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Self-Dual Metrics on 4-dimensional Circle Bundles

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Abstract

A circle bundle P over an oriented 3-manifold is endowed with a bundle metric g in terms of a connection γ. We investigate the self-duality of g in terms of Yang-Mills condition on γ and base metric curvature conditions, and also verify the circle bundle version of Joyce's theorem with respect to Einstein-Weyl structures together with the generalized monopole solutions.

§1. Introduction Let $\pi : P \longrightarrow M$ be a circle principal bundle over an oriented Riemannian 3-manifold (M, h) and γ a connection on P.

We have then a bundle metric g on P in terms of the base metric h and γ;

$$g = \gamma^2 + \pi^* h. \tag{1}$$

Namely, with respect to this metric the vertical subspace $V_u \cong \mathbf{R}$, $u \in P$, is measured through γ by the standard inner product of $\mathbf{R}$ and the horizontal subspace $H_u \cong T_{\pi(u)}M$ by h such that V_u and H_u are required to be orthogonal.

The subject of this paper is to investigate the self-duality of a bundle metric on an oriented 4-manifold P. Here P is orientable, because $T_uP = V_u \oplus H_u$ has the canonical orientation. So we fix an orientation of P in this way.

Consider a metric g on a general oriented 4-manifold X. We say that g is self-dual when the half part of the Weyl conformal curvature tensor vanishes.

To define the self-duality of metric more precisely, we provide on the bundle $\Omega^2(X)$ of 2-forms the Hodge operator $*$ which is an involution so that $\Omega^2(X)$ decomposes into the ± 1 eigensubbundles $\Omega_{\pm}$;

$$\Omega^2(X) = \Omega^2_+ \oplus \Omega^2_-. \tag{2}$$

The Weyl conformal curvature tensor W is regarded as an endomorphism of $\Omega^2(X)$ which commutes with Hodge operator. So, W maps $\Omega^2_\pm$ into $\Omega^2_\pm$ and we have the splitting;

$$W = \begin{pmatrix} W^+ & 0 \\ 0 & W^- \end{pmatrix}. \tag{3}$$

A metric g on X is self-dual (anti-self-dual) when $W^- = 0$ ($W^+ = 0$, respectively)([1], [5]). If both W^+ and W^- vanish, g is a conformally flat metric. Note that the self-duality equation is the equation on a conformal structure represented by a metric.

Consider the trivial case where P and γ are trivial. In this case a bundle metric reduces to a product metric. The second author showed in his master thesis [13] that a product metric is self-dual if and only if the base manifold (M, h) is of constant curvature. This constant curvature statement holds even for a circle bundle with a flat connection. In fact, over a small neighborhood in M such a bundle with a flat connection turns out after a gauge transformation to be a product bundle with a trivial connection (see [10] Theorem 9.1). So the following question can be posed: are there a non-trivial circle bundle $\pi : P \longrightarrow M$ and a non-flat connection γ such that the bundle metric is self-dual ?

We consider this question in the case that the connection is Yang-Mills.

We then have the following

THEOREM 1. Let $g = \gamma^2 + \pi^* h$ be a bundle metric on a circle bundle $\pi : P \longrightarrow M$ where γ is a Yang-Mills connection relative to the base metric h.

We assume that g is self-dual with respect to the orientation of P. If (M, h) has constant scalar curvature σ and the Ricci curvature satisfies

$$6\, Ric_h(Z, Z) + \sigma\, h(Z, Z) \leq 0 \tag{4}$$

for any tangent vector Z of M, then γ and (M, h) must be flat and of constant curvature, respectively. Moreover the self-dual metric g is conformally flat.

Theorem 1 asserts under a certain nonpositive Ricci curvature condition on the base metric that every self-dual bundle metric is essentially given as a product metric on a product circle bundle over a 3-dimensional space form.

The self-dual bundle metrics above defined admit the right action of S^1 as isometries.

