially normal operators are normal operators plus compact operators, and it is also closely related to the notion of quasidiagonality of operators (cf. Davidson [Dv, IX. Corollary 7.4 and IX.8], Brown-Douglas-Fillmore [BDF1-2], Salinas [Sl], O'Donovan [Od] and Voiculescu [Vl]).

In this paper we show that $C^*$-algebras with Fredholm operators having index nonzero are not approximately subhomogeneous (ASH). Consequently, using Z'ep and Rosenberg's results ([Zp],[Rs]), we obtain that the unitizations of the group $C^*$-algebras of the $ax+b$ groups over real or complex fields and some semi-direct products of $\mathbb{R}^n$ by $\mathbb{R}$ are not ASH. As an application we have that the unitizations of group $C^*$-algebras of all simply connected, solvable Lie groups of non type R are not ASH.

As a remark, these results could be deduced from a point of view of quasidiagonality (cf. [Sl, Remark 2.3]).

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**Notation.** We denote by $\mathbb{B}(H)$ the $C^*$-algebra of all bounded operators on a separable, infinite dimensional Hilbert space $H$, and by $\mathbb{F}_m(H)$ the set of all Fredholm operators on $H$ with index $m \in \mathbb{Z}$ (cf. [Bl], [Mp], [Pd1-2] and [Wo] for general references).

**Theorem 1.** *Let $\mathfrak{A}$ be a $C^*$-algebra in $\mathbb{B}(H)$. If $\mathfrak{A} \cap \mathbb{F}_m(H) \neq \emptyset$ for some $m \neq 0$, then $\mathfrak{A}$ is not approximately subhomogeneous (i.e, not an ASH-algebra).*

*Proof.* Suppose that $\mathfrak{A}$ is approximately subhomogeneous, i.e, $\mathfrak{A} = \varinjlim \mathfrak{A}_n$ with $\mathfrak{A}_n$ finite direct sums of subhomogeneous $C^*$-algebras $\mathfrak{A}_{n,k}$ $(1 \leq k \leq l_n < \infty)$ by definition. Since $\mathbb{F}_m(H)$ is open in $\mathbb{B}(H)$ (cf.[Pd2, Proposition 3.3.18]), and $\mathfrak{A} \cap \mathbb{F}_m(H) \neq \emptyset$, we have $\mathfrak{A}_n \cap \mathbb{F}_m(H) \neq \emptyset$ for some $n$. Let $a = (a_k)_{k=1}^{l_n} \in \mathfrak{A}_n \cap \mathbb{F}_m(H)$ with $\mathfrak{A}_n = \oplus_{k=1}^{l_n} \mathfrak{A}_{n,k}$. Now let $c \in \mathfrak{A}_n$ with $\mathrm{Index}(c) = m \neq 0$. We consider the universal atomic representation $\Phi$ of $\mathfrak{A}_n$ acting on the universal Hilbert space $\tilde{H}$ (cf. [Tk]) such that

$$\Phi(c) = (\pi(c)) \in \oplus_{k=1}^{l_n} \oplus_{\pi \in \hat{\mathfrak{A}}_{n,k}} \pi(\mathfrak{A}_{n,k})$$

where $\hat{\mathfrak{A}}_n = \cup_{k=1}^{l_n} \hat{\mathfrak{A}}_{n,k}$, and $\hat{\mathfrak{A}}_n$, $\hat{\mathfrak{A}}_{n,k}$ are the spaces of all irreducible representations of $\mathfrak{A}_n$ and $\mathfrak{A}_{n,k}$ up to unitary equivalence respectively, and $\pi(\mathfrak{A}_{n,k})$ for any $\pi \in \hat{\mathfrak{A}}_{n,k}$ is isomorphic to a matrix algebra over $\mathbb{C}$ (Note that $\mathfrak{A}_{n,k}$ is subhomogeneous means the dimension of elements of $\hat{\mathfrak{A}}_{n,k}$ is bounded). Since $c = U\Phi(c)U^*$ for $U$ an unitary from $H$ to $\tilde{H}$, it follows that

$$\mathrm{Index}(c) = \mathrm{Index}(U\Phi(c)U^*) = \mathrm{Index}(U) + \mathrm{Index}(\Phi(c)) + \mathrm{Index}(U^*) = \mathrm{Index}(\Phi(c)) < \infty$$

in a generalized sense from the identification between $H$ and $\tilde{H}$. Note $\dim\ker(\pi(c)) = \dim\ker(\pi(c)^*)$ from an elementary fact of linear algebra theory, where $\ker(\cdot)$ means the kernel. Then $\dim\ker(\Phi(c)) = \dim\ker(\Phi(c)^*)$ is deduced. Thus, $\mathrm{Index}(\Phi(c)) = 0$, which is the contradiction. □

*Remark.* In particular, we may take $\mathfrak{A}_{n,k}$ as the so-called dimension drop algebras (cf.[JS]). As a special case, $\mathfrak{A}_{n,k}$ may be the algebra of continuous functions $f$ on the interval $[0,1]$ such that $f(0), f(1) \in \mathbb{C}$ and $f(t) \in M_l(\mathbb{C})$ for $0 < t < 1$ and for some $l$.

