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A NOTE ON UNIQUENESS IN AN INVERSE PROBLEM FOR A SEMILINEAR PARABOLIC EQUATION

SHIN-ICHI NAKAMURA

ABSTRACT. Consider the mixed problem for a semilinear parabolic equation $u_t - \Delta u + a(u) = 0$. Isakov proved the uniqueness result of the function a by prescribing any initial and lateral Dirichlet data and measuring lateral Neumann data and final data under the condition $a(0) = 0$. In this note we shall study the case $a(0) \neq 0$.

1. Introduction. Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with a C^2-boundary $\partial\Omega$ and set $Q_T \equiv \Omega \times (0,T)$ in $\mathbb{R}^{n+1}$. Let H be the subspace of function g on $\partial Q_T \backslash \{t = T\}$ which belongs to $C^{2,1}(\partial\Omega \times [0,T]) \cap C^1(\bar{\Omega} \times \{0\})$ and which have $C^{\lambda,\lambda/2}(\bar{Q}_T)$ extensions. We now consider the mixed problem:

$$u_t - \Delta u + a(u) = 0 \quad \text{in } Q_T, \tag{1.1}$$

$$u = g \in H \qquad \text{on } \partial Q_T \backslash \{t = T\}, \tag{1.2}$$

where $a(s) \in C^2(\mathbb{R})$ satisfies the conditions:

$$a(s) \text{ and } a_{ss}(s) \text{ are bounded on } \mathbb{R}, \tag{1.3a}$$

$$0 \leq a_s \leq M, \tag{1.3b}$$

where M is a positive constant.

Under the condition (1.3b), there is a unique solution $u \in H^{2,1}(Q_T) \cap C(\bar{Q}_T)$ to the problem (1.1)-(1.2)(Theorem 6.1 in [3, p. 452] and [2]). (The norms and the properties of the function spaces can be found in [2] or [3].) So we may define

$$h = u \quad \text{on } \Omega \times \{T\}, \; h = \partial_\nu u \quad \text{on } \partial\Omega \times (0,T),$$

here ν denotes the unit exterior normal to $\partial\Omega$. We are interested in uniqueness results of the function a from the map:

$$\Lambda(a) : g \longmapsto h.$$

Let $\Lambda_j = \Lambda(a^j)$ $(j = 1, 2)$. The following theorem can be derived from Theorem 1 in [2].

Theorem I. *Assume that, for* $a = a^j$ $(j = 1, 2)$,

$$a^j(0) = 0. \tag{1.4}$$

If $\Lambda_1 = \Lambda_2$ *on* H, *then* $a^1 = a^2$.

In this note we shall study the assumption (1.4). Define u^j as a solution to the problem (1.1)-(1.2) with $a = a^j$ $(j = 1, 2)$. The following lemma will be proved in section 2 by modyfying the methods for a semilinear elliptic equation in [4]:

Lemma. *If* $\Lambda_1 = \Lambda_2$ *on* H, *then*

$$a^1(0) = a^2(0). \tag{1.5}$$

Combining this lemma with the above theorem I, we can remove the assumption (1.4) to derive the following theorem:

Theorem. *If* $\Lambda_1 = \Lambda_2$ *on* H, *then* $a^1 = a^2$.

2. Proof of Lemma. Denote by $Q_\tau \equiv \Omega \times (0, \tau)$ for any τ $(0 < \tau \leq T)$. It is easily seen that for any $\phi \in H^{2,1}(Q_\tau)$ we have

$$\begin{aligned} 0 &= \int_{Q_\tau} (u_t - \Delta u + a(u))\phi \, dxdt \\ &= \int_\Omega [u\phi]_0^\tau \, dx - \int_{\partial\Omega \times (0,\tau)} (\phi \partial_\nu u - u \partial_\nu \phi) \, dSdt - \int_{Q_\tau} u(\phi_t + \Delta\phi) \, dxdt \\ &\quad + \int_{Q_\tau} a(u)\phi \, dxdt. \end{aligned}$$

This implies

$$\begin{aligned} \int_{Q_\tau} a^j(u^j)\phi \, dxdt = -\int_\Omega [u^j\phi]_0^\tau \, dx &+ \int_{\partial\Omega \times (0,\tau)} (\phi \partial_\nu u^j - u^j \partial_\nu \phi) \, dSdt \\ &+ \int_{Q_\tau} u^j(\phi_t + \Delta\phi) \, dxdt, \end{aligned} \tag{2.1}$$

where u^j is a solution to the problem (1.1)-(1.2) with $a = a^j$ $(j = 1, 2)$. By using (2.1), if $\Lambda_1 = \Lambda_2$ and $\phi(x, \tau) = 0$ then we obtain