So we might be able to consider more general setting, namely self-dual 4-manifolds admitting a non-trivial Killing field. Those 4-manifolds have been considered by Jones and Tod ([7]) and Joyce ([8]) in terms of Einstein-Weyl 3-manifolds with an additional generalized monopole solution.

As a conformal generalization of Einstein manifolds, an Einstein-Weyl manifold is defined as a manifold M being endowed with $(D, [h])$ for a torsion-free affine connection D and a conformal structure $[h]$ represented by a metric h which satisfies $Dh = \omega \otimes h$ for a 1-form ω and the symmetrized Ricci tensor sym$Ric^D = \Lambda h$ for a $\Lambda \in C^\infty(M)$ (see, for more precise definition, for instances [4], [7], [12] and [6]).

The following has been shown by Jones and Tod.

THEOREM(Jones and Tod [7]). Let P be a self-dual 4-manifold with a nowhere vanishing conformal Killing field K. Then the orbit space $M = P/K$ of all trajectories generated by K carries a structure of Einstein-Weyl 3-manifold with a generalized monopole solution.

Conversely, given an Einstein-Weyl 3-manifold M with a generalized monopole solution, then the product manifold $M \times \mathbf{R}$ admits a self-dual metric with a nowhere vanishing conformal Killing field K such that the orbit space of all trajectories generated by the field K is just the Einstein-Weyl 3-manifold M having a generalized monopole solution.

This correspondence was also obtained by N. Hitchin in the context of twitor spaces. In fact, given a self-dual 4-manifold admitting a conformal Killing vector field H, its twistor space admits a holomorphic field so that one can consider the space T of trajectories of H in the twistor space. And it was shown by Hitchin that T carries the mini-twistor space of an Einstein-Weyl 3-manifold $(M, D, [h])$, i.e., the space of oriented geodesics in $(M, D, [h])$ and vice versa. See for this [4] and also [11].

The second part of Theorem of Jones and Tod was also shown by Joyce in a more direct form (see [8], Proposition 2.2.3).

Even Joyce treated the product bundle $P = M \times \mathbf{R}$, the statement of his result still holds for bundle metrics on non-trivial circle bundles. Although Joyce did not mention it explicitly, we can actually show in our terminology the equivalence between Einstein-Weyl 3-manifolds together with generalized monopole solutions and self-dual bundle metrics on circle bundles.

THEOREM 2. Let M be an oriented 3-manifold with a metric h and $\pi : P \longrightarrow M$ a circle bundle with a connection γ.

Then, a bundle metric $g = \pi^* h + \frac{1}{f^2}\gamma^2$ (f is a positive function on M) is self-dual if and only if there exists a torsion free affine connection D satisfying $Dh = -2\omega \otimes h$ for a 1-form ω so that (D, h) is Einstein-Weyl and f is a solution to the generalized monopole equation:

$$df - f\omega = + *_h (d\gamma). \tag{5}$$

We postpone its proof in §5.

In Theorem 2 we specialize for a bundle metric $g = \pi^* h + \frac{1}{f^2}\gamma^2$ as $f = 1$ and the connection γ to be Yang-Mills, i.e., the 1-form $*_h d\gamma$ is closed. So in this specialized case, namely for the bundle metrics considered in Theorem 1 the generalized monopole equation reads as

$$\omega = - *_h d\gamma \tag{6}$$

and hence ω is closed, that is, ω is locally exact and then the Einstein-Weyl structure (D, h) is locally trivial, i.e., D coincides locally with the Levi-Civita connection of a conformal change of h([6]). Theorem 1 implies then that under the curvature conditions this closed 1-form ω vanishes. So the base manifold (M, h) must be an Einstein space and then a space form because of $\dim M = 3$. Further the curvature form $d\gamma$ turns out to be zero.

