Nihonkai Math. J.
Vol.13(2002), 57-66

UNIT TANGENT BUNDLE OVER TWO-DIMENSIONAL REAL PROJECTIVE SPACE

TATSUO KONNO

ABSTRACT. In this paper, we prove that the unit tangent bundle over the two-dimensional real projective space is isometric to a lens space. Also, we characterize the Killing vector fields and the geodesics of this bundle in terms of the geometry of the base space.

1. INTRODUCTION

It has been proven by Klingenberg and Sasaki [1] that the unit tangent bundle over a unit two-sphere is isometric to the three-dimensional real projective space of constant curvature 1/4. The authors also studied the geodesics of the bundle. In this paper, we prove that the unit tangent bundle over two-dimensional real projective space is isometric to a lens space. Then we describe the Killing vector fields on this bundle in terms of the geometry of the base space, and characterize the geodesics of the bundle using the Killing vector fields on the base space.

Let $S^n(k)$ denote an n-dimensional sphere of radius $1/\sqrt{k}$ in the Euclidean $(n+1)$-space. We assume that the center of $S^n(k)$ is fixed at the origin of the Euclidean space, and that $S^n(k)$ is endowed with the standard metric. The sectional curvatures of $S^n(k)$ are constant, k. We then denote by $\boldsymbol{RP}^n(k)$ the real projective space that is given by identifying the antipodal points of $S^n(k)$. In order to define lens spaces, we identify the point (x^1, x^2, x^3, x^4) of $\boldsymbol{R}^4$ with the point $(x^1+\sqrt{-1}x^2, x^3+\sqrt{-1}x^4)$ of $\boldsymbol{C}^2$. For any relatively prime integers p, q satisfying $1 \leq q < p$, an isometry of $S^3(k)$ is given by

$$(Z^1, Z^2) \longmapsto (Z^1 e^{2\pi\sqrt{-1}/p}, Z^2 e^{2\pi\sqrt{-1}q/p}), \quad \text{where } (Z^1, Z^2) \in S^3(k) \subset \boldsymbol{C}^2.$$

Let $\Gamma(p,q)$ denote the transformation group generated by this isometry. The quotient space $S^3(k)/\Gamma(p,q)$ is then called the lens space of type (p,q).

For a Riemannian manifold (M, g), let $U(M)$ denote the unit tangent bundle over M, whose total space is the tangent vectors of length one. We assume the Sasaki metric on $U(M)$. Let $\mathfrak{i}(M)$ denote the Lie algebra of the Killing vector fields on M. Then we have

1991 *Mathematics Subject Classification.* Primary 53C25; Secondary 53C22;.
Key words and phrases. unit tangent bundle, lens space, Killing vector field, geodesic.

Theorem 1.1. (i) *The unit tangent bundle* $U(\boldsymbol{RP}^2(1))$ *over the two-dimensional real projective space* $\boldsymbol{RP}^2(1)$ *is isometric to the lens space* $S^3(1/4)/\Gamma(4,1)$.

(ii) *The Lie algebra of the Killing vector fields on this space is given by*

$$\mathfrak{i}(U(\boldsymbol{RP}^2(1))) = \mathrm{span}\,\{X^L,\ F\,;\ X \in \mathfrak{i}(\boldsymbol{RP}^2(1))\},$$

where X^L *denotes the natural lift of* X, *and* F *denotes the geodesic spray on* $U(\boldsymbol{RP}^2(1))$. *Hence, the dimension of the lens space* $S^3(1/4)/\Gamma(4,1)$ *is five.*

(iii) *Any integral curve of the natural lift* X^L *is a geodesic in* $U(\boldsymbol{RP}^2(1))$. *Conversely, every geodesic in the unit tangent bundle can be obtained in this way.*

The first part of Theorem 1.1 can be proven by considering the isometry of $\boldsymbol{RP}^3(1)$ as $SO(3) \times SO(3)$ ([5]). But, we characterize $U(\boldsymbol{RP}^2(1))$ in terms of the geometry of the base space. This allows us to know the relation between the total space and the base space.

2. The Killing Vector Fields on $U(S^2(1))$

In this section we describe the general forms of Killing vector fields on $U(S^2(1))$, and study certain properties of the Lie algebra $\mathfrak{i}(U(S^2(1)))$. These shall be used in the proof of Theorem 1.1.

