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Transcendental entire solution of some q-difference equation

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Abstract

We treat linear q-difference equations with polynomial coefficients, in which $q = e^{2\pi\lambda i}, \lambda \in (0,1) \setminus \mathbb{Q}$. Supposing that there is a transcedental entire solution $f(z)$ for this equation, we will show that $f(z)$ takes any finite value infinitely often in any sector.

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1 Introduction

We consider here a q-difference equation

(1.1) $$b_p(z)f(q^p z) + \cdots + b_0(z)f(z) = \mathfrak{b}(z), \quad b_j(z),\ \mathfrak{b}(z) \in \mathbb{C}[z],$$

with $b_j(z) = \sum_{k=0}^{B_j} b_k^{(j)} z^k \quad (b_{B_j}^{(j)} \neq 0), 0 \leq j \leq p$, in which we suppose that $|q| = 1$, i.e., $q = e^{2\pi i\lambda}$. Further we suppose that

(1.2) $$q = e^{2\pi i\lambda}, \quad \lambda \in (0,1) \setminus \mathbb{Q}.$$

The equation (1.1) with q in (1.2) may have transcendental entire solution. In fact, Driver et al. [2] p. 474 showed that there exists a pair (q, A), q in (1.2) and $|A| = 1$, such that the equation

(1.3) $$qzf(qz) + (1 - Az)f(z) = 1$$

has a transcendental entire solution $f(z)$. See also [6].

By the way, Ramis [7] questioned whether (1.1) with q in (1.2) would have transcendental entire solution which also satisfies a linear differential equation.

Here we will consider some properties of solutions of (1.1) with q in (1.2).

First, we introduce some notations: Put $B^* = \max_{0 \leq j \leq p} B_j$ $(B_j = \deg[b_j(z)])$ and $j_1 < \cdots < j_\tau$ be such that $B^* = B_{j_t}$ $(1 \leq t \leq \tau)$ with $B_j < B^*$ $(j \neq j_t)$. Write $b_t = b_{B^*}^{(j_t)}$ and set

(1.4) $$\phi(z) = \sum_{t=1}^{\tau} b_t z^{j_t - j_1} = 0.$$

Lemma 1.1 *Let* (1.1) *be with* $q = e^{2\pi i\lambda}, \lambda \in (0,1)$, *and* $\phi(z)$ *be as in* (1.4). *Suppose* (1.1) *admits a transcendental entire solution* $f(z)$. *Then the equation* (1.4) *has at least one root of modulus* 1.

Thus, if (1.1) has the only one coefficient of the highest degree, then $\phi(z)$ is a non-zero const., and (1.1) cannot have any transcendental entire solution. E.g., $zf(qz) - f(z) = q^2z^3 - z^2$ has no transcendental entire solution, while it has a polynomial solution $f(z) = z^2$. It may be $\lambda \in \mathbb{Q}$ or $\lambda \notin \mathbb{Q}$ in the lemma.

In (1.4), write $j_\tau - j_1 = \iota$. Denote by $\xi_j, 1 \le j \le \iota$ the roots of (1.4). By Lemma 1.1, at least one of ξ_j has modulus 1. Let $|\xi_j| = 1$ for $1 \le j \le \kappa$, and $|\xi_j| \ne 1$ for $\kappa < j \le \iota$. We write $\xi_j (1 \le j \le \kappa)$ also as $q_j (1 \le j \le \kappa)$.

For (1.3), we have $\phi(z) = qz - A$ and $\xi_1 = q_1 = A/q$, that is, $\iota = \kappa = 1$.

We will show an intersting property of solution of (1.3), i.e.,

Theorem 1.2 *Suppose* (1.1), *with* λ *in* (1.2), *admits a transcendental entire solution* $f(z)$. *Further assume that* $\phi(z)$ *in* (1.4) *has only one root of the modulus* 1, *i.e.,* $\kappa = 1$ (ι *may be* > 1). *Then, in any sector,* $f(z)$ *takes any finite value infinitiely often.*

When $\lambda \in \mathbb{Q}$, the theorem does not hold. In fact, consider $f(z) = \cos z$ which satisfies $f(-z) = f(z)$, with $\lambda = 1/2$. Thus, value distributions of solutions depend heavily on the irrationality of λ.