**Corollary 2.** *Under the same situation as Theorem 1, $\mathfrak{A}$ is not AH.*

*Proof.* Note that AH-algebras are ASH. □

*Remark.* The homogeneous algebras $\mathfrak{A}_{n,k}$ for AH-algebras are usually taken as matrix algebras over commutative $C^*$-algebras. But homogeneous algebras are not always of this type.

Theorem 1 implies

**Corollary 3.** *For an ASH-algebra $\mathfrak{A}$ in $\mathbb{B}(H)$, we have $\mathfrak{A} \cap \mathbb{F}_m(H) = \emptyset$ for any $m \neq 0$.*

The next corollary will be useful later:

**Corollary 4.** *Let $\mathfrak{A}$ be a $C^*$-algebra with a quotient $\mathcal{Q}$ in $\mathbb{B}(H)$. If $\mathcal{Q} \cap \mathbb{F}_m(H) \neq \emptyset$ for some $m \neq 0$, then $\mathfrak{A}$ is not ASH.*

*Proof.* Note that ASH-algebras are closed under quotients since so are subhomogeneous algebras. □

*Remark.* Note that the quasidiagonality for operator algebras is not closed under quotients in general (cf. [Sl, Remark 3.4]).

Now recall that Z'ep [Zp] and Rosenberg [Rs] studied the structure of group $C^*$-algebras of the real $ax+b$ group, and the $ax+b$ group over complex (or nonarchimedean fields) and certain semi-direct products of $\mathbb{R}^n$ by $\mathbb{R}$ with hyperbolic orbits by actions of $\mathbb{R}$. We say that one of these groups is in the class ZR.

**Corollary 5.** *Let $G$ be a group in the class ZR. Then the unitizations $C^*(G)^+$ of the group $C^*$-algebra $C^*(G)$ of $G$ is not ASH.*

*Proof.* Z'ep and Rosenberg showed that the unitizations $C^*(G)^+$ for $G$ in the class ZR has an irreducible representation $\pi$ such that $\pi(C^*(G)^+)$ contains a Fredholm operator with index nonzero ([Zp], [Rs]). Thus Corollary 4 implies the conclusion. □

*Remark.* Note that $C^*(G)^+$ of $G$ above is finite and of type I.

**Corollary 6.** *Let $G$ be a locally compact group with an amenable closed normal subgroup $H$ whose quotient $G/H$ is isomorphic to a group in the class ZR. Then the unitizations $C^*(G)^+$ and $C_r^*(G)^+$ are not ASH, where $C_r^*(G)$ is the reduced group $C^*$-algebra of $G$.*

*Proof.* Note that we have the following quotients and equality:

$$C^*(G)^+ \to C_r^*(G)^+ \to C_r^*(G/H)^+ = C^*(G/H)^+$$

since $H$ is amenable for the second quotient and $G/H$ is amenable for the equality (cf.[Kn, p.1349], [Dx, Chapter 18] and [FD, 12.20, p.1172]) since $G/H$ is a group in the class ZR. Using Corollary 4, we complete the proof. □

*Remark.* When a simply connected solvable Lie group $G$ has the $ax+b$ group as a quotient, it is of non type R in the sense of [AM]. We can take $G$ to be of non type I. For example, we may let $G = M \times K$ for $M$ the Mautner group (cf. a remark of Proposition 10) and $K$ in the class ZR.

