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THE STRUCTURE OF NORM-ACHIEVED TOEPLITZ AND HANKEL OPERATORS

Dedicated to Professor Tsuyosi Andô on his 70th birthday

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Abstract. In this paper, we shall study the structure of norm-achieved Toeplitz and Hankel operators and give their applications for the case where they are paranormal operators. And also we shall prove some property of continuous functions on the unit circle.

A bounded measurable function $\varphi \in L^\infty$ on the circle induces the multiplication operator on L^2 called the Laurent operator L_φ given by $L_\varphi f = \varphi f$ for $f \in L^2$. And the Laurent operator induces in a natural way twin operators on H^2 called Toeplitz operator T_φ given by $T_\varphi f = PL_\varphi f$ for $f \in H^2$, where P is the orthogonal projection from L^2 onto H^2 and Hankel operator H_φ given by $H_\varphi f = J(I-P)L_\varphi f$ for $f \in H^2$, where J is the unitary operator on L^2 defined by $J(z^{-n}) = z^{n-1}$, $n = 0, \pm 1, \pm 2, \cdots$.

The following results are well known.

Proposition 1. For $f \in L^2$, let $f^*(z) = \overline{f(\overline{z})}$ where the bar denotes the complex conjugate. Then $\|f^*\|_2 = \|f\|_2$ and $f^* \in L^2$. Particularly, if $f \in H^2$, then $f^* \in H^2$ also. Moreover, for $\varphi \in L^\infty$, $\|\varphi^*\|_\infty = \|\varphi\|_\infty$ and $\varphi^* \in L^\infty$. Particularly, if φ is inner, then φ^* is also inner.

Proposition 2. ([1]) Let $\mathcal{M}$ be an invariant subspace of L_z. Then, in the case where $L_z\mathcal{M} = \mathcal{M}$, there exists a characteristic function χ_E of some measurable subset E of the unit circle such that $\mathcal{M} = L_{\chi_E}L^2$ and, in the case where $L_z\mathcal{M} \subset \mathcal{M}$, there exists a unitary Laurent operator L_g uniquely, except a constant multiple of absolute value one, such that $\mathcal{M} = L_gH^2$.

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Corollary 1. Let $\mathcal{M}^{\perp} = L^2 \ominus \mathcal{M}$ be an invariant subspace of L_z^*. Then, in the case where $L_z^*\mathcal{M}^{\perp} = \mathcal{M}^{\perp}$, there exists a characteristic function χ_F of some measurable subset F of the unit circle such that $\mathcal{M}^{\perp} = L_{\chi_F} L^2$ and, in the case where $L_z^*\mathcal{M}^{\perp} \subset \mathcal{M}^{\perp}$, there exists a unitary Laurent operator L_q uniquely, except a constant multiple of absolute value one, such that $\mathcal{M}^{\perp} = L_q\overline{H^2}$.

Proof. Since $\mathcal{M}$ is invariant under L_z, for the g and E in Proposition 2, let $q = \overline{z}g$ and let $F = \{\mu \in \mathbb{C} : |\mu| = 1\} \setminus E$. Then L_q is a unitary Laurent operator and $I - L_{\chi_E} = L_{\chi_F}$ and, by Proposition 1, we have the conclusion because $L_zH^2 = H_0^2 \overset{\text{def}}{=} \{f \in H^2 : f(0) = 0\}$. □

Corollary 2. ([5]) If φ is non-analytic (i.e., $\varphi \notin H^{\infty}$), then the only invariant subspace of L_{φ} which is contained in H^2 is $\{o\}$ itself.

Proof. Let $\mathcal{M}^{\perp} = \vee\{L_{\varphi}^{*n}f : f \in L^2 \ominus H^2,\ n = 0, 1, 2, \cdots\}$. Then it is the smallest invariant subspace of L_{φ}^* which includes $L^2 \ominus H^2$. Hence we have only to prove $\mathcal{M}^{\perp} = L^2$. Since L_z commutes with L_{φ} and since $L^2 \ominus H^2$ is invariant under L_z^*, $\mathcal{M}^{\perp}$ is invariant under L_z^*. If $\mathcal{M}^{\perp}$ reduces L_z, then $z^{n-1} = L_z^n\overline{z} \in \mathcal{M}^{\perp}$ $(n = 1, 2, \cdots)$ because $\overline{z} \in L^2 \ominus H^2 \subseteq \mathcal{M}^{\perp}$ and hence $\mathcal{M}^{\perp} = L^2$. If $\mathcal{M}^{\perp}$ is a non-reducing invariant subspace of L_z^*, then $L_z^*\mathcal{M}^{\perp} \subset \mathcal{M}^{\perp}$ because L_z is unitary and, by Corollary 1, $\mathcal{M}^{\perp} = L_q\overline{H^2}$ for some unitary Laurent operator L_q and $L_qL_{\varphi}^*\overline{H^2} = L_{\varphi}^*\mathcal{M}^{\perp} \subseteq \mathcal{M}^{\perp} = L_q\overline{H^2}$ and hence $L_{\varphi}^*\overline{H^2} \subseteq \overline{H^2}$. Since $1 \in \overline{H^2}$, $\overline{\varphi} \in \overline{H^2}$ and $\varphi \in H^2 \cap L^{\infty} = H^{\infty}$. This contradicts the hypothesis that φ is non-analytic. □

Corollary 3. If φ is non-co-analytic (i.e., $\overline{\varphi} \notin H^{\infty}$), then the only invariant subspace of L_{φ} which is contained in $L^2 \ominus H_0^2$ is $\{o\}$ itself.