Let $SO(S^2(1))$ be the bundle of all oriented orthonormal frames over $S^2(1)$, and let θ and ω denote the canonical form and the Riemannian connection form on $SO(S^2(1))$, respectively. If we consider the Riemannian metric G on $SO(S^2(1))$ as defined by

$$G(Z,W) = {}^t\theta(Z)\theta(W) + \frac{1}{2}\mathrm{trace}\,({}^t\omega(Z)\omega(W)) \quad \text{for } Z, W \in T_uSO(S^2(1)),\ u \in SO(S^2(1)),$$

then the bundle $SO(S^2(1))$ is isometric to the unit tangent bundle $U(S^2(1))$ ([3], Theorem 2). In fact, the isometric mapping of $SO(S^2(1))$ onto $U(S^2(1))$ is given by

$$SO(S^2(1)) \ni (X_1, X_2) \longmapsto X_2\,. \tag{2.1}$$

In the following, we identify $SO(S^2(1))$ with $U(S^2(1))$ in this manner.

We present several lemmas concerning the Lie algebra $\mathfrak{i}(SO(S^2(1)))$. The next lemma follows from Theorem 1.2 in [3].

Lemma 2.1. *The Lie algebra of the Killing vector fields on* $SO(S^2(1))$ *is given by*

$$\mathfrak{i}(SO(S^2(1))) = \mathrm{span}\,\{X^L,\ A^*,\ B(\xi)\,;\ X \in \mathfrak{i}(S^2(1)),\ A \in \mathfrak{o}(2),\ \xi \in \boldsymbol{R}^2\},$$

where $\mathfrak{o}(2)$ *is the Lie algebra of the special orthogonal group* $SO(2)$, A^* *denotes the fundamental vector field corresponding to* A *in* $\mathfrak{o}(2)$, *and* $B(\xi)$ *denotes the standard horizontal vector field corresponding to* ξ *in* $\boldsymbol{R}^2$.

For X, Y in $\mathfrak{i}(S^2(1))$, A, C in $\mathfrak{o}(2)$, and ξ, η in $\boldsymbol{R}^2$, it follows that

$$[X^L, Y^L] = [X, Y]^L, \quad [A^*, C^*] = 0, \quad [X^L, A^*] = 0,$$

$$[B(\xi), B(\eta)] = -(\xi \wedge \eta)^*, \quad [X^L, B(\xi)] = 0, \quad [A^*, B(\xi)] = B(A\xi),$$

where $(\xi \wedge \eta)$ is the element of $\mathfrak{o}(2)$ defined by $(\xi \wedge \eta)\zeta = (\eta, \zeta)\xi - (\xi, \zeta)\eta$, for $\zeta \in \boldsymbol{R}^2$.

Lemma 2.2. *The Lie algebra $\mathfrak{i}(SO(S^2(1)))$ is decomposed as a direct sum:*

$$\mathfrak{i}(SO(S^2(1))) = \mathcal{I} \oplus \mathcal{J},$$

where we put

$$\mathcal{I} = \text{span}\,\{X^L;\ X \in \mathfrak{i}(S^2(1))\}, \ \textit{and}\ \mathcal{J} = \text{span}\,\{A^*,\ B(\xi);\ A \in \mathfrak{o}(2),\ \xi \in \boldsymbol{R}^2\}.$$

Proof. From Lemma 2.1, the subspaces $\mathcal{I}$ and $\mathcal{J}$ are ideals in $\mathfrak{i}(SO(S^2(1)))$. So Lemma 2.2 follows from the definition of direct sum. □

$\mathcal{I}$ and $\mathcal{J}$ are vector spaces over $\boldsymbol{R}$, and each of them admits a natural inner product $\langle\, , \rangle$. In fact, for X^L and Y^L in $\mathcal{I}$, set $\langle X^L,\ Y^L\rangle = G(X^L,\ Y^L)_u$, where u is an arbitrary point of $SO(S^2(1))$. This is independent of the choice of u. Because, from Lemma 2.1, we have that

$$A^*G(X^L, Y^L) = G([A^*, X^L], Y^L) + G(X^L, [A^*, Y^L]) = 0,$$

$$B(\xi)G(X^L, Y^L) = G([B(\xi), X^L], Y^L) + G(X^L, [B(\xi), Y^L]) = 0$$

and since these formulas hold for any A in $\mathfrak{o}(2)$ and ξ in $\boldsymbol{R}^2$, the function $G(X^L, Y^L)$ is constant on $SO(S^2(1))$, and hence the inner product $\langle\, , \rangle$ is well defined. On the other hand, the natural inner product $\langle\, , \rangle$ on $\mathcal{J}$ is defined by

$$\langle A^* + B(\xi), C^* + B(\eta)\rangle = G(A^* + B(\xi), C^* + B(\eta))_u$$

for A, C in $\mathfrak{o}(2)$, and ξ, η in $\boldsymbol{R}^2$. Indeed, we have

$$G(A^* + B(\xi), C^* + B(\eta))_u = \frac{1}{2}\text{trace}\,({}^tA \cdot C) + {}^t\xi \cdot \eta.$$

This is also independent of the choice of u in $SO(S^2(1))$.