**Theorem 7.** *Let $G$ be a simply connected solvable Lie group of non type R. Then the unitization $C^*(G)^+$ is not ASH.*

*Proof.* By [AM, Proposition 2.2, p.172], $G$ has a quotient isomorphic to one of the following:

$$\begin{cases} \text{The real (proper) } ax+b \text{ group: } A = \mathbb{R} \rtimes_\alpha \mathbb{R}, \\ S_c = \mathbb{R}^2 \rtimes_{\alpha^c} \mathbb{R}, \quad S = \mathbb{R}^2 \rtimes_\beta \mathbb{R}^2, \end{cases}$$

where the actions $\alpha$, $\alpha^c$, $\beta$ are defined by

$$\begin{cases} \alpha_t(s) = e^t s, \quad t, s \in \mathbb{R}, \\ \alpha_t^c = e^{ct} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \quad \beta_{(s,t)} = e^s \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \end{cases}$$

for $c \in \mathbb{R} \setminus \{0\}$. Note that Rosenberg's methods calculating Fredholm index of certain elements of certain quotients of the unitizations of the $C^*$-algebras of the real or complex $ax+b$ groups and the semi-direct products in the class ZR are applicable to those of $A$, $S$ and $S_c$ respectively. In fact, the groups $A$, $S$ and $S_c$ are similar as the complex $ax+b$ group or the semi-direct products in the class ZR having wandering orbits by their actions. In particular, note that $C^*(S)$ is isomorphic to the crossed product $C^*(\mathbb{C}) \rtimes_\beta \mathbb{C}$ with the identification between $\mathbb{R}^2$ and $\mathbb{C}$ via $(s,t) = s + it$. Therefore, we can show that the unitizations $C^*(A)^+$, $C^*(S_c)^+$ and $C^*(S)^+$ have quotients containing Fredholm operators with index nonzero. Since $G$ is amenable and of non type R, there is a quotient map from $C^*(G)^+$ to either $C^*(A)^+$, $C^*(S_c)^+$ or $C^*(S)^+$. Thus Corollary 4 deduces the conclusion. □

*Remark.* See [Sd4-6] for the algebraic structure of group $C^*$-algebras of some connected or disconnected Lie groups, that is, they have finite composition series with their subquotients either AH-algebras or continuous fields of AH-algebras.

Theorem 7 tempts us to raise the following question:

**Question.** *Let $G$ be a simply connected solvable Lie group of type R. Then the unitization $C^*(G)^+$ is not ASH? In particular, we may take $G$ as the real Heisenberg group or the Mautner group.*

On the other hand, it follows that

**Theorem 8.** *Let $\mathfrak{A}$ be a liminal $C^*$-algebra in $\mathbb{B}(H)$. Then $\mathfrak{A}$ or its unitization $\mathfrak{A}^+$ have no intersection with $\mathbb{F}_m(H)$ for any $m \neq 0$.*

*Proof.* When $\mathfrak{A}$ is nonunital, we consider $\mathfrak{A}^+$. Let $\Phi$ be the universal atomic representation of $\mathfrak{A}^+$ (or $\mathfrak{A}$ when it is unital). Then $\Phi(\mathfrak{A}^+) \cong (\oplus_{\pi \in \hat{\mathfrak{A}}} \pi(\mathfrak{A}^+)) \oplus \mathbb{C}1$, where $\hat{\mathfrak{A}}$ is the space of all irreducible representations of $\mathfrak{A}$, identified with $(\mathfrak{A}^+)^\wedge \setminus \{1\}$, and 1 is the trivial representation of $\mathfrak{A}^+$. Note that $\pi(\mathfrak{A}^+)$ is isomorphic to either $\mathbb{K}^+$ or a matrix algebra over $\mathbb{C}$, and $\pi(\mathfrak{A})$ is isomorphic to a matrix algebra over $\mathbb{C}$ when $\mathfrak{A}$ is unital.

When $\pi(\mathfrak{A}^+) \cong \mathbb{K}^+ \equiv \mathbb{K} + \mathbb{C}1_\pi$, if $c$ is a Fredholm operator in $\mathfrak{A}^+$, we have $\mathrm{Index}(\pi(c)) = \mathrm{Index}(1_\pi) = 0$. Therefore, $\mathrm{Index}(c) = 0$ follows from the same argument as in the proof of Theorem 1. □

*Remark.* Compare this theorem with the result [Sl, Theorem 2.2].

**Corollary 9.** *Let $G$ be a connected, nilpotent Lie group or a real connected, semi-simple Lie group. Then the unitization of $C^*(G)$ in $\mathbb{B}(H)$ has no intersection with $\mathbb{F}_m(H)$ for any $m \neq 0$.*

*Proof.* It is known that $C^*(G)$ is liminal (cf.[Dx, 13.11.12]).

*Remark.* The Hilbert space $H$ may be taken as the universal representation space of $C^*(G)$ or $L^2(G)$ the space of all square integrable functions on $G$ with the convolution. The real Heisenberg group is in the case of Corollary 9. Note that any connected nilpotent Lie group is of type R [AM]. Thus Corollary 9 suggests that the Question is negative.

