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ESTIMATING COMMON FIXED POINTS OF TWO NONEXPANSIVE MAPPINGS BY STRONG CONVERGENCE

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ABSTRACT. In this paper, we introduce an iteration scheme defined by

$$x_0 = x \in C,\ x_{n+1} = \alpha_n x + (1-\alpha_n)U_n x_n,$$
$$U_n = \gamma_n T(\beta_n S + (1-\beta_n)I) + (1-\gamma_n)I,\ n = 0,1,2,\ldots,$$

where S and T are nonexpansive mappings from a closed convex subset of a Banach space into itself and $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are in [0,1]. This scheme contains all four schemes given by Mann, Ishikawa, Das–Debata and Halpern. Using this scheme we approximate common fixed points of S and T.

1. INTRODUCTION

Let C be a closed convex subset of a Banach space E. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A number of iteration schemes have been introduced to approximate the fixed points of nonexpansive mappings. Mann [6] introduced the following iteration scheme :

$$(1) \qquad x_1 \in C,\ x_{n+1} = \alpha_n T x_n + (1-\alpha_n)x_n,$$

for all $n = 1, 2, \ldots$, where $\{\alpha_n\}$ is in $[0, 1]$. Ishikawa [4] gave the following iteration scheme:

$$(2) \qquad x_1 \in C,\ x_{n+1} = \alpha_n T(\beta_n T x_n + (1-\beta_n)x_n) + (1-\alpha_n)x_n,$$

for all $n = 1, 2, \ldots$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $[0, 1]$. Das and Debata [2] defined $\{x_n\}$ using two mappings S and T as follows:

$$(3) \qquad x_1 \in C,\ x_{n+1} = \alpha_n S(\beta_n T x_n + (1-\beta_n)x_n) + (1-\alpha_n)x_n$$

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for $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$. Halpern's scheme is as under.

$$x_1 \in C,\ x_{n+1} = \alpha_n x + (1-\alpha_n)Tx_n, \tag{4}$$

for all $n = 1, 2, \ldots$, where $\{\alpha_n\}$ is in $[0,1]$. Many authors including Reich [7], Wittmann [12], Tan and Xu [11], Takahashi and Tamura [10] and Shioji and Takahashi [8] have considered the convergence of these iteration schemes for approximating the fixed points of nonexpansive mappings. We introduce a new iteration scheme defined by:

$$\begin{cases} x_0 = x \in C, \\ x_{n+1} = \alpha_n x + (1-\alpha_n)U_n x_n, \\ U_n = \gamma_n T(\beta_n S + (1-\beta_n)I) + (1-\gamma_n)I, \end{cases} \tag{5}$$

for all $n = 0, 1, 2, \ldots$, with $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ in $[0,1]$.

This scheme contains all the four schemes given above: for $\alpha_n = 0$, our scheme reduces to Das–Debata type (3), for $\alpha_n = 0$, $S = T$ to Ishikawa-type (2), for $\alpha_n = 0$, $S = I$ to Mann-type (1) and finally for $\gamma_n = 1$, $S = I$ to Halpern-type (4). We shall use this scheme to prove a strong convergence theorem to approximate the common fixed points of the two nonexpansive mappings S and T.

2. Preliminaries And Notation

We shall use the following notations in this paper.

$$\mathbb{N} = \{1, 2, 3, \ldots\},$$
$$\mathbb{N}_+ = \{0, 1, 2, \ldots\}.$$

$\mathbb{R}$ will stand for set of real numbers. We only consider the real vector spaces. For a Banach space E, let E' be its topological dual. The value of $y \in E'$ at $x \in E$ will be denoted by $\langle x, y\rangle$. We reserve J for the duality mapping of E into $2^{E'}$, defined as :

$$Jx = \left\{y \in E' : \langle x, y\rangle = \|x\|^2 = \|y\|^2\right\}, \quad x \in E.$$

Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{6}$$

exists uniformly for $x \in U$. E is termed as uniformly smooth if the limit in (6) exists uniformly for $x, y \in U$. The fact that if the norm of E is uniformly Gâteaux differentiable then the duality mapping is single-valued and norm to weak* uniformly continuous on each bounded subset of E is quite well-known. See, for instance, [1].

Suppose that μ is a continuous linear functional defined on l^∞ and $(a_0, a_1, \ldots) \in l^\infty$. We write $\mu_n(a_n)$ instead of $\mu((a_0, a_1, \ldots))$. If μ satisfies $\|\mu\| = \mu_n(1) = 1$ and $\mu_n(a_{n+1}) = \mu_n(a_n)$ for all $(a_0, a_1, \ldots) \in l^\infty$ then μ is called a Banach limit. Finally, we denote by $F(T)$ the set of fixed points of a mapping T.

We now state the following propositions obtained in [8].