Let D denotes the Levi-Civita connection of the Riemannian manifold $(SO(S^2(1)), G)$. Then we have

Lemma 2.3. *Each α in $\mathcal{I}$ or $\mathcal{J}$ satisfies $D_\alpha \alpha = 0$. Conversely, if α in $\mathfrak{i}(SO(S^2(1)))$*

satisfies $D_\alpha \alpha = 0$, *then it belongs to either* $\mathcal{I}$ *or* $\mathcal{J}$.

Proof. For any X in $\mathfrak{i}(S^2(1))$, A in $\mathfrak{o}(2)$, and ξ in $\boldsymbol{R}^2$, we have from Lemma 2.1 that

$$\begin{aligned} G(D_{X^L}X^L, A^*) &= X^L G(X^L, A^*) - G(X^L, D_{X^L}A^*) \\ &= X^L G(X^L, A^*) = G([X^L, X^L], A^*) + G(X^L, [X^L, A^*]) = 0, \\ G(D_{X^L}X^L, B(\xi)) &= X^L G(X^L, B(\xi)) - G(X^L, D_{X^L}B(\xi)) \\ &= X^L G(X^L, B(\xi)) = G([X^L, X^L], B(\xi)) + G(X^L, [X^L, B(\xi)]) = 0. \end{aligned}$$

From these formulas, we can see that $D_{X^L}X^L = 0$ because vector fields such as A^* and $B(\xi)$ give the basis of each tangent space of $SO(S^2(1))$.

For any A in $\mathfrak{o}(2)$ and ξ, η in $\boldsymbol{R}^2$, the Killing vector fields A^*, $B(\xi)$ and $B(\eta)$ have the constant norms, so we have that

$$\begin{aligned} & G(D_{(A^*+B(\xi))}(A^* + B(\xi)), A^*) \\ & = (A^* + B(\xi))G(A^* + B(\xi), A^*) - G(A^* + B(\xi), D_{(A^*+B(\xi))}A^*) \\ & = A^*G(A^*, A^*) + A^*G(B(\xi), A^*) + B(\xi)G(A^*, A^*) + B(\xi)G(B(\xi), A^*) \\ & \quad -G(A^*, D_{A^*}A^*) - G(A^*, D_{B(\xi)}A^*) - G(B(\xi), D_{A^*}A^*) - G(B(\xi), D_{B(\xi)}A^*) = 0 \end{aligned}$$

and that

$$\begin{aligned} & G(D_{(A^*+B(\xi))}(A^* + B(\xi)), B(\eta)) \\ & = (A^* + B(\xi))G(A^* + B(\xi), B(\eta)) - G(A^* + B(\xi), D_{(A^*+B(\xi))}B(\eta)) \\ & = - G(A^*, D_{B(\xi)}B(\eta)) - G(B(\xi), D_{A^*}B(\eta)) = 0. \end{aligned}$$

Hence we obtain that $D_{(A^*+B(\xi))}(A^* + B(\xi)) = 0$, also. Thus, we have proven the first part of Lemma 2.3.

In order to prove the second part, we assume that $D_\alpha \alpha = 0$ for α in $\mathfrak{i}(SO(S^2(1)))$ with $\alpha \neq 0$. By Lemma 2.1, there exist X in $\mathfrak{i}(S^2(1))$, A in $\mathfrak{o}(2)$, and ξ in $\boldsymbol{R}^2$ such that $\alpha = X^L + A^* + B(\xi)$. From the first part of Lemma 2.3, we find that $D_\alpha \alpha$ is

$$D_{(A^*+B(\xi))}X^L + D_{X^L}(A^* + B(\xi)) = 0. \tag{2.2}$$

Let C be a non-zero element of $\mathfrak{o}(2)$. Then, if $A = 0$, equation (2.2) implies

$$G(D_{B(\xi)}X^L, C^*) + G(D_{X^L}B(\xi), C^*) = 0. \tag{2.3}$$

On the other hand, if $A \neq 0$, by calculating the inner product of A^* and the left hand side of (2.2) we obtain

$$\begin{aligned} & G(D_{A^*}X^L + D_{B(\xi)}X^L + D_{X^L}A^* + D_{X^L}B(\xi), A^*) \\ & \quad = G(D_{B(\xi)}X^L, A^*) + G(D_{X^L}B(\xi), A^*). \end{aligned}$$

Since A is a scalar multiple of C, we also obtain equation (2.3) in this case.