Proof. Let $\mathcal{M} = \vee\{L_{\varphi}^{*n}f : f \in H_0^2,\ n = 0, 1, 2, \cdots\}$. Then it is the smallest invariant subspace of L_{φ}^* which includes H_0^2. Hence we have only to prove $\mathcal{M} = L^2$. Since L_z commutes with L_{φ}^* and since H_0^2 is invariant under L_z, $\mathcal{M}$ is invariant under L_z. If $\mathcal{M}$ reduces L_z, then $\overline{z}^{n-1} = L_z^{*n}z \in \mathcal{M}$ $(n = 1, 2, \cdots)$ because $z \in H_0^2 \subseteq \mathcal{M}$ and hence $\mathcal{M} = L^2$. If $\mathcal{M}$ is a non-reducing invariant subspace of L_z, then $L_z\mathcal{M} \subset \mathcal{M}$ because L_z is unitary and, by Proposition 2, $\mathcal{M} = L_gH^2$ for some unitary Laurent operator L_g and $L_gL_{\varphi}^*H^2 = L_{\varphi}^*\mathcal{M} \subseteq \mathcal{M} = L_gH^2$ and hence

$L_\varphi^* H^2 \subseteq H^2$. Since $1 \in H^2$, $\overline{\varphi} \in H^2$ and $\overline{\varphi} \in H^2 \cap L^\infty = H^\infty$. This contradicts the hypothesis that φ is non-co-analytic. $\square$

Proposition 3. ([5]) If φ is a non-constant function in L^∞, then $\sigma_p(T_\varphi) \cap \overline{\sigma_p(T_\varphi^*)} = \emptyset$ where $\sigma_p(T_\varphi)$ denotes the point spectrum of T_φ.

Proposition 4. ([2]) $T_\varphi T_\psi$ is a Toeplitz operator if and only if $\overline{\varphi}$ or $\psi \in H^\infty$. And, in this case, $T_\varphi T_\psi = T_{\varphi\psi}$.

Proposition 5.

(1) ([2]) $A \in \mathcal{B}(H^2)$ is a Toeplitz operator if and only if $T_z^* A T_z = A$. And, in particular, $A \in \mathcal{B}(H^2)$ is an analytic Toeplitz operator (i.e., $A = T_\varphi$ for some $\varphi \in H^\infty$) if and only if $T_z A = A T_z$.

(2) (Nehari) $A \in \mathcal{B}(H^2)$ is a Hankel operator if and only if $T_z^* A = A T_z$. Moreover we can choose the symbol $\varphi \in L^\infty$ of $A = H_\varphi$ such that $\|A\| = \|\varphi\|_\infty$.

Proposition 6. ([4]) Let q be a non-constant inner function, and let Q be the orthogonal projection from L^2 onto $K = H^2 \ominus T_q H^2$. If $A \in \mathcal{B}(K)$ commutes with $Q L_z Q$, then there is a function ψ in H^∞ such that $\|\psi\|_\infty = \|A\|$ and $A = Q L_\psi Q$.

Proposition 7. T_φ and H_φ have the following properties ;

(1) $T_z^* T_\varphi T_z = T_\varphi, \quad T_z^* H_\varphi = H_\varphi T_z$

(Hence $\mathcal{N}_{H_\varphi} \overset{\text{def}}{=} \{x \in H^2 : H_\varphi x = o\}$ is invariant under T_z

and $\mathcal{N}_{H_\varphi} = \{o\}$ or $\mathcal{N}_{H_\varphi} = T_q H^2$, where q is inner)

(2) $T_\varphi^* = T_{\overline{\varphi}}, \quad H_\varphi^* = H_{\varphi^*}$

(3) $T_{\alpha\varphi+\beta\psi} = \alpha T_\varphi + \beta T_\psi, \quad H_{\alpha\varphi+\beta\psi} = \alpha H_\varphi + \beta H_\psi$ for $\alpha, \beta \in \mathbb{C}$

(4) $T_\varphi = O$ if and only if $\varphi = o$,

$H_\varphi = O$ if and only if $(I - P)\varphi = o$ (i.e., $\varphi \in H^\infty$)

(5) $\|T_\varphi\| = \|L_\varphi\| = \|\varphi\|_\infty, \quad \|H_\varphi\| = \min\{\|\varphi + \psi\|_\infty : \psi \in H^\infty\} = \mathrm{dist}(\varphi, H^\infty)$.

Now we state here the relations between these twin operators.

Proposition 8. (see [6]) $H_\psi^* H_\varphi = T_{\overline{\psi}\varphi} - T_{\overline{\psi}} T_\varphi$ and

$$H_{\overline{\varphi}}^* H_{\overline{\varphi}} - H_\varphi^* H_\varphi = T_\varphi^* T_\varphi - T_\varphi T_\varphi^*.$$

Proposition 9. (see [6]) For any $\psi \in H^\infty$, $H_\varphi T_\psi = H_{\varphi\psi}$ and $T_\psi^* H_\varphi = H_\varphi T_{\psi^*}$.

Proposition 10. (see [6]) The following assertions are equivalent;

(1) $\mathcal{N}_{H_\varphi} \neq \{o\}$

(2) $[H_\varphi H^2]^{\sim L^2} \neq H^2$

(3) $\varphi = \bar{g}h$ for some inner function g and $h \in H^\infty$ such that g and h have no common non-constant inner factor and that $\mathcal{N}_{H_\varphi} = T_g H^2$.