Proposition 1. *Let $a \in \mathbb{R}$ and $(a_0, a_1, \ldots) \in l^\infty$. Then $\mu_n(a_n) \leq a$ for all Banach limits μ if and only if for each $\epsilon > 0$, there exists $p_0 \in \mathbb{N}$ such that $\frac{a_n + a_{n+1} + \cdots + a_{n+p-1}}{p} < a + \epsilon$ for all $p \geq p_0$ and $n \in \mathbb{N}_+$.*

Proposition 2. *Let $a \in \mathbb{R}$ and $(a_0, a_1, \ldots) \in l^\infty$ such that $\mu_n(a_n) \leq a$ for all Banach limits μ. If $\overline{\lim}_{n\to\infty}(a_{n+1} - a_n) \leq 0$ then $\overline{\lim}_{n\to\infty} a_n \leq a$.*

3. Main Theorem

We are now in a position to give our Main Theorem. For its proof, we shall mainly follow the technique used in [8].

Main Theorem. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm and C a closed, convex subset of E. Let S and T be nonexpansive mappings from C into itself such that $F(T) \cap F(S) \neq \phi$. Further, let $\{x_n\}$ defined by (5) satisfy*

$$\begin{cases} 0 \leq \alpha_n \leq 1,\ \alpha_n \to 0,\ \sum_{n=0}^{\infty} \alpha_n = \infty,\ \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \\ 0 < \beta_n < 1,\ \sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \\ 0 < \gamma_n \leq 1,\ \sum_{n=0}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty \end{cases}$$

for all $n = 0, 1, 2, \ldots$. Then $\{x_n\}$ converges strongly to a common fixed point of S and T.

If C satisfies certain additional conditions then we can prove the following theorem on lines similar to those of [9].

Theorem 1. *Let E be a Banach space with a uniformly Gâteaux differentiable norm and C a weakly compact, convex subset of E. Let $x \in C$ and suppose that z_t is a unique element of C for $0 < t < 1$ satisfying $z_t = tx + (1-t)Tz_t$ where T is a nonexpansive mapping of C into itself. If each nonempty T-invariant, closed, convex subset of C contains a fixed point of T then $\{z_t\}$ converges to a fixed point of T.*

In order to prove our Main Theorem, we first prove the following lemmas.

Lemma 1. *Under the conditions of our Main Theorem,*

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Proof. Note that $F(T) \cap F(S) \subset F(U_n)$ for all $n \in \mathbb{N}$. Also note that U_n is nonexpansive. Moreover, it can be verified that $\{x_n\}$, $\{U_n x_n\}$, $\{Tx_n\}$ and $\{Sx_n\}$ are bounded. Next, set $L = \sup\{\|x\| : x \in C\}$. Then

$$\|x_{n+1} - x_n\| \leq 2L|\alpha_n - \alpha_{n-1}| + \|U_n x_n - U_{n-1}x_n\| + (1-\alpha_n)\|x_n - x_{n-1}\|$$

for all $n \in \mathbb{N}$. Let $t_n = 2L(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}|)$ for all $n \in \mathbb{N}$. As $\|U_n x_n - U_{n-1}x_n\| \leq 2L(|\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}|)$ can be shown to be true, we have $\|x_{n+1} - x_n\| \leq t_n + (1-\alpha_n)\|x_n - x_{n-1}\|$ for all $n \in \mathbb{N}$. This gives

$$\begin{aligned}
\|x_{n+m+1} - x_{n+m}\| &\leq \sum_{k=m+1}^{n+m} t_k + \left(\prod_{k=m+1}^{n+m}(1-\alpha_k)\right)\|x_{m+1} - x_m\| \\
&\leq \sum_{k=m+1}^{n+m} t_k + \exp\left(-\sum_{k=m+1}^{n+m}\alpha_k\right)\|x_{m+1} - x_m\|
\end{aligned}$$

for all $m, n \in \mathbb{N}$. Now using $\sum_{k=0}^{\infty}\alpha_k = \infty$, $\sum_{k=1}^{\infty} t_k < \infty$ and the boundedness of $\{x_n\}$, we conclude that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0.$$

□

Lemma 2. *Under the conditions of our Main Theorem,*

$$\overline{\lim_{n\to\infty}}\langle x - z, J(x_n - z)\rangle \leq 0.$$

Proof. Let $n \in \mathbb{N}$ and assume that $\{z_{nt}\}$ converges strongly to z in $F(U_n)$ as $t \to 0$ where z_{nt} is a unique element of C for $0 < t < 1$ and satisfies $z_{nt} = tx + (1-t)U_n z_{nt}$. Let μ be a Banach limit. Then $\mu_n(\alpha_n) = 0$ and the nonexpansiveness of U_n gives

$$\mu_n\|x_n - U_n z_{nt}\|^2 \leq \mu_n\|x_n - z_{nt}\|^2.$$

Now $(1-t)(x_n - U_n z_{nt}) = (x_n - z_{nt}) - t(x_n - x)$ yields

$$(1-t)^2\mu_n\|x_n - U_n z_{nt}\|^2 \geq (1-2t)\mu_n\|x_n - z_{nt}\|^2 + 2t\mu_n\langle x - z_{nt}, J(x_n - z_{nt})\rangle.$$

This implies that

$$\tfrac{t}{2}\mu_n\|x_n - z_{nt}\|^2 \geq \mu_n\langle x - z_{nt}, J(x_n - z_{nt})\rangle.$$

Let $t \to 0$. Then because E has uniformly Gâteaux differentiable norm, J is norm to weak* uniformly continuous and so we have

$$\mu_n\langle x - z, J(x_n - z)\rangle \leq 0.$$

On the other hand, by Lemma 1,

$$\lim_{n\to\infty}|\langle x - z, J(x_{n+1} - z)\rangle - \langle x - z, J(x_n - z)\rangle| = 0.$$

Hence by Proposition 2,

$$\overline{\lim_{n\to\infty}} \langle x - z, J(x_n - z)\rangle \le 0.$$

□

We now give the proof of our Main Theorem.