When $\xi \neq 0$, we can show that $X = 0$ as follows: From (2.3) and Lemma 2.1, we have that

$$\begin{aligned}0 &= G(D_{B(\xi)}X^L, C^*) + G(D_{X^L}B(\xi), C^*)\\ &= B(\xi)G(X^L, C^*) - \{G(X^L, D_{B(\xi)}C^*) + G(B(\xi), D_{X^L}C^*)\}\\ &= B(\xi)G(X^L, C^*) = G([B(\xi), X^L], C^*) + G(X^L, [B(\xi), C^*])\\ &= -G(X^L, B(C\xi)) = -G(X^H, B(C\xi)),\end{aligned}$$

where X^H is the horizontal lift of X. It follows from this formula that the assumption that $\xi \neq 0$ implies the existence of a function f such that $X^H = f \cdot B(\xi)$. Then, we have

$$0 = [C^*, X^H] = [C^*, f \cdot B(\xi)] = (C^* f) \cdot B(\xi) + f \cdot [C^*, B(\xi)] = (C^* f) \cdot B(\xi) + f \cdot B(C\xi).$$

Since $B(\xi)$ and $B(C\xi)$ are linearly independent, we obtain $f = 0$ and $X = 0$.

When $\xi = 0$ and $A \neq 0$, we can show that $X = 0$ as follows: Let η be an arbitrary element of $\boldsymbol{R}^2$. From (2.2) we have that

$$\begin{aligned}0 &= G(D_{A^*}X^L, B(\eta)) + G(D_{X^L}A^*, B(\eta))\\ &= -G(D_{B(\eta)}X^L, A^*) - G(A^*, D_{X^L}B(\eta))\\ &= -B(\eta)G(X^L, A^*) + G(X^L, D_{B(\eta)}A^*) + G(D_{X^L}A^*, B(\eta))\\ &= -B(\eta)G(X^L, A^*) = -G([B(\eta), X^L], A^*) - G(X^L, [B(\eta), A^*])\\ &= G(X^L, B(A\eta)) = G(X^H, B(A\eta)).\end{aligned}$$

Since η is an arbitrary element of $\boldsymbol{R}^2$, the vector field X^H vanishes from the assumption of $A \neq 0$, and hence $X = 0$. We have proven the second part of Lemma 2.3. □

Let Φ be an isometry of $SO(S^2(1))$ and α an element of $\mathcal{I}$ or $\mathcal{J}$. Then we have

$$\Phi(\alpha) \in \mathrm{i}(SO(S^2(1))), \quad \text{and} \quad D_{\Phi(\alpha)}\Phi(\alpha) = 0.$$

Hence $\Phi(\alpha)$ belongs to $\mathcal{I}$ or $\mathcal{J}$. Furthermore, we can show that

Lemma 2.4. *Let Φ be an isometry of $SO(S^2(1))$. Then the image $\Phi(\mathcal{I})$ of $\mathcal{I}$ coincides with either $\mathcal{I}$ or $\mathcal{J}$, and the restricted mapping $\Phi|_{\mathcal{I}}$ provides an isometric linear mapping of $\mathcal{I}$ into $\mathrm{i}(SO(S^2(1)))$. Similar facts hold for $\mathcal{J}$.*

Proof. We only prove this lemma for $\mathcal{I}$ because the case of $\mathcal{J}$ can be shown in a similar way. The mapping $\Phi|_{\mathcal{I}}$ is a linear mapping of $\mathcal{I}$ into $\mathrm{i}(SO(S^2(1)))$, and preserves the inner products on $\mathcal{I}$ because the differential Φ_* preserves the inner products of the tangent spaces. Let $\{X^L, Y^L, Z^L\}$ be an orthonormal basis of $\mathcal{I}$. At each point u of $SO(S^2(1))$, we have

$$G(\Phi(X^L), \Phi(Y^L))_u = 0, \quad G(\Phi(Y^L), \Phi(Z^L)) = 0 \quad \text{and} \quad G(\Phi(Z^L), \Phi(X^L)) = 0.$$

According to these orthogonalities, the space spanned by $\Phi(X^L)$, $\Phi(Y^L)$ and $\Phi(Z^L)$ must coincide with $\mathcal{I}$ or $\mathcal{J}$, because there exists a point u of $SO(S^2(1))$ for any non-zero α in $\mathcal{I}$ and β in $\mathcal{J}$ such that $G(\alpha, \beta)_u \neq 0$. In fact, if

$$G(X^L, A^* + B(\xi)) = 0 \quad \text{and} \quad A^* + B(\xi) \neq 0$$

hold for some X in $\mathfrak{i}(S^2(1))$, A in $\mathfrak{o}(2)$ and ξ in $\boldsymbol{R}^2$, then we can derive that $X^L = 0$ from these conditions as follows:

When $\xi \neq 0$, let C be a non-zero element of $\mathfrak{o}(2)$. It then follows from Lemma 2.1 that

$$G(X^L, B(C\xi)) = G(X^L, [C^*, B(\xi)]) = C^* G(X^L, A^* + B(\xi)) = 0.$$

From this, we can similarly obtain that $G(X^L, B(C^2\xi)) = 0$. Since $C\xi$ and $C^2\xi$ are linearly independent, we know that the horizontal part of X^L vanishes, and hence $X^L = 0$.