Proposition 11. If φ and ψ are in H^∞, then $T_\varphi H^2 \subseteq T_\psi H^2$ if and only if there exists a $g \in H^\infty$ uniquely, up to a unimodular constant, such that $T_\varphi = T_\psi T_g = T_{\psi g}$. And then $\varphi = \psi g$. Particularly, if φ and ψ are inner, then g is also inner.

Concerning the range inclusion of Hankel operators, we have the following.

Theorem 1. The following assertions are equivalent;

(1) $H_{\varphi_1} H^2 \subseteq H_{\varphi_2} H^2$

(2) $H_{\varphi_1} H_{\varphi_1}^* \leq \lambda^2 H_{\varphi_2} H_{\varphi_2}^*$ for some $\lambda \geq 0$

(3) There exists a function $h \in H^\infty$ such that $\|h\|_\infty \leq \lambda$ for some $\lambda \geq 0$ and that $H_{\varphi_1} = H_{\varphi_2} T_h = H_{\varphi_2 h}$.

To prove this theorem, we need the following lemma. We denote the set of all bounded linear operators on a Hilbert space $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$.

Lemma. ([3]) For $A,\ B \in \mathcal{B}(\mathcal{H})$, the following assertions are equivalent;

(1) $A\mathcal{H} \subseteq B\mathcal{H}$

(2) $AA^* \leq \lambda^2 BB^*$ for some $\lambda \geq 0$

(3) There exists a $C \in \mathcal{B}(\mathcal{H})$ such that $A = BC$.

In particular, there exists a $C \in \mathcal{B}(\mathcal{H})$ uniquely such that

$$(a)\ \ \|C\|^2 = \inf\{\mu : AA^* \leq \mu BB^*\}$$

$$(b)\ \ \mathcal{N}_A = \mathcal{N}_C \quad \text{and} \quad (c)\ \ C\mathcal{H} \subseteq [B^*\mathcal{H}]^\sim.$$

Proof of Theorem 1. By Lemma, we have only to prove (2) implies (3).

If $H_{\varphi_1}H_{\varphi_1}^* \leq \lambda^2 H_{\varphi_2}H_{\varphi_2}^*$ for some $\lambda \geq 0$, then, by Lemma, there exists a $A \in \mathcal{B}(H^2)$ uniquely such that $H_{\varphi_1} = H_{\varphi_2}A$ and that

$$(a) \quad \|A\|^2 = \inf\{\mu : H_{\varphi_1}H_{\varphi_1}^* \leq \mu H_{\varphi_2}H_{\varphi_2}^*\} \leq \lambda^2$$
$$(b) \quad \mathcal{N}_{H_{\varphi_1}} = \mathcal{N}_A \quad \text{and} \quad (c) \quad AH^2 \subseteq [H_{\varphi_2}^*H^2]^{\sim L^2}.$$

And then, by Proposition 7 (1), $\mathcal{N}_{H_{\varphi_2}} = T_qH^2$, where q is a zero function or an inner function and, by Proposition 9, we have, for any $\psi \in H^\infty$,

$$\begin{aligned} A^*T_\psi^*H_{\varphi_2}^* &= A^*H_{\varphi_2}^*T_{\psi^*} = H_{\varphi_1}^*T_{\psi^*} \\ &= T_\psi^*H_{\varphi_1}^* = T_\psi^*A^*H_{\varphi_2}^* \end{aligned}$$

and hence

$$(A^*T_\psi^* - T_\psi^*A^*)[H_{\varphi_2}^*H^2]^{\sim L^2} = \{o\}. \tag{i}$$

Since

$$\langle (T_qA - AT_q)H^2,\ H_{\varphi_2}^*H^2\rangle = \langle H^2,\ (T_qA - AT_q)^*H_{\varphi_2}^*H^2\rangle = 0 \ \text{ by (i)},$$

$(T_qA - AT_q)H^2 \subseteq \mathcal{N}_{H_{\varphi_2}} = T_qH^2$ and $\mathcal{N}_{H_{\varphi_2}}$ is invariant under A and hence $[H_{\varphi_2}^*H^2]^{\sim L^2}$ is invariant under A^*. Since $[H_{\varphi_2}^*H^2]^{\sim L^2}$ is invariant under T_z^* by Proposition 7 (2) and (1) and since

$$(A^*T_z^* - T_z^*A^*)[H_{\varphi_2}^*H^2]^{\sim L^2} = \{o\} \ \text{ by (i)},$$

the restriction $A^*|_{[H_{\varphi_2}^*H^2]^{\sim L^2}}$ commutes with the restriction $T_z^*|_{[H_{\varphi_2}^*H^2]^{\sim L^2}}$ and hence $(A^*|_{[H_{\varphi_2}^*H^2]^{\sim L^2}})^*$ commutes with $QL_zQ = (T_z^*|_{[H_{\varphi_2}^*H^2]^{\sim L^2}})^*$, where Q is the orthogonal projection from L^2 onto $[H_{\varphi_2}^*H^2]^{\sim L^2}$. And, by Proposition 6, there is a function h in H^∞ such that

$$\|h\|_\infty = \|(A^*|_{[H_{\varphi_2}^*H^2]^{\sim L^2}})^*\| = \|A^*|_{[H_{\varphi_2}^*H^2]^{\sim L^2}}\| \leq \|A^*\| = \|A\| \leq \lambda$$