$$\begin{aligned} &\int_{Q_\tau} (a^1(u^1) - a^2(u^1)) \, \phi \, dxdt \\ &= \int_{Q_\tau} (a^1(u^1) - a^2(u^2)) \, \phi \, dxdt + \int_{Q_\tau} (a^2(u^2) - a^2(u^1)) \, \phi \, dxdt \\ &= -\int_{Q_\tau} [(u^1 - u^2)\phi]_0^\tau \, dx + \int_{\partial\Omega \times (0,\tau)} (\partial_\nu (u^1 - u^2)\phi - (u^1 - u^2)\partial_\nu \phi) \, dSdt \\ &\quad + \int_{Q_\tau} (u^1 - u^2)(\phi_t + \Delta\phi) \, dxdt + \int_{Q_\tau} (a^2(u^2) - a^2(u^1)) \, \phi \, dxdt \\ &= \int_{Q_\tau} \{(u^1 - u^2)(\phi_t + \Delta\phi) - (a^2(u^1) - a^2(u^2))\phi\} \, dxdt \\ &= \int_{Q_\tau} (u^1 - u^2)(\phi_t + \Delta\phi - p(x, t)\phi) \, dxdt, \end{aligned} \tag{2.2}$$

here we have set

$$p(x,t) = \int_0^1 a_s^2(u^2 + \theta(u^1 - u^2))\, d\theta.$$

Let us consider the following mixed problem to derive (1.5) from (2.2).

$$\psi_t + \Delta\psi - p(x,t)\psi = 0 \quad \text{in } Q_\tau, \tag{2.3}$$

$$\psi(x,\tau) = 0 \quad \text{on } \Omega, \tag{2.4}$$

$$\psi(x,t) = h(x,t) \quad \text{on } \partial\Omega \times (0,\tau), \tag{2.5}$$

where $h(x,t) \in C^2(\partial\Omega \times [0,\tau])$ satisfies the condition $h(x,\tau) = 0$. From the assumptions (1.3a) and (1.3b), we see that $p(x,t) \geq 0$ is Lipschitz with respect to x and t. Hence there exists a unique solution $\psi \in H^{2,1}(Q_\tau)$ to the problem (2.3)-(2.5) (Theorem 9.1 in [3], p.341).

Substituting $\phi = \psi$ into (2.2), we obtain

$$I_\tau \equiv \int_{Q_\tau} (a^1(u^1) - a^2(u^1))\, \psi \, dxdt = 0. \tag{2.6}$$

If $a^1(0) \neq a^2(0)$, then then there exist ϵ_0, $\epsilon_1 > 0$ such that $a^1(s) - a^2(s) > \epsilon_0$ or $a^2(s) - a^1(s) > \epsilon_0$ for $|s| \leq \epsilon_1$. We can choose $h(x,t)$ so that $\psi > 0$ in Q_τ by the maximum principle. From (1.3a) and Lemma 1.1 in [1], we can easily seen that

$$\max_{Q_\tau} |u^1| \leq \max_{Q_\tau} |v| + C\tau, \tag{2.7}$$

where C is a positive constant and v is a solution to the problem:

$$v_t - \Delta v = 0 \quad \text{in } Q_\tau,$$
$$v = g \in H \quad \text{on } \partial Q_\tau \backslash \{t = \tau\}.$$

By (2.7) and the maximum principle, we will be able to take g and τ such that $|u^1| \leq \epsilon_1$ on Q_τ. Hence we have $I_\tau > 0$. This contradicts (2.6). Thus we may conclude that $a^1(0) = a^2(0)$. The proof is completed.

2. Proof of theorem. In the proof of Theorem I stated in Introduction, it was proved that $a_s^1(s) = a_s^2(s)$ if $\Lambda_1 = \Lambda_2$ on H ((1.13) in [2]). By integrating this equality from 0 to s and using $a^1(0) = a^2(0)$, we obtain $a^1 = a^2$.

References

1. P. DuChateau and W. Rundell, *Unicity in an inverse problem for an unknown reaction term in a reaction-diffusion equation*, Journal of Differential Equations **59** (1985), 155–164.
2. V. Isakov, *On uniqueness in inverse problems for semilinear parabolic equations*, Arch. Rat. Mech. Anal. **124** (1993), 1–12.
3. O. A. Ladyzhenskaja, V. A. Solonikov, and N. N. Uralceva, *Linear and Quasilinear Equations of Parabolic Type*, A. M. S., 1968.
4. S. Nakamura, *An inverse problem for a semilinear elliptic equation*, Comm. in P. D. E. **24** (1999), 743–748.

Department of Mathematics, TOUA University. 2-1 Ichinomiya Shimonoseki Yamaguchi, 751-0807, JAPAN

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