REMARK 1. LeBrun constructed in [11] on a non trivial circle bundle over a hyperbolic 3-space self-dual metrics as $g = \frac{1}{V}\gamma^2 + V\pi^* h$, where $V > 0$ are fundamental solutions of the Laplace equation $\Delta V = 0$ so that $\frac{1}{2\pi} * dV$ are integral homology classes and that $*dV = d\gamma$. Because of the conformal invariance g is self-dual if its conformal change $\frac{1}{V} g = \frac{1}{V^2}\gamma^2 + \pi^* h$ is self-dual. In LeBrun's construction the hyperbolic 3-space structure is nothing but the Einstein-Weyl 3-structure so that the equation $*dV = d\gamma$ is the generalized monopole equation.

REMARK 2. If the base manifold is compact, then any self-dual bundle metric on a circle bundle P is automatically conformally flat. This is because the S^1 action is free on P so that the signature $\tau(P)$ must be zero (see [3], p.722).

REMARK 3. A Riemannian 3-manifold (M, h) satisfying the curvature inequality (4) in Theorem 1 must be of nonpositive scalar curvature. On the other hand, a nonpositively curved 3-manifold necessarily satisfies (4). So, let Σ be a hyperbolic 2-space. Then a Riemannian product metric on $M = \Sigma \times S^1$ has constant scalar curvature and satisfies (4).

COROLLARY of THEOREM 1. Let P be an arbitrary circle bundle over $M = \Sigma \times S^1$. Then P admits no self-dual bundle metric $g = \gamma^2 + \pi^* h$ associated to a Yang-Mills connection γ.

This corollary is shown as follows. In fact, if we suppose that some bundle metric g be self-dual, then from Theorem 1 the base manifold (M, h) must be of constant curvature, and hence this contradicts to a Riemannian product structure $M = \Sigma \times S^1$.

§2. Weyl conformal curvature tensor on a circle bundle

Let (M, h) be an oriented Riemannian 3-manifold and $\pi : P \longrightarrow M$ be a circle bundle. Let γ be a connection on P and consider a bundle metric $g = \gamma^2 + \pi^* h$.

We take an orthonormal local frame $\{e_0, \cdots, e_3\}$ of TP compatible with the orientation of P in such a way that $\{\pi_* e_1, \pi_* e_2, \pi_* e_3\}$ is an orthonormal frame of TM, associated to the orientation of M. The dual frame is given as $\{\theta^0 = \gamma, \theta^1, \theta^2, \theta^3\}$, where $\{\theta^1, \theta^2, \theta^3\}$ is the pull back of the dual frame of $\{\pi_* e_1, \pi_* e_2, \pi_* e_3\}$.

Since the structure group of P is abelian, the curvature form $\Gamma = d\gamma$ of γ is a real valued 2-form defined over M;

$$\Gamma = \frac{1}{2} \sum_{s,t} A_{st} \theta^s \wedge \theta^t, \quad A_{st} = -A_{ts}. \tag{7}$$

Here and in what follows, the Roman indices i, j, k, ℓ, s, t run from 1 to 3, while the Greek indices $\alpha, \beta, \gamma, \delta$ from 0 to 3. In addition, we define over M tensors $A_{ij,k}$, B_{ij} and C by

$$\begin{aligned} \sum_s A_{ij,s} \theta^s &= dA_{ij} - \sum A_{is} \omega_j^s - \sum_s A_{sj} \omega_i^s, \\ B_{ij} &= \sum_s A_{si} A_{sj}, \\ C &= \sum_s B_{ss} = \sum_{s,t} (A_{st})^2 \end{aligned} \tag{8}$$

where ω^i_j is the Levi-Civita connection of (M, h). The tensor $A_{ij,k}$ is the covariant derivative of Γ and $\frac{1}{2}C$ is the square norm of Γ.