and $(A^*|_{[H_{\varphi_2}^*H^2]^{\sim L^2}})^* = QL_hQ$. And then, for any $f \in H^2$, we have

$$\begin{aligned} H_{\varphi_1}^*f &= A^*H_{\varphi_2}^*f = QL_h^*H_{\varphi_2}^*f \\ &= QT_h^*H_{\varphi_2}^*f = H_{\varphi_2}^*T_{h^*}f = T_h^*H_{\varphi_2}^*f \end{aligned}$$

by Proposition 9 and $H^*_{\varphi_1} = T^*_h H^*_{\varphi_2}$ and hence $H_{\varphi_1} = H_{\varphi_2} T_h = H_{\varphi_2 h}$ by Proposition 9. □

As a special case of Theorem 1, we have the following.

Theorem 2. H_φ is hyponormal (i.e., $H_\varphi H^*_\varphi \leq H^*_\varphi H_\varphi$) if and only if $H_\varphi = H^*_\varphi T_h$ for some $h \in H^\infty$ such that $\|h\|_\infty \leq 1$.

Proof. Since $H^*_\varphi H_\varphi = H_{\varphi^*} H^*_{\varphi^*}$ by Proposition 7 (2), the hyponormality of H_φ is equivalent to the existence of a function $h \in H^\infty$ such that $\|h\|_\infty \leq 1$ and that $H_\varphi = H_{\varphi^*} T_h = H^*_\varphi T_h$ by Theorem 1 and by Proposition 7 (2). □

Corollary 4. Every hyponormal Hankel operator is normal.

Proof. If H_φ is hyponormal, then $H_\varphi = H^*_\varphi T_h$ for some $h \in H^\infty$ such that $\|h\|_\infty \leq 1$ by Theorem 2 and, by Propositions 7 (2) and 9,

$$H_{\varphi^*} = H^*_\varphi = T^*_h H_\varphi = H_\varphi T_{h^*} = H^*_{\varphi^*} T_{h^*}.$$

Since $h^* \in H^\infty$ and $\|h^*\|_\infty = \|h\|_\infty$ by Proposition 1, $H^*_\varphi = H_{\varphi^*}$ is also hyponormal by Theorem 2. Therefore H_φ is normal. □

Remark. It is known that every normal Hankel operator is a scalar (of absolute value one) multiple of a Hermitian Hankel operator.

Theorem 3. T_φ is hyponormal if and only if $H_\varphi = T^*_h H_{\overline{\varphi}}$ for some function $h \in H^\infty$ such that $\|h\|_\infty \leq 1$.

Proof. By Proposition 8, T_φ is hyponormal if and only if $H^*_\varphi H_\varphi \leq H^*_{\overline{\varphi}} H_{\overline{\varphi}}$ and, by Proposition 7 (2), it is equivalent to $H_{\varphi^*} H^*_{\varphi^*} \leq H_{\overline{\varphi}^*} H^*_{\overline{\varphi}^*}$ and hence, by Theorem 1, the hyponormality of T_φ is equivalent to the existence of a function $h \in H^\infty$ such that $\|h\|_\infty \leq 1$ and that $H_{\varphi^*} = H_{\overline{\varphi}^*} T_h$. And, by Proposition 7 (2), the result follows. □

Theorem 4. The following assertions are equivalent;

(1) $\{f \in H^2 : \|T_\varphi f\|_2 = \|T_\varphi\| \|f\|_2\} \neq \{o\}$ (i.e., T_φ is norm-achieved)

(2) $\frac{\varphi}{\|T_\varphi\|} = g$ for some $g \in L^\infty$ such that $|g| = 1$ a.e. and that $0 \in \sigma_p(H_g)$

(3) $\frac{\varphi}{\|T_\varphi\|} = \bar{q}h$ for some inner functions q and h such that q and h have no common non-constant inner factor.
In this case, $\{f \in H^2 : \|T_\varphi f\|_2 = \|T_\varphi\|\|f\|_2\} = \mathcal{N}_{H_g}$.

Proof. (1) $\to$ (2) ; If $\|T_\varphi f\|_2 = \|T_\varphi\|\|f\|_2$ for some non-zero $f \in H^2$, then we have, for $g = \frac{\varphi}{\|T_\varphi\|}$,

$$\|f\|_2 = \|T_{\frac{\varphi}{\|T_\varphi\|}} f\|_2 = \|T_g f\|_2 = \|PL_g f\|_2 \leq \|L_g f\|_2 \leq \|f\|_2$$

because $\|L_g\| = \|T_g\| = \frac{\|T_\varphi\|}{\|T_\varphi\|} = 1$. Hence $T_g^* T_g f = f$ and $PL_g f = L_g f$ and hence $H_g f = J(I-P)L_g f = o$ (i.e., $0 \in \sigma_p(H_g)$). Since, by Proposition 8, $H_g^* H_g = T_{|g|^2} - T_{\bar{g}} T_g$, we have $T_{|g|^2} f = f$ (i.e., $1 \in \sigma_p(T_{|g|^2})$) and, by Proposition 3, $|g|^2$ is constant and $|g| = 1$ a.e.