Proof of Main Theorem. Let $z \in F(U_n)$. Since

$$(1-\alpha_n)(U_n x_n - z) = (x_{n+1} - z) - \alpha_n(x - z),$$

we have

$$\|(1-\alpha_n)(U_n x_n - z)\|^2 \ge \|x_{n+1} - z\|^2 - 2\alpha_n \langle x - z, J(x_{n+1} - z)\rangle$$

which implies

$$\begin{aligned}\|x_{n+1} - z\|^2 &\le (1-\alpha_n)\|x_n - z\|^2 \\ &\quad + 2(1-(1-\alpha_n))\langle x - z, J(x_{n+1} - z)\rangle \text{ for all } n \in \mathbb{N}.\end{aligned}$$

Let $\epsilon > 0$. By Lemma 2, there exists $m \in \mathbb{N}$ such that

$$\langle x - z, J(x_n - z)\rangle \le \epsilon/2 \text{ for all } n \ge m.$$

Then

$$\begin{aligned}&\|x_{n+m} - z\|^2 \\ &\le \left(\prod_{k=m}^{n+m-1}(1-\alpha_k)\right)\|x_m - z\|^2 + \left(1 - \prod_{k=m}^{n+m-1}(1-\alpha_k)\right)\epsilon \\ &\le \exp\left(-\sum_{k=m}^{n+m-1}\alpha_k\right)\|x_m - z\|^2 + \epsilon.\end{aligned}$$

Hence by $\sum_{k=0}^{\infty}\alpha_k = \infty$, we get

$$\overline{\lim_{n\to\infty}}\|x_n - z\|^2 = \overline{\lim_{n\to\infty}}\|x_{n+m} - z\|^2 \le \epsilon.$$

Since $\epsilon > 0$ is arbitrary so $\{x_n\}$ converges strongly to $z \in F(U_n)$.

Finally, since E is strictly convex, $0 < \beta_n < 1$ and $0 < \gamma_n \le 1$ therefore $z \in F(T) \cap F(S)$. For, let $z \in F(U_n)$ but $z \notin F(T) \cap F(S)$. Let $w \in F(T) \cap F(S)$, $w \ne z$. Now $z = U_n z$ gives $z = T(\beta_n S z + (1-\beta_n)z)$

so

$$\begin{aligned}\|z-w\| &= \|T(\beta_n Sz+(1-\beta_n)z)-w\|\\ &\le \|\beta_n Sz+(1-\beta_n)z-w\|\\ &= \|\beta_n(Sz-w)+(1-\beta_n)(z-w)\|\\ &\le \beta_n\|Sz-w\|+(1-\beta_n)\|z-w\|\\ &\le \beta_n\|z-w\|+(1-\beta_n)\|z-w\|\\ &= \|z-w\|.\end{aligned}$$

Also because S is nonexpansive, $\|Sz-w\| \le \|z-w\|$. Thus

$$\|z-w\| = \|Sz-w\| = \|\beta_n(Sz-w)+(1-\beta_n)(z-w)\|.$$

Since E is strictly convex therefore $z-w = Sz-w$ implies $z = Sz$. Then by $z = T(\beta_n Sz+(1-\beta_n)z)$, we also obtain $z = Tz$. Consequently, $\{x_n\}$ converges strongly to a common fixed point of S and T. Hence the proof. □

Remark. Note that for $\beta_n = 0$ and $\gamma_n = 1$, (5) reduces to

$$x_0 = x,\ x_{n+1} = \alpha_n x + (1-\alpha_n)Tx_n,\ n = 0,1,2,\dots,$$

and hence the "Theorem" of Shioji and Takahashi [8] can be deduced from our above Main Theorem as the following corollary.

Corollary. *Let E be a Banach space with a uniformly Gâteaux differentiable norm and C a closed, convex subset of E. Let T be nonexpansive mapping from C into itself such that $F(T) \neq \phi$. Suppose that $\{x_n\}$ is defined by*

$$x_0 = x,\ x_{n+1} = \alpha_n x + (1-\alpha_n)Tx_n,\ n = 0,1,2,\dots,$$

where $0 \le \alpha_n \le 1$ forall$n = 0,1,2,\dots$, $\alpha_n \to 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$ and $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$. Moreover, assume that $\{z_t\}$ converges strongly to $z \in F(T)$ as $t \to 0$ where z_t is a unique element of C for $0 < t < 1$ and satisfies $z_t = tx + (1-t)Tz_t$. Then $\{x_n\}$ converges strongly to $z \in F(T)$.

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