More generally, it follows that

**Proposition 10.** *Let $\mathfrak{A}$ be a $C^*$-algebra in $\mathbb{B}(\oplus_{\lambda\in\Lambda}H_\lambda)$ by a direct sum representation $\oplus_{\lambda\in\Lambda}\pi_\lambda$, where $H_\lambda$ are representation spaces of $\pi_\lambda$. If $\pi_\lambda(\mathfrak{A})$ has no intersection with $\mathbb{F}_m(H_\lambda)$ for any $m \neq 0$ and $\lambda$, then $\mathfrak{A}$ has no intersection with $\mathbb{F}_m(\oplus_{\lambda\in\Lambda}H_\lambda)$ for any $m \neq 0$.*

*Proof.* When $a \in \mathfrak{A}$ is a Fredholm operator, note that

$$\mathrm{Index}(a) = \mathrm{Index}(\oplus_{\lambda\in\Lambda}\pi_\lambda(a)) = \sum_{\lambda\in\Lambda}\mathrm{Index}(\pi_\lambda(a)) = 0. \qquad \square$$

*Remark.* We may take $\mathfrak{A}$ as a $C^*$-algebra of continuous fields on a locally compact Hausdorff space with its fibers AH or ASH-algebras. In particular, the group $C^*$-algebra of the discrete Heisenberg group $H_3^{\mathbb{Z}} = \mathbb{Z}^2 \rtimes \mathbb{Z}$ is in the case of Proposition 10. In fact, $C^*(H_3^{\mathbb{Z}})$ is regarded as a $C^*$-algebra of continuous fields on $\mathbb{T}$ with its fibers given by the crossed products $\{C(\mathbb{T}) \rtimes_{\theta_z} \mathbb{Z}\}_{z\in\mathbb{T}}$ (rotation algebras), where the action $\theta_z$ of $\mathbb{Z}$ on $\mathbb{T}$ is defined by $\theta_z(w) = zw$ for $w \in \mathbb{T}$.

*Remark.* We also have that the group $C^*$-algebra of the Mautner group is in the case of Proposition 10. Recall that the Mautner group $M_5$ is defined by the semi-direct product $\mathbb{C}^2 \rtimes_\alpha \mathbb{R}$, where $\alpha_t(e^{2\pi it}z, e^{2\pi i\theta t}w)$ for $t \in \mathbb{R}$, $z, w \in \mathbb{C}$ and $\theta$ an irrational number (cf. [AM]). Then $C^*(M_5)$ is isomorphic to the crossed product $C_0(\mathbb{C}^2) \rtimes_{\hat{\alpha}} \mathbb{R}$ with $\hat{\alpha}$ the dual action of $\alpha$ through the Fourier transform, and regarded as a $C^*$-algebra of continuous fields on $[0,\infty)^2$ with its fibers $\mathfrak{B}_t$ given by $\mathfrak{B}_{(0,0)} = C_0(\mathbb{R})$ and $\mathfrak{B}_t = C_0(\mathbb{R}) \otimes C(\mathbb{T}) \rtimes \mathbb{R} \cong C_0(\mathbb{R}\times\mathbb{T}) \otimes \mathbb{K}$ for $t \in \{0\}\times(0,\infty) \cup (0,\infty)\times\{0\}$, and $\mathfrak{B}_t = C_0(\mathbb{R}^2) \otimes C(\mathbb{T}^2) \rtimes \mathbb{R} \cong C_0(\mathbb{R}^2) \otimes \mathbb{K} \otimes (C(\mathbb{T}) \rtimes_\theta \mathbb{Z})$ for $t \in (0,\infty)^2$, which is deduced from the natural quotient map from $\mathbb{C}^2$ to $[0,\infty)^2$ defined by $\mathbb{C}^2 \ni (0,0) \mapsto (0,0) \in [0,\infty)^2$ and $(\mathbb{C}\setminus\{0\}) \times \{0\} \ni (z,0) \mapsto (|z|,0) \in (0,\infty)\times\{0\}$ and $\{0\} \times (\mathbb{C}\setminus\{0\}) \ni (0,w) \mapsto (0,|w|) \in \{0\}\times(0,\infty)$ and $(\mathbb{C}\setminus\{0\})^2 \ni (z,w) \mapsto (|z|,|w|) \in (0,\infty)^2$ (cf. [Le], [Sd4]). Since $M_5$ is of type R, this remark suggests that the Question is negative.

From this point of view, we propose the following conjecture:

**Conjecture.** *Let $G$ be a simply connected solvable Lie group of type R. Then $C^*(G)$ is ASH?*

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