(2) $\to$ (1) ; Since $\|T_g\| = \frac{\|T_\varphi\|}{\|T_\varphi\|} = 1$ and since, by Proposition 8, $H_g^* H_g = I - T_{\bar{g}} T_g$, we have $T_g^* T_g f = f$ for all $f \in \mathcal{N}_{H_g}$ and hence $\|T_g f\|_2 = \|f\|_2$. Therefore $\|T_\varphi f\|_2 = \|T_\varphi\|\|T_g f\|_2 = \|T_\varphi\|\|f\|_2$.

The assertion $\{f \in H^2 : \|T_\varphi f\|_2 = \|T_\varphi\|\|f\|_2\} = \mathcal{N}_{H_g}$ is clear. In fact, (1) implies that $\{f \in H^2 : \|T_\varphi f\|_2 = \|T_\varphi\|\|f\|_2\} \subseteq \mathcal{N}_{H_g}$ and (2) implies the converse inclusion.

The equivalence between (2) and (3) follows from Propositions 7, 9 and 10.□

In the case of Hankel operators, we have the following.

Theorem 5. The following assertions are equivalent;

(1) $\{f \in H^2 : \|H_\varphi f\|_2 = \|H_\varphi\|\|f\|_2\} \neq \{o\}$ (i.e., H_φ is norm-achieved)

(2) $\frac{\varphi}{\|H_\varphi\|} = g + \psi$ for some $\psi \in H^\infty$ and $g \in L^\infty$ such that $|g| = 1$ a.e. and that $0 \in \sigma_p(T_g)$.
In this case, $\{f \in H^2 : \|H_\varphi f\|_2 = \|H_\varphi\|\|f\|_2\} = \mathcal{N}_{T_g}$.

Proof. (1) $\to$ (2) ; By Proposition 5, there exists a $g \in L^\infty$ such that $H_{\frac{\varphi}{\|H_\varphi\|}} = H_g$ and $\|H_g\| = \|g\|_\infty$. And then $H_{\frac{\varphi}{\|H_\varphi\|} - g} = O$ and $\psi = \frac{\varphi}{\|H_\varphi\|} - g \in H^\infty$ by Proposition 7. If $\|H_\varphi f\|_2 = \|H_\varphi\|\|f\|_2$ for some non-zero $f \in H^2$, then we have $\|f\|_2 = \|H_{\frac{\varphi}{\|H_\varphi\|}} f\|_2 = \|H_g f\|_2 = \|(I-P)L_g f\|_2 \leq \|L_g f\|_2 \leq \|f\|_2$ because $\|L_g\| = \|g\|_\infty = \|H_g\| = \|H_{\frac{\varphi}{\|H_\varphi\|}}\| = \frac{\|H_\varphi\|}{\|H_\varphi\|} = 1$. Hence $H_g^* H_g f = f$ and $(I-P)L_g f = L_g f$ and hence $T_g f = PL_g f = o$ (i.e., $0 \in \sigma_p(T_g)$). Since, by

Proposition 8, $H_g^*H_g = T_{|g|^2} - T_{\overline{g}}T_g$, we have $T_{|g|^2}f = f$ (i.e., $1 \in \sigma_p(T_{|g|^2})$) and, by Proposition 3, $|g|^2$ is constant and $|g| = 1$ a.e.

(2) → (1) ; By Proposition 7, $\|H_g\| = \|H_{\frac{\varphi}{\|H_\varphi\|}}\| = \frac{\|H_\varphi\|}{\|H_\varphi\|} = 1$. Since, by Proposition 8, $H_g^*H_g = I - T_{\overline{g}}T_g$, we have $H_g^*H_gf = f$ for all $f \in \mathcal{N}_{T_g}$ and hence $\|H_gf\|_2 = \|f\|_2$. Therefore, by Proposition 7, $\|H_\varphi f\|_2 = \|H_{\|H_\varphi\|g}f\|_2 = \|H_\varphi\|\|H_gf\|_2 = \|H_\varphi\|\|f\|_2$.

The last assertion of the theorem is clear. In fact, (1) implies that $\{f \in H^2 : \|H_\varphi f\|_2 = \|H_\varphi\|\|f\|_2\} \subseteq \mathcal{N}_{T_g}$ and (2) implies the converse inclusion.□

Theorem 6. ([5]) For a T_φ such that $\|T_\varphi\| = 1$, if

$$\{f \in H^2 : \|T_\varphi^n f\|_2 = \|f\|_2,\ n = 0, 1, 2, \cdots\} \neq \{o\},$$

then T_φ is an isometry.

Proof. For a non-zero $f \in \{f \in H^2 : \|T_\varphi^n f\|_2 = \|f\|_2,\ n = 0, 1, 2, \cdots\}$, we have $\|f\|_2 = \|T_\varphi f\|_2 = \|PL_\varphi f\|_2 \leq \|L_\varphi f\|_2 \leq \|f\|_2$ because $\|L_\varphi\| = \|T_\varphi\| = 1$ by Proposition 7. This implies that $T_\varphi f = PL_\varphi f = L_\varphi f$ and

$$\|f\|_2 = \|T_\varphi^2 f\|_2 = \|T_\varphi L_\varphi f\|_2 = \|PL_\varphi^2 f\|_2 \leq \|L_\varphi^2 f\|_2 \leq \|f\|_2$$

and hence $T_\varphi^2 f = PL_\varphi^2 f = L_\varphi^2 f$. Similarly, we have $T_\varphi^n f = PL_\varphi^n f = L_\varphi^n f$ for all $n \geq 0$.