Though the condition $\kappa = 1$ is very restrictive, the equation (1.3) satisfies this condition.

2 Proof of Lemma 1.1

We write the entire solution $f(z)$ as

$$f(z) = \sum_{n=0}^{\infty} \alpha_n z^n. \tag{2.1}$$

Denote by $\nu(r)$ the central index of $f(z)$, [3] p.318.

We have, by (1.1) and (2.1), for any $n > \deg[\mathfrak{b}(z)]$,

$$\sum_{t=1}^{\tau} b_t q^{j_t\, n} \alpha_n z^n z^{B^*} + \sum_{k=0}^{B^*-1} \left(\sum_{j=0}^{p} b_k^{(j)} q^{j(n+B^*-k)} \right) \alpha_{n+B^*-k} z^{n+B^*-k} z^k = 0.$$

Write $\beta_{(k,n)}(z) = \alpha_{n+B^*-k} z^{n+B^*-k} / \alpha_n z^n$, when $\alpha_n \ne 0$. Then $|\beta_{(k,\nu(r))}(z)| \le 1$ for $|z| = r$. Thus

$$\begin{aligned} &\sum_{t=1}^{\tau} b_t q^{(j_t - j_1)\,\nu(r)} + q^{-j_1 \nu(r)} \sum_{k=0}^{B^*-1} \left(\sum_{j=0}^{p} b_k^{(j)} q^{j(\nu(r)+B^*-k)} \right) \beta_{(k,\nu(r))}(z) z^{k-B^*} \\ &= \phi(q^{\nu(r)}) + O(1/z) = 0, \end{aligned} \tag{2.2}$$

hence any accumulation value of $\{q^{\nu(r)}\}$ is a root of (1.4). □

3 Proof of Theorem 1.2

Let $f(z)$ be the transcendental entire solution of (1.1). Let $j_1 < \cdots < j_\tau$ and b_t $(1 \le t \le \tau)$ be as in (1.4). Write $\iota = j_\tau - j_1$ as before.

Put $f_0(z) = f(z)$ and

$$f_j(z) = f_{j-1}(qz) - \xi_j f_{j-1}(z) \quad \text{for } 1 \le j \le \iota.$$

Since we can write (1.1) as $\sum_{t=1}^{\tau} b_t z^{B^*} f(q^{j_t} z) = \sum_{j=0}^{p} b_j^*(z) f(q^j z) + \mathfrak{b}(z)$ with some polynomials $b_j^*(z)$, where $\deg[b_j^*(z)] < B^*$, we have

$$b_\tau f_\iota(z) = \mathfrak{s}_{p-j_1}(z) f(q^{p-j_1} z) + \cdots + \mathfrak{s}_{-j_1}(z) f(q^{-j_1} z) + \mathfrak{s}(z),$$

with rational coefficients $\mathfrak{s}_j(z), j = -j_1, \ldots, p - j_1$, which is $O(1/|z|)$ for large $|z|$, and a ratioal function $\mathfrak{s}(z)$. Let $M(r, f) = \max_{|z|=r} |f(z)|$.

Thus $f_\iota(z) = O(\frac{1}{|z|} M(|z|, f)) = o(M(|z|, f))$. If $\iota = \kappa = 1$, then

$$(Q) \qquad f_1(z) = f(qz) - q_1 f(z) = O(|z|^{-1} M(|z|, f)).$$

If $\iota > \kappa$, then $|\xi_\iota| \neq 1$. Hence we have

$$|f_{\iota-1}(z)||1 - |\xi_\iota|| \le |f_\iota(z)| = O(|z|^{-1} M(|z|, f)).$$

Repeating this procedure, using the fact that $|\xi_j| \neq 1$, $1 < j \le \iota$, we obtain the above inequality (Q) also.

Hence, if $\zeta = \zeta(r)$ is a point such that $|\zeta| = r$, $|f(\zeta)| = M(r, f)$, then we have

$$f(q\zeta) = q_1 f(\zeta)(1 + o(1)),$$

generally for $N \in \mathbb{N}$,

$$(3.1) \quad f(q^k \zeta) = q_1^k f(\zeta)(1 + o(1)), \quad |f(q^k \zeta)| = M(r, f)(1 + o(1)), \quad 1 \le k \le N.$$

Suppose there is a sector $\Delta(\alpha, \beta, r_0) = \{z \; ; \; \alpha < \arg[z] < \beta, \; |z| > r_0\}$, in which $f(z)$ has no zeros. Let $r_0 < r_1 < r_2$ and $0 < \epsilon < \frac{1}{4}$.