The curvature tensor $K_{\alpha\beta\gamma\delta}$ of the metric g is calculated by the aid of the structure equations;

$$\begin{aligned} d\theta^\alpha &= -\sum_\beta \tilde{\omega}^\alpha_\beta \wedge \theta^\beta, \\ d\tilde{\omega}^\alpha_\beta &+ \sum_\gamma \tilde{\omega}^\alpha_\gamma \wedge \tilde{\omega}^\gamma_\beta = \tilde{\Omega}^\alpha_\beta, \end{aligned} \tag{9}$$

where

$$\tilde{\Omega}^\alpha_\beta = \frac{1}{2}\sum_{\gamma,\delta} K_{\alpha\beta\gamma\delta}\theta^\gamma \wedge \theta^\delta, \; K_{\alpha\beta\gamma\delta} = -K_{\alpha\beta\delta\gamma} \tag{10}$$

and represented as

$$\begin{aligned} K_{ijik} &= R_{ijik} - \frac{3}{4}A_{ij}A_{ik}, \\ K_{0i0j} &= \frac{1}{4}B_{ij}, \\ K_{0ijk} &= \frac{1}{2}A_{jk,i}, \end{aligned} \tag{11}$$

where R_{ijkl} is the curvature tensor of (M, h) (see for these formulas also [9], §3). Further the Ricci tensor $K_{\alpha\beta}$ and the scalar curvature κ of (P, g) are

$$\begin{aligned} K_{ij} &= R_{ij} - \frac{1}{2}B_{ij}, \\ K_{0i} &= -\frac{1}{2}\sum_s A_{si,s}, \\ \kappa &= \sigma - \frac{1}{4}C, \end{aligned} \tag{12}$$

where R_{ij} and σ denote the Ricci tensor and the scalar curvature of (M, h).

The Weyl conformal curvature tensor W of (P, g) is

$$\begin{aligned} W_{\alpha\beta\gamma\delta} &= K_{\alpha\beta\gamma\delta} + \frac{1}{2}(K_{\beta\gamma}\delta_{\alpha\delta} - K_{\beta\delta}\delta_{\alpha\gamma} - K_{\alpha\gamma}\delta_{\beta\delta} + K_{\alpha\delta}\delta_{\beta\gamma}) \\ &+ \frac{\kappa}{6}(\delta_{\beta\delta}\delta_{\alpha\gamma} - \delta_{\beta\gamma}\delta_{\alpha\delta}). \end{aligned} \tag{13}$$

and from (11) and (12)

$$\begin{aligned} W_{0i0j} &= \frac{1}{4}B_{ij} - \frac{1}{2}(R_{ij} - \frac{1}{2}B_{ij} + \frac{1}{4}C\delta_{ij}) + \frac{1}{6}(\sigma - \frac{1}{4}C)\delta_{ij} \\ &= \frac{1}{2}B_{ij} - \frac{1}{6}C\delta_{ij} - \frac{1}{2}T_{ij}, \\ W_{0ijk} &= \frac{1}{2}A_{jk,i} + \frac{1}{4}\sum_s A_{sj,s}\delta_{ik} - \frac{1}{4}\sum_s A_{sk,s}\delta_{ij}, \end{aligned} \tag{14}$$

where T_{ij} denotes the trace-free Ricci tensor of (M, h);

$$T_{ij} = R_{ij} - \frac{1}{3}\sigma\delta_{ij}. \tag{15}$$

We consider W as an endomorphism of Ω^2;

$$W(\theta^\gamma \wedge \theta^\delta) = \sum_{\alpha<\beta} W_{\alpha\beta\gamma\delta}\theta^\alpha \wedge \theta^\beta. \tag{16}$$

We take now the basis of $\Omega^2_\pm$;

$$\begin{aligned} f_1^\pm &= \theta^0 \wedge \theta^1 \pm \theta^2 \wedge \theta^3, \\ f_2^\pm &= \theta^0 \wedge \theta^2 \pm \theta^3 \wedge \theta^1, \\ f_3^\pm &= \theta^0 \wedge \theta^3 \pm \theta^1 \wedge \theta^2 \end{aligned} \tag{17}$$

with respect to which W^+ and W^- have trace-free symmetric 3×3-matrix representations.

In what follows, we will adopt the convention that indices i, j, k appeared in the propositions and formulae mean an even permutation of $1, 2, 3$.