Let $\mathcal{N} = \vee\{L_\varphi^n f\ :\ n = 0, 1, 2, \cdots\}$. Then $\mathcal{N} \neq \{o\}$ is an invariant subspace of L_φ contained in H^2 and, by Corollary 2, φ is analytic, i.e., $\varphi \in H^\infty$. Since, by Theorem 4, $\varphi = \overline{q}h$ for some inner functions q and h such that q and h have no common non-constant inner factor, $h = q\varphi$ and $q = e^{i\theta_0}1$ for some $\theta_0 \in [0, 2\pi)$ and hence $\varphi = e^{-i\theta_0}h$ is inner. Therefore T_φ is an isometry. □

Theorem 7. For a H_φ such as $\|H_\varphi\| = 1$, if

$$\{f \in H^2 : \|H_\varphi^n f\|_2 = \|f\|_2,\ n = 0, 1, 2, \cdots\} \neq \{o\},$$

then H_φ is normal.

Proof. Since, by Theorem 5, $\varphi = g + \psi$ for some $\psi \in H^\infty$ and $g \in L^\infty$ such that $|g| = 1$ a.e. and that $0 \in \sigma_p(T_g)$. And hence $H_\varphi = H_g$ by Proposition 7.

For a non-zero $f \in \{f \in H^2 : \|H_\varphi^n f\|_2 = \|f\|_2,\ n = 0,1,2,\cdots\}$, we have $\|f\|_2 = \|H_g f\|_2 = \|J(I-P)L_g f\|_2 = \|(I-P)L_g f\|_2 \le \|L_g f\|_2 \le \|f\|_2$ because $\|L_g\| = \|g\|_\infty = 1$. This implies that $(I-P)L_g f = L_g f$ and $H_g f = JL_g f$. Since $\|f\|_2 = \|H_g^2 f\|_2 = \|(I-P)L_g JL_g f\|_2 \le \|L_g JL_g f\|_2 \le \|f\|_2$, we have $(I-P)L_g JL_g f = L_g JL_g f = JL_{\overline{g}^*}L_g f$ and $H_g^2 f = L_{\overline{g}^*}L_g f$. Analogously, since $\|f\|_2 = \|H_g^3 f\|_2 = \|(I-P)L_g L_{\overline{g}^*}L_g f\|_2 \le \|L_g L_{\overline{g}^*}L_g f\|_2 \le \|f\|_2$, we have $(I-P)L_g L_{\overline{g}^*}L_g f = L_g L_{\overline{g}^*}L_g f$ and $H_g^3 f = JL_g L_{\overline{g}^*}L_g f$. Similarly, we have $(I-P)(L_g L_{\overline{g}^*})^n L_g f = (L_g L_{\overline{g}^*})^n L_g f$ and $(I-P)J(L_{\overline{g}^*}L_g)^n f = J(L_{\overline{g}^*}L_g)^n f$ for all $n \ge 0$.

Let $\mathcal{N} = \vee\{L_{g\overline{g}^*}^n L_g f : n = 0,1,2,\cdots\}$. Then $\mathcal{N} \ne \{o\}$ is an invariant subspace of $L_{g\overline{g}^*}$ contained in $L^2 \ominus H^2$ and, by Corollary 3, $g\overline{g}^*$ is co-analytic, i.e., $\overline{g}g^* \in H^\infty$ and hence $u = \overline{g}g^*$ is inner because $|\overline{g}g^*| = |g|\ |g^*| = 1$ a.e. Since $u^*u = g\overline{g}^*\overline{g}g^* = 1$, $\overline{u} = u^* \in H^\infty$ and hence u is a constant of absolute value one because $u \in H^\infty \cap \overline{H^\infty} = \{\lambda 1 : \lambda \in \mathbb{C}\}$. Therefore $g^* = e^{i\theta_0}g$ for some $\theta_0 \in [0, 2\pi)$ and we have the conclusion. □

We say that a bounded linear operator A on a Hilbert space $\mathcal{H}$ is paranormal if $\|Ax\|^2 \le \|A^2x\|\ \|x\|$ for all $x \in \mathcal{H}$. It is known that every hyponormal operator is paranormal.

Theorem 8. If T_φ is norm-achieved paranormal, then T_φ is a scalar multiple of an isometry.

Proof. We may assume that $\|T_\varphi\| = 1$.

Let $\mathcal{M} = \{f \in H^2 : \|T_\varphi f\|_2 = \|f\|_2\}$. Then , by the hypothesis, $\mathcal{M} \ne \{o\}$ and, by the paranormality of T_φ, $T_\varphi\mathcal{M} \subseteq \mathcal{M}$. In fact, if $f \in \mathcal{M}$, then we have

$$\|f\|_2^2 \ge \|f\|_2\ \|T_\varphi^2 f\|_2 \ge \|T_\varphi f\|_2^2 = \|f\|_2^2$$

and $\|T_\varphi^2 f\|_2 = \|f\|_2 = \|T_\varphi f\|_2$ and hence $T_\varphi f \in \mathcal{M}$. Therefore

$$\{f \in H^2 : \|T_\varphi^n f\|_2 = \|f\|_2,\ n = 0,1,2,\cdots\} \ne \{o\}$$

and T_φ is an isometry by Theorem 6. □

In the case of Hankel operators, we have the following.

Theorem 9. If H_φ is norm-achieved paranormal, then H_φ is normal.

Proof. We may assume that $\|H_\varphi\| = 1$.