There is δ^* such that $|\log|f(r_j e^{2\pi i \theta})| - \log|f(r_j e^{2\pi i \theta'})|| < \epsilon, j = 1, 2$, whenever $\theta, \theta' \in (\alpha, \beta), |\theta - \theta'| < \delta^*$.

Let $\alpha < \theta_1 < \theta_2 < \beta$. Let $\delta > 0$ be $\delta < \min(\theta_1 - \alpha, \beta - \theta_2, \frac{1}{4}(\theta_2 - \theta_1), \delta^*)$, and let $C(\theta, \delta) \; : \; z = z_\delta(r; \theta) = r e^{2\pi i \psi_\delta(r, \theta)}, 0 \le r < \infty$, be a curve such that $|f(z_\delta(r; \theta)))| = \sup_{|\vartheta - \theta| \le \delta} |f(r e^{2\pi i \vartheta})|$. The local maximum curve $C(\theta, \delta)$ is obtained by finding local maximum points of $U(r, \vartheta) = f(r e^{2\pi i \vartheta}) \overline{f(r e^{2\pi i \vartheta})}$ in $\theta - \delta \le \vartheta \le \theta + \delta$, by differentiation or others. It is a locally analytic curve.

By the way, the maximum curve, which is nothing but a trace of $\zeta = \zeta(r)$, was given in a classical work of O. Blumenthal [1].

Take $N_1 \in \mathbb{N}$ so large that, each arc $\gamma(r, \theta, \delta) = \{z = r e^{2\pi i \vartheta}; \theta - \delta < \vartheta < \theta + \delta\}$ with opening 2δ contains at least one $rq^k, 1 \le k \le N_1$, by the uniform distribution property of $k\lambda, k \in \mathbb{N}$ [5].

Let $\zeta(r), |\zeta(r)| = r$, be such that $|f(\zeta(r))| = M(r,f)$. Since $\gamma(r, \theta_j, \delta), j = 1, 2$, contain at least one $q^k\zeta(r)$, we have that $|f(z_\delta(r, \theta_j))| = M(r,f)(1+o(1)) > M(r,f)(1-\epsilon)$. Further, for any $\theta \in [\theta_1, \theta_2]$, the arc $\gamma(r_j, \theta, \delta)$ contains $q^k\zeta(r_j)$, and $|f(q^k\zeta(r_j))| > M(r_j, f)(1-\epsilon)$ by (3.1). Therefore for any $\theta_1 \leq \theta \leq \theta_2$, there is k such that

$$\log|f(r_j e^{2\pi i\theta})| > \log|f(q^k\zeta(r_j))| - \epsilon > \log M(r_j, f) - 2\epsilon, \quad j = 1, 2.$$

$u(r) = \log M(r,f)$ is convex with respect to $\log r$, hence $\nabla^2 u \geq 0$, and can be considered as subharmonic [4] p.41. Since $\log|f(z)|$ is harmonic, we get that

$$\log|f(z)| > \log M(r,f) - 2\epsilon \quad \text{if } r_1 \leq |z| \leq r_2,\ \theta_1 + \delta \leq \arg[z] \leq \theta_2 - \delta.$$

For any z, $r_1 \leq |z| \leq r_2$, there is a ζ, $\arg[\zeta] \in (\theta_1 + \delta, \theta_2 - \delta)$, such that $z = q^k\zeta$ for some $k, 1 \leq k \leq N_1$. Hence $f(z) \neq 0$ for $r_1 \leq |z| \leq r_2$. Since $r_1 < r_2$ are arbitrary, we see that $f(z) \neq 0$ for $|z| > r_0$. Further we see that $f(z) \to \infty$ as $z \to \infty$, which shows that the point at infinity is not essential singularity for $f(z)$, hence $f(z)$ is a polynomial, a contradiction.

For any $a \in \mathbb{C}$, we have only to consider $f(z) - a$ for $f(z)$. □

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