PROPOSITION 2.1. The components W^-_{ij} of W^- are

$$\begin{aligned} W^-_{ii} &= \frac{1}{2}B_{ii} - \frac{1}{6}C - \frac{1}{2}T_{ii} - \frac{1}{2}A_{jk,i}, \\ W^-_{ij} &= \frac{1}{2}B_{ij} - \frac{1}{2}T_{ij} - \frac{1}{2}A_{jk,j} + \frac{1}{4}\sum_s A_{sk,s}. \end{aligned} \tag{18}$$

Proof. We have

$$\begin{aligned} W^-_{ii} &= W_{0i0i} - W_{0ijk} = \frac{1}{2}B_{ii} - \frac{1}{6}C - \frac{1}{2}T_{ii} - \frac{1}{2}A_{jk,i}, \\ W^-_{ij} &= W_{0i0j} - W_{0jjk} = \frac{1}{2}B_{ij} - \frac{1}{2}T_{ij} - \frac{1}{2}A_{jk,j} + \frac{1}{4}\sum_s A_{sk,s}. \end{aligned} \tag{19}$$

This is because, on a general Riemannian 4-manifold

$$W(\theta^\alpha \wedge \theta^\beta - \theta^\gamma \wedge \theta^\delta) = \sum_{\lambda<\mu}(W_{\alpha\beta\lambda\mu} - W_{\gamma\delta\lambda\mu})\theta^\lambda \wedge \theta^\mu \tag{20}$$

and

$$W_{\alpha\beta\alpha\beta} = W_{\gamma\delta\gamma\delta}, \;\; W_{\alpha\beta\gamma\beta} = -W_{\alpha\delta\gamma\delta}, \tag{21}$$

where $\alpha, \beta, \gamma, \delta$ are distinct indices. Q.E.D.

We assume that a connection γ is Yang-Mills, namely, the curvature form $\Gamma = d\gamma$ is coclosed; $\delta\,\Gamma = 0$.

Then, since $\sum_s A_{sk,s} = 0$, the anti-self-dual part W^- has from Proposition 2.1 the components

$$\begin{aligned} W^-_{ii} &= -\frac{1}{2}A_{jk,i} + \frac{1}{2}B_{ii} - \frac{1}{6}C - \frac{1}{2}T_{ii},\; i = 1,2,3 \\ W^-_{ij} &= -\frac{1}{2}A_{jk,j} + \frac{1}{2}B_{ij} - \frac{1}{2}T_{ij},\; i \neq j. \end{aligned} \tag{22}$$

Thus, we have

PROPOSITION 2.2 Let $g = \gamma^2 + \pi^* h$ be a bundle metric on P. Then g is self-dual if and only if the covariant derivative of the curvature form Γ satisfies

$$A_{ij,k} = B_{kk} - \frac{1}{3}C - T_{kk}, \;\; A_{ij,i} = B_{ki} - T_{ki}. \tag{23}$$

We have then

PROPOSITION 2.3 Assume that a bundle metric $g = \gamma^2 + \pi^* h$ is self-dual with respect to the orientation of P. If (M, h) has constant scalar curvature, then under the condition that γ is a Yang-Mills connection the second covariant derivatives of the curvature form of γ satisfy

$$\sum_s A_{ij,ss} = -\frac{1}{12}C_k, \tag{24}$$

where $dC = \sum_s C_s \theta^s$.

Proof. Since $\dim M = 3$, the Yang-Mills equation $\sum_s A_{si,s} = 0$ is equivalent to $A_{ji,j} = -A_{ki,k}$ for an even permutation $\{i, j, k\}$ of $\{1, 2, 3\}$.

We take covariant derivative of formulae (23) in Proposition 2.2 to have

$$\begin{aligned}\sum_s A_{ij,ss} &= (B_{ki,i} - T_{ki,i}) + (B_{jk,j} - T_{jk,j}) + (B_{kk,k} - \frac{1}{3}C_k - T_{kk,k}) \quad (25)\\ &= \sum_s B_{sk,s} - \frac{1}{3}C_k - \sum_s T_{sk,s}.\end{aligned}$$

The last term vanishes from the assumption that (M, h) has constant scalar curvature together with the secon Bianchi identity.