When $\xi = 0$ and $A \neq 0$, let η be a non-zero element of $\boldsymbol{R}^2$. It then follows from Lemma 2.1 that

$$G(X^L, B(A\eta)) = G(X^L, [A^*, B(\eta)]) = -B(\eta) G(X^L, A^* + B(\xi)) = 0.$$

From this, we obtain that $G(X^L, B(A^2\eta)) = 0$ and that $X^L = 0$ in the same way as the former case. □

Lemma 2.5. (i) *There exists an isometry of $SO(S^2(1))$ that maps $\mathcal{I}$ onto $\mathcal{J}$, and $\mathcal{I}$ is isomorphic to $\mathcal{J}$ as Lie algebra.*

(ii) *Let a be a positive number, and set $\mathcal{I}_a = \{\alpha \in \mathcal{I};\ \langle \alpha, \alpha \rangle = a^2\}$. Then, there exists an isometry subgroup of $SO(S^2(1))$ that acts transitively on $\mathcal{I}_a$. The same holds for $\mathcal{J}$.*

Proof. (i) Since the Riemannian manifold $(SO(S^2(1)), G)$ is isometric to the real projective space $\boldsymbol{RP}^3(1/4)$, it is sufficient to show (i) for $\boldsymbol{RP}^3(1/4)$. Since the Lie algebra $\mathfrak{i}(\boldsymbol{RP}^3(1/4))$ can be identified with the Lie algebra $\mathfrak{i}(S^3(1/4))$, we will prove (i) for $\mathfrak{i}(S^3(1/4))$.

We define a linearly independent system α_1, α_2, α_3 of the Lie algebra $\mathfrak{o}(4)$ of the special orthogonal group $SO(4)$ by

(2.4)

$$\alpha_1 = \begin{pmatrix} & -1 & & 0 \\ 1 & & & \\ & & & -1 \\ 0 & & 1 & \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & & 1 & \\ & & & -1 \\ -1 & & & \\ & 1 & & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & 0 \end{pmatrix}.$$

For each α_i, $1 \leq i \leq 3$, a one-parameter group of transformations Ψ_i of $S^3(1/4)$ can be given by

$$\Psi_i(t, x) = \left(\exp \frac{t}{2}\alpha_i\right) x \qquad \text{for } t \in \boldsymbol{R} \text{ and } x \in S^3(1/4).$$

Each Ψ_i is an isometry of $S^3(1/4)$ for any fixed t. Using the standard coordinate system of $\boldsymbol{R}^4$, Ψ_i is represented by

$$\Psi_1(t,x) = (x^1\cos\frac{t}{2}-x^2\sin\frac{t}{2}, x^1\sin\frac{t}{2}+x^2\cos\frac{t}{2}, x^3\cos\frac{t}{2}-x^4\sin\frac{t}{2}, x^3\sin\frac{t}{2}+x^4\cos\frac{t}{2}),$$

$$\Psi_2(t,x) = (x^1\cos\frac{t}{2}+x^3\sin\frac{t}{2}, -x^4\sin\frac{t}{2}+x^2\cos\frac{t}{2}, x^3\cos\frac{t}{2}-x^1\sin\frac{t}{2}, x^2\sin\frac{t}{2}+x^4\cos\frac{t}{2}),$$

$$\Psi_3(t,x) = (x^1\cos\frac{t}{2}+x^4\sin\frac{t}{2}, x^2\cos\frac{t}{2}+x^3\sin\frac{t}{2}, x^3\cos\frac{t}{2}-x^2\sin\frac{t}{2}, x^4\cos\frac{t}{2}-x^1\sin\frac{t}{2}),$$

where $x = (x^1,x^2,x^3,x^4)$. We can easily see that each orbit of $\Psi_i(t,x_0)$, $t \in \boldsymbol{R}$, through any point x_0 in $S^3(1/4)$ is a geodesic of $S^3(1/4)$. Next, we consider a Killing vector field X_i on $S^3(1/4)$ that is defined by

$$S^3(1/4) \ni x \longmapsto \frac{d}{dt}(\Psi_i(t,x))\Big|_{t=0}. \tag{2.5}$$

In other words, $\Psi_i(t,\cdot)$ is a (local) one-parameter group of (local) transformations generated by X_i. Using the standard coordinate system of $\boldsymbol{R}^4$, we have