Let $\mathcal{M} = \{f \in H^2 : \|H_\varphi f\|_2 = \|f\|_2\}$. Then, by the same reason as in the proof of Theorem 8, $H_\varphi \mathcal{M} \subseteq \mathcal{M} \neq \{o\}$ and

$$\{f \in H^2 : \|H_\varphi^n f\|_2 = \|f\|_2,\ n = 0, 1, 2, \cdots\} \neq \{o\}$$

and hence H_φ is normal by Theorem 7. □

Since, for any $f \in H^2$, $\|H_{\overline{\varphi}} f\|_2^2 = \|H_\varphi f\|_2^2 + \langle (H_{\overline{\varphi}}^* H_{\overline{\varphi}} - H_\varphi^* H_\varphi) f,\ f\rangle$, any two intersection of the following three sets $\mathcal{N}_{H_\varphi}$, $\mathcal{N}_{H_{\overline{\varphi}}}$ and $\mathcal{N}_{H_{\overline{\varphi}}^* H_{\overline{\varphi}} - H_\varphi^* H_\varphi}$ is contained in the rest set. And if T_φ is hyponormal, then, by Proposition 8, $\langle (H_{\overline{\varphi}}^* H_{\overline{\varphi}} - H_\varphi^* H_\varphi) f,\ f\rangle = \|(T_\varphi^* T_\varphi - T_\varphi T_\varphi^*)^{\frac{1}{2}} f\|_2^2$ and we have easily that

$$\mathcal{N}_{H_{\overline{\varphi}}} = \mathcal{N}_{H_\varphi} \cap \mathcal{N}_{H_{\overline{\varphi}}^* H_{\overline{\varphi}} - H_\varphi^* H_\varphi} = \mathcal{N}_{H_\varphi} \cap \mathcal{N}_{T_\varphi^* T_\varphi - T_\varphi T_\varphi^*}.$$

Theorem 10. If T_φ is hyponormal and if $\mathcal{N}_{H_{\overline{\varphi}}} \subsetneq \mathcal{N}_{T_\varphi^* T_\varphi - T_\varphi T_\varphi^*}$, then, for some inner function q, $T_q^* T_\varphi$ is normal or equal to T_φ^*.

Proof. Since, by Theorem 3 and by Proposition 9,

$$H_\varphi = T_h^* H_{\overline{\varphi}} = H_{\overline{\varphi} h^*} \text{ for some function } h \in H^\infty \text{ such that } \|h\|_\infty \leq 1, \tag{1}$$

we have, by Proposition 7,

$$\varphi = \overline{\varphi} h^* + u \text{ for some } u \in H^\infty \tag{2}$$

and $T_\varphi^* T_\varphi - T_\varphi T_\varphi^* = H_{\overline{\varphi}}^* H_{\overline{\varphi}} - H_\varphi^* H_\varphi = H_{\overline{\varphi}}^* (I - T_h T_h^*) H_{\overline{\varphi}}$ by Proposition 8 and hence

$$(I - T_h T_h^*) H_{\overline{\varphi}} \mathcal{N}_{T_\varphi^* T_\varphi - T_\varphi T_\varphi^*} = \{o\} \tag{3}$$

because $\|T_h^*\| = \|T_h\| = \|h\|_\infty \leq 1$. Since $H_{\overline{\varphi}} \mathcal{N}_{T_\varphi^* T_\varphi - T_\varphi T_\varphi^*} \neq \{o\}$ by the assumption, there exists, by (3), a non-zero vector $y \in H_{\overline{\varphi}} \mathcal{N}_{T_\varphi^* T_\varphi - T_\varphi T_\varphi^*}$ such that $T_h T_h^* y = y$ and we have $\|T_h^* y\|_2 = \|y\|_2$ and hence T_h^* is norm-achieved. Thus, by Theorem 4, $|\overline{h}| = 1$ a.e. and h is inner because $h \in H^\infty$ by (1). Therefore h^* is also inner by Proposition 1.

And then $\varphi = (\varphi \overline{h}^* + \overline{u})h^* + u = \varphi + \overline{u}h^* + u$ by (2) and

$$\overline{u}h^* + u = 0. \tag{4}$$

Since $H_{\overline{u}} T_{h^*} = H_{\overline{u}h^*} = H_{-u} = O$ by Propositions 7 and 9, $\{o\} \neq T_{h^*}H^2 \subseteq \mathcal{N}_{H_{\overline{u}}}$ and, by Proposition 10,

$\overline{u} = \overline{q}k$ for some inner function q and $k \in H^\infty$
such that q and k have no common non-constant inner factor
and that $\mathcal{N}_{H_{\overline{u}}} = T_q H^2$. (5)

And since $u = \overline{k}q$ by (5), $H_{\overline{k}} T_q = H_{\overline{k}q} = H_u = O$ and

$$\{o\} \neq T_q H^2 \subseteq \mathcal{N}_{H_{\overline{k}}} \tag{6}$$

and hence, by Proposition 10,

$\overline{k} = \overline{q'}k'$ for some inner function q' and $k' \in H^\infty$
such that q' and k' have no common non-constant inner factor
and that $\mathcal{N}_{H_{\overline{k}}} = T_{q'} H^2$. (7)