On the other hand, the Yang-Mills equation $A_{ji,i} = -A_{jk,k}$ together with the second Bianchi identity on Γ, i.e., $d\Gamma = 0$, implies

$$\begin{aligned}\sum_s B_{sk,s} &= (A_{ji}A_{jk})_{,i} + (A_{ij}A_{ik})_{,j} + ((A_{ik})^2 + (A_{jk})^2)_{,k} \qquad (26)\\ &= A_{ji,i}A_{jk} + A_{ji}A_{jk,i} + A_{ij,j}A_{ik}\\ &+ A_{ij}A_{ik,j} + 2A_{ik,k}A_{ik} + 2A_{jk,k}A_{jk}\\ &= \frac{1}{4}C_k.\end{aligned}$$

from which the proposition follows. Q.E.D.

Since it holds

$$A_{ij}B_{ki} + A_{jk}B_{ii} + A_{ki}B_{ji} = 0 \qquad (27)$$

for any even permutation $\{i, j, k\}$, we have by applying Proposition 2.2

$$\begin{aligned}C_i &= 4(A_{ij}A_{ij,i} + A_{jk}A_{jk,i} + A_{ki}A_{ki,i}) \qquad (28)\\ &= -\frac{4}{3}A_{jk}C - 4(A_{ij}T_{ki} + A_{jk}T_{ii} + A_{ki}T_{ij}).\end{aligned}$$

Consequently we have the following

PROPOSITION 2.4 Assume $g = \gamma^2 + \pi^* h$ is a self-dual metric on P. If (M, h) has constant scalar curvature and γ is Yang-Mills, then the curvature form $\Gamma = \frac{1}{2}\sum A_{ij}\theta^i \wedge \theta^j$ satisfies the following equations.

$$\sum_s A_{ij,ss} = \frac{1}{9}A_{ij}C + \frac{1}{3}(A_{ki}T_{jk} + A_{ij}T_{kk} + A_{jk}T_{ik}). \qquad (29)$$

§3. The Bochner-Weitzenböck formula for 2-form Laplacian

Let φ be a 2-form on an oriented 3-manifold (M, h). Then the Bochner-Weitzenböck formula of $\Delta\varphi = (\delta d + d\delta)\varphi$ is

$$(\Delta\varphi)_{ij} = -\sum_s \varphi_{ij,ss} - \sum_{s,t} R_{stij}\varphi_{st} + \sum_s (R_{si}\varphi_{sj} + R_{sj}\varphi_{is}). \tag{30}$$

To get this formula is a routine business. So, consult (3.10) and also (3.8) in [2] where the Bochner-Weitzenböck formula was derived for a vector bundle valued 2-form.

Let x be an arbitrary point of M. Diagonalize the Ricci tensor at x. So,

$$(\Delta\varphi)_{ij} = -\sum_s \varphi_{ij,ss} + (R_{ii} + R_{jj} - 2R_{ijij})\varphi_{ij}, \tag{31}$$

because

$$\sum_{s,t} R_{stij}\varphi_{st} = 2(R_{ijij}\varphi_{ij} + R_{jkij}\varphi_{jk} + R_{kiij}\varphi_{ki}) \tag{32}$$

and, the both $R_{ki} = R_{jkji}$ and $R_{kj} = R_{ikij}$ vanish at x.

PROPOSITION 3.1 Let γ be a Yang-Mills connection on a circle bundle P. Then the curvature form $\Gamma = \frac{1}{2}\sum_{s,t} A_{st}\theta^s \wedge \theta^t$ of γ satisfies

$$\sum_s A_{ij,ss} = (R_{ii} + R_{jj} - 2R_{ijij})A_{ij}. \tag{33}$$

Proof. Since γ is Yang-Mills, $\Delta\Gamma = 0$. So, the equation (33) follows from (31). Q.E.D.