$$\begin{aligned} 2X_1 &= -x^2\frac{\partial}{\partial x^1} + x^1\frac{\partial}{\partial x^2} - x^4\frac{\partial}{\partial x^3} + x^3\frac{\partial}{\partial x^4}, \\ 2X_2 &= x^3\frac{\partial}{\partial x^1} - x^4\frac{\partial}{\partial x^2} - x^1\frac{\partial}{\partial x^3} + x^2\frac{\partial}{\partial x^4}, \\ 2X_3 &= x^4\frac{\partial}{\partial x^1} + x^3\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^3} - x^1\frac{\partial}{\partial x^4}. \end{aligned} \tag{2.6}$$

The vector fields X_1, X_2 and X_3 are orthonormal at each point of $S^3(1/4)$, and hence the space spanned by X_1, X_2 and X_3 corresponds to $\mathcal{I}$ or $\mathcal{J}$ (cf. the proof of Lemma 2.4). Now, we define the isometry Φ of $S^3(1/4)$ by

$$\Phi(x_1,x_2,x_3,x_4) = (x_1,x_3,x_2,x_4),$$

where Φ also yields an isometry of $\boldsymbol{RP}^3(1/4)$. Then,

$$X_1,\ X_2,\ X_3,\ \Phi(X_1),\ \Phi(X_2),\ \Phi(X_3)$$

provide a linearly independent system of $\mathfrak{i}(S^3(1/4))$. The elements of $\mathfrak{o}(4)$ corresponding to $\Phi(X_1)$, $\Phi(X_2)$, $\Phi(X_3)$ are respectively given by

$$\beta_1 = \begin{pmatrix} 0 & & -1 & \\ & & & -1 \\ 1 & & & \\ & 1 & & 0 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} & 1 & & 0 \\ -1 & & & \\ & & & -1 \\ 0 & & 1 & \end{pmatrix}, \quad \beta_3 = \begin{pmatrix} 0 & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & 0 \end{pmatrix}. \tag{2.7}$$

Then, (2.4) and (2.7) provide a basis of $\mathfrak{o}(4)$. Hence, the isometry corresponding to Φ maps $\mathcal{I}$ onto $\mathcal{J}$. Thus, we have proven (i) of Lemma 2.5.

(ii) Let a be any positive number. We note that there is a one-to-one correspondence between $\mathcal{I}_a$ and the set of all axes of rotation of $S^2(1)$. Since the isometry group of $S^2(1)$ acts transitively on the set of such axes, statement (ii) of Lemma 2.5 holds for $\mathcal{I}$. Together with (i) of Lemma 2.5, statement (ii) also holds for $\mathcal{J}$. □

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1. In the following, the vector fields on $U(S^2(1))$ are identified with that of $SO(S^2(1))$ by (2.1). First we show that the unit tangent bundle $U(S^2(1))$ is isometric to the lens space $S^3(1/4)/\Gamma(4,1)$. As stated in § 1, each point (x^1, x^2, x^3, x^4) of $\boldsymbol{R}^4$ is identified with the point $(x^1 + \sqrt{-1}x^2, x^3 + \sqrt{-1}x^4)$ of $\boldsymbol{C}^2$. We define the one-parameter group of transformations φ_t of $S^3(1/4)$ by

$$\varphi_t(Z^1, Z^2) = (Z^1 e^{t\sqrt{-1}/2}, Z^2 e^{t\sqrt{-1}/2}), \quad t \in \boldsymbol{R}.$$

The transformation group generated by φ_π is $\Gamma(4,1)$, which is defined in § 1. Using the standard coordinate system of $\boldsymbol{R}^4$, we have

$$\varphi_t(x) = (x^1 \cos\frac{t}{2} - x^2 \sin\frac{t}{2}, x^1 \sin\frac{t}{2} + x^2 \cos\frac{t}{2}, x^3 \cos\frac{t}{2} - x^4 \sin\frac{t}{2}, x^3 \sin\frac{t}{2} + x^4 \cos\frac{t}{2}),$$