Thus $T_q H^2 \subseteq T_{q'} H^2$ by (6) and (7) and, by Proposition 11, there exists an inner function g uniquely, up to a unimodular constant, such that

$$q = q'g. \tag{8}$$

Since q and k have no common non-constant inner factor by (5), q' is constant and k is also constant because $\overline{k} \in H^\infty$ by (7) and hence, by (5),

$$u = \lambda q \quad \text{for some} \quad \lambda \in \mathbb{C}. \tag{9}$$

Then, by (4), we have $\overline{\lambda}\overline{q}h^* + \lambda q = o$ and

$$\overline{\lambda} h^* + \lambda q^2 = o. \tag{10}$$

If $\lambda = 0$, then $u = o$ by (9) and $\varphi = \overline{\varphi} h^*$ by (2) and hence $T_{h^*}^* T_\varphi = T_\varphi^*$.
If $\lambda \neq 0$, then $h^* = -\frac{\lambda}{\overline{\lambda}} q^2$ by (10) and, by (2) and (9),

$$\varphi \overline{q} = \left\{ \overline{\varphi} \left(-\frac{\lambda}{\overline{\lambda}} q^2 \right) + \lambda q \right\} \overline{q} = -\frac{\lambda}{\overline{\lambda}} \overline{\varphi} q + \lambda 1$$

and hence $\psi = \varphi \overline{q} = -\frac{\lambda}{\overline{\lambda}} \overline{\psi} + \lambda 1$. Therefore $T_\psi = T_q^* T_\varphi = -\frac{\lambda}{\overline{\lambda}} T_\psi^* + \lambda I$ is normal because T_ψ commutes with T_ψ^*. □

Let $\mathcal{A}$ be the uniform closure of polynomials in z and let $\mathcal{C}$ be the set of continuous complex-valued functions on $\{z \in \mathbb{C} : |z| = 1\}$. Then $\mathcal{C}$ is the uniform closure of polynomials in z and $\overline{z}$ by the Stone-Weierstrass theorem.

The following results are well known.

Proposition 12. If $\varphi \in \mathcal{C}$, then $\mathrm{dist}(\varphi,\ H^\infty) = \mathrm{dist}(\varphi,\ \mathcal{A})$.

Proposition 13. **(Hartman)** H_φ is compact if and only if $\varphi \in \mathcal{C} + H^\infty$.

By the analogous method as in the proof of Theorem 5, we have the following.

Theorem 11. If $\varphi \in \mathcal{C} + H^\infty$, then there exists a $u \in H^\infty$ uniquely such that $|\varphi - u| = \mathrm{dist}(\varphi,\ H^\infty)$ a.e.

Proof. Since $\|H_\varphi\| = \min\{\|\varphi - \psi\|_\infty : \psi \in H^\infty\} = \mathrm{dist}(\varphi,\ H^\infty)$ by Proposition 7, there exists a $u_1 \in H^\infty$ such that $\|H_\varphi\| = \|\varphi - u_1\|_\infty$.

Firstly, we shall prove the uniqueness of the existence of such a u_1. If there exists an another $u_2 \in H^\infty$ such that $\|H_\varphi\| = \|\varphi - u_2\|_\infty$, then

$$H_{\varphi - u_1} = H_{\varphi - u_2} = H_\varphi$$

by Proposition 7 and it is compact by Proposition 13 and norm-achieved and hence there exists a non-zero vector $f \in H^2$ such that, for each $j = 1,\ 2$,

$$\begin{aligned}\|H_\varphi\|\ \|f\|_2 = \|H_\varphi f\|_2 = \|H_{\varphi - u_j} f\|_2 &= \|(I - P)L_{\varphi - u_j} f\|_2 \\ \leq \|L_{\varphi - u_j} f\|_2 \leq \|L_{\varphi - u_j}\|\ \|f\|_2 &= \|H_\varphi\|\ \|f\|_2\end{aligned}$$

because $\|L_{\varphi - u_j}\| = \|\varphi - u_j\|_\infty = \|H_\varphi\|$. Then

$$H^*_{\varphi - u_j} H_{\varphi - u_j} f = \|H_\varphi\|^2 f \quad \text{and} \quad (I - P)L_{\varphi - u_j} f = L_{\varphi - u_j} f$$

and hence $T_{\varphi - u_j} f = o$, i.e., $0 \in \sigma_p(T_{\varphi - u_j})$. Since

$$T_{u_1 - u_2} f = T_{\varphi - u_2} f - T_{\varphi - u_1} f = o$$

and since $u_1 - u_2 \in H^\infty$, $T^*_{u_1 - u_2} f = o$ and $u_1 - u_2$ is constant by Proposition 3 and hence $u_1 - u_2 = o$.

Since $H^*_{\varphi-u_1}H_{\varphi-u_1} = T_{|\varphi-u_1|^2} - T^*_{\varphi-u_1}T_{\varphi-u_1}$ by Proposition 8,

$$\|H_\varphi\|^2 f = T_{|\varphi-u_1|^2} f, \quad \text{i.e.,} \ \|H_\varphi\|^2 \in \sigma_p(T_{|\varphi-u_1|^2})$$

and, by Proposition 3, $|\varphi - u_1|^2$ is constant and hence

$$|\varphi - u_1| = \|H_\varphi\| = \text{dist}(\varphi, \ H^\infty) \text{ a.e.}$$ □

Corollary 5. For every $\varphi \in \mathcal{C}$, there exists a $u \in H^\infty$ such that $|\varphi - u|$ is non-zero constant.

Proof. In the case where $\varphi \in \mathcal{C} \setminus \mathcal{A}$, the existence of such a u follows from Theorem 11 and Proposition 12. And, in the other case, for example, we may take $u = \varphi - z$. □

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