§4. The proof of Theorem 1

Let (M, h) be of constant scalar curvature. Suppose that the connection γ is Yang-Mills and the bundle metric $g = \gamma^2 + \pi^* h$ is self-dual.

If we diagonalize the Ricci tensor at $x \in M$, the equation (29) becomes at x

$$\sum_s A_{ij,ss} = (\frac{1}{9}C + \frac{1}{3}T_{kk})A_{ij}. \tag{34}$$

Combining this with (33) we have

$$\frac{1}{9}CA_{ij} = (R_{ii} + R_{jj} - 2R_{ijij} - \frac{1}{3}T_{kk})A_{ij} \tag{35}$$

the RHS of which reduces to

$$(\frac{2}{3}R_{kk} + \frac{1}{9}\sigma)A_{ij}, \tag{36}$$

because $R_{ii} + R_{jj} - 2R_{ijij} = R_{kiki} + R_{jkjk} = R_{kk}$ and $T_{kk} = R_{kk} - \frac{1}{3}\sigma$. Consequently, we have

$$\frac{1}{9}CA_{ij} = (\frac{2}{3}R_{kk} + \frac{1}{9}\sigma)A_{ij}. \tag{37}$$

Suppose $\Gamma \neq 0$ at some point $x \in M$. So (37) holds at x for the diagonalized Ricci tensor. Since $\Gamma \neq 0$, we can assume without loss of generality $A_{ij} \neq 0$ for some indices i, j at x.

We have then

$$C = 6R_{kk} + \sigma \tag{38}$$

and $C = \sum(A_{st})^2 > 0$ at x. The curvature assumption on (M, h), however, implies $6R_{kk} + \sigma \leq 0$. This causes a contradiction. Thus Γ vanishes identically on M.

That (M, h) is of constant curvature follows from the equation (22). In fact, since γ is flat, $A_{ij,k}, B_{ij}, C$ all vanish and hence $T = 0$, that is, the base metric h is Einstein. On 3-manifold M this implies that h is of constant curvature.

Finally we will show that the bundle metric is conformally flat. Because γ is flat and $T = 0$ for (M, h), W vanishes from formulae (14).

§5. The proof of Theorem 2

Set $F = f^{-1}$ in the bundle metric form. Then $g = F^2\gamma^2 + \pi^* h$.

To verify the equivalence in Theorem 2 we need to adapt Joyce's terminology to our calculation framework. So, we set $\mu = -\omega$. The generalized monopole equation $-\frac{dF}{F^2} - \frac{1}{F}\omega = *d\gamma$ reduces to $-\frac{dF}{F} + \mu = F * d\gamma$.

It suffices then to show that the metric $g = F^2\gamma + \pi^* h$ is self-dual if and only if the 1-form μ and the positive function F fulfill

$$\begin{aligned} Ric_h + \frac{1}{2}(\nabla^s\mu &+ 2\mu\otimes\mu) = \Lambda h, \\ \mu &= d\log F + F(*d\gamma) \end{aligned} \tag{39}$$

($\nabla^s\mu(X,Y) = (\nabla_X\mu)(Y) + (\nabla_Y\mu)(X)$). Here the first equation represents the Einstein-Weyl equation (see [6]).

We adopt an orthonormal dual frame similarly as in §2;
$\{\theta^0 = F\gamma, \theta^1, \theta^2, \theta^3\}$.

Then by the aid of the structure equations the curvature tensor $K_{\alpha\beta\gamma\delta}$ of the metric $g = F^2\gamma^2 + \pi^* h$ is represented as

$$\begin{aligned} K_{ijik} &= R_{ijik} - \frac{3}{4}F^2 A_{ij}A_{ik}, \\ K_{0i0j} &= -F^{-1}F_{i,j} + \frac{1}{4}F^2 B_{ij}, \\ K_{0ijk} &= \frac{1}{2}FA_{jk,i} + F_iA_{jk} - \frac{1}{2}(F_jA_{ki} + F_kA_{ij}), \end{aligned} \tag{40}$$

where $F_{i,j}$ is the second covariant derivative of F.

We