where $x = (x^1, x^2, x^3, x^4) \in \boldsymbol{R}^4$. This is $\Psi_1(t, x)$ that was defined in the proof of Lemma 2.5. Let us recall from (2.5) and (2.6) that φ_t $(= \Psi_1(t, \cdot))$ is the one-parameter group of transformations that is generated by the Killing vector field X_1 on $S^3(1/4)$. X_1 is automatically regarded as a Killing vector field on $\boldsymbol{RP}^3(1/4)$, and we denote this vector field by $\tilde{X}_1$. Any integral curve of $\tilde{X}_1$ is a closed geodesic with length 2π in $\boldsymbol{RP}^3(1/4)$. Let $\tilde{\varphi}_t$ be the one-parameter group of transformations of $\boldsymbol{RP}^3(1/4)$ generated by $\tilde{X}_1$, where t is the arc length parameter of the closed geodesic. Let us recall that X_1 corresponds to an element of ideal $\mathcal{I}$ or $\mathcal{J}$, which were both definded in § 2. Then we also know that $\tilde{X}_1$ corresponds to an element of $\mathcal{I}$ or $\mathcal{J}$. By Lemma 2.5, there exists an isometric mapping of $\boldsymbol{RP}^3(1/4)$ onto $U(S^2(1))$ that maps the vector field $\tilde{X}_1$ to the vector field $B(e_2)$ on $U(S^2(1))$. Let $\tilde{\varphi}_t$ also denote the one-parameter group of transformations of $U(S^2(1))$ generated by $B(e_2)$. Since the transformation group $\Gamma(4,1)$ is generated by φ_π, we have that

$$S^3(1/4)/\Gamma(4,1) = S^3(1/4)/\langle\varphi_\pi\rangle \overset{\text{isome.}}{\approx} \boldsymbol{RP}^3(1/4)/\langle\tilde{\varphi}_\pi\rangle \overset{\text{isome.}}{\approx} U(S^2(1))/\langle\tilde{\varphi}_\pi\rangle,$$

where $\langle\varphi_\pi\rangle$ and $\langle\tilde{\varphi}_\pi\rangle$ are the transformation groups that are generated by φ_π and $\tilde{\varphi}_\pi$, respectively. Since $\tilde{\varphi}_\pi$ identifies the antipodal points of the base space $S^2(1)$, it follows that $U(S^2(1))/\langle\tilde{\varphi}_\pi\rangle$ is isometric to $U(\boldsymbol{RP}^2(1))$. Consequently, the unit tangent bundle $U(\boldsymbol{RP}^2(1))$ is isometric to the lens space $S^3(1/4)/\Gamma(4,1)$. We have proven the first part of Theorem 1.1.

We next describe the Killing vector field on the unit tangent bundle $U(\boldsymbol{R}P^2(1))$ in terms of the geometry of the base space. Let W be a Killing vector field on $U(\boldsymbol{R}P^2(1))$. We know from Theorem 1.2 in [3] that, for any point u of $U(\boldsymbol{R}P^2(1))$, there exist a neighborhood V of u, X in $\mathfrak{i}(S^2(1))$, A in $\mathfrak{o}(2)$, and ξ in $\boldsymbol{R}^2$ such that the formula

$$W = X^L + A^* + B(\xi) \tag{3.1}$$

holds on V. With the other expression

$$W = Y^L + C^* + B(\eta), \qquad Y \in \mathfrak{i}(S^2(1)),\ C \in \mathfrak{o}(2),\ \eta \in \boldsymbol{R}^2 \tag{3.2}$$

on the open set, we have from (3.1) and (3.2) that $(X-Y)^L+(A-C)^*+B(\xi-\eta)=0$ on V. By the same calculation that for Lemma 2.1, we obtain that $X=Y$, $A=C$ and that $\xi=\eta$. Hence, formula (3.1) holds on $U(\boldsymbol{R}P^2(1))$. By presuming that $\xi = a^1 e_1 + a^2 e_2$ for a_1, a_2 $\in\boldsymbol{R}$, we have that

$$\begin{aligned} (\tilde{\varphi}_\pi)_* W - W \circ (\tilde{\varphi}_\pi) &= (\tilde{\varphi}_\pi)_*(X^L + A^* + B(\xi)) - (X^L + A^* + B(\xi)) \circ (\tilde{\varphi}_\pi) \\ &= (\tilde{\varphi}_\pi)_*(X^L) + (\tilde{\varphi}_\pi)_*(A^*) + a^1(\tilde{\varphi}_\pi)_*(B(e_1)) + a^2(\tilde{\varphi}_\pi)_*(B(e_2)) \\ &\quad -(X^L)\circ(\tilde{\varphi}_\pi) - A^*\circ(\tilde{\varphi}_\pi) - a^1 B(e_1)\circ(\tilde{\varphi}_\pi) - a^2 B(e_2)\circ(\tilde{\varphi}_\pi). \end{aligned} \tag{3.3}$$

Since $\tilde{\varphi}_t$ is the one-parameter group of transformations generated by $B(e_2)$, the vector field $B(e_2)$ is invariant for all $\tilde{\varphi}_t$. Likewise, since $\tilde{\varphi}_t$ is commutable with any element of the one-parameter group of transformations generated by X^L, the vector field X^L is also invariant for all $\tilde{\varphi}_t$. Thus, we now have that

$$(\tilde{\varphi}_\pi)_*(X^L) = (X^L)\circ(\tilde{\varphi}_\pi), \tag{3.4}$$

$$(\tilde{\varphi}_\pi)_*(B(e_2)) = B(e_2)\circ(\tilde{\varphi}_\pi). \tag{3.5}$$

On the other hand, since $\tilde{\varphi}_\pi$ identifies the antipodal points of the base space $S^2(1)$, we have that

$$(\tilde{\varphi}_\pi)_*(A^*) = -A^*\circ(\tilde{\varphi}_\pi), \tag{3.6}$$

$$(\tilde{\varphi}_\pi)_*(B(e_1)) = -B(e_1)\circ(\tilde{\varphi}_\pi), \tag{3.7}$$

where we regard A^* and $B(e_1)$ as vector fields on $U(S^2(1))$ by (2.1). By substituting formulas (3.4), (3.5), (3.6) and (3.7) for formula (3.3), we obtain that

$$(\tilde{\varphi}_\pi)_* W - W\circ(\tilde{\varphi}_\pi) = -2\{A^* + a^1 B(e_1)\}\circ(\tilde{\varphi}_\pi).$$

However, W is the vector field on $U(\boldsymbol{R}P^2(1))$, and is invariant by $\tilde{\varphi}_\pi$. Hence, formula above must vanishes on $U(\boldsymbol{R}P^2(1))$. For this to hold, we must have that $A=0$ and

$a^1 = 0$, that is to say, $W = X^L + a^2 B(e_2)$ from (3.1). Conversely, every vector field of that form is invariant by $\tilde{\varphi}_\pi$, and is a Killing vector field on $U(\boldsymbol{R}P^2(1))$. Note that the vector field $B(e_2)$ corresponds to the geodesic spray F on $U(\boldsymbol{R}P^2(1))$. Then, we have that

$$\mathfrak{i}(U(\boldsymbol{R}P^2(1))) = \operatorname{span}\{X^L,\ F;\ X \in \mathfrak{i}(\boldsymbol{R}P^2(1))\}.$$

Thus, we have described the Killing vector fields on $U(\boldsymbol{R}P^2(1))$.

Finally, we represent the geodesics in the unit tangent bundle $U(\boldsymbol{R}P^2(1))$ by using the Killing vector fields on the base space. Let A be in $\mathfrak{o}(2)$ and ξ be in $\boldsymbol{R}^2$. From the first part of Lemma 2.3, any integral curve of $A^* + B(\xi)$ is a geodesic in $SO(S^2(1))$. Since each tangent space of $SO(S^2(1))$ is spanned by

$$\{A^*_u + B(\xi)_u;\ A \in \mathfrak{o}(2),\ \xi \in \boldsymbol{R}^2\}, \qquad u \in SO(S^2(1)),$$

any geodesic of $SO(S^2(1))$ can be given by an integral curve of a vector field of the form $A^* + B(\xi)$. Therefore, it follows from (i) of Lemma 2.5 that any geodesic in $SO(S^2(1))$ can be given by an integral curve of the natural lift of a Killing vector field on $S^2(1)$, and that the converse is also true. Since $U(\boldsymbol{R}P^2(1))$ is identified with $SO(S^2(1))/\langle\tilde{\varphi}_\pi\rangle$, we have obtained the last statement of Theorem 1.1. □

Acknowledgments. The author is grateful to Professor Katsuei Kenmotsu and Seiki Nishikawa for their valuable advice and encouragement in seminars.

REFERENCES

[1] Klingenberg, W. and Sasaki, S., On the tangent sphere bundle of a 2-sphere, Tohoku Math. J. **27** (1975), 49–56.

[2] Konno, T. and Tanno, S., Geodesics and Killing vector fields on the tangent sphere bundle, Nagoya Math. J. **151** (1998), 91–97.

[3] Konno, T., Decomposition of Killing vector fields on tangent sphere bundles, Tohoku Math. J. **52** (2000), 349–366.

[4] Nagy, P. T., Geodesics on the tangent sphere bundle of a Riemannian manifold, Geom. Dedicata **7** (1978), 233–243.

[5] Thurston, W. P., Three-dimensional geometry and topology, Princeton Math. Ser. 35, Vol. 1, Princeton Univ. Press, Princeton, 1997.

COLLEGE OF ENGINEERING, NIHON UNIVERSITY, GENERAL EDUCATION,
KORIYAMA, FUKUSIMA 963-8642, JAPAN.
E-mail address: konno@ge.ce.nihon-u.ac.jp

Received December 20, 2001 Revised March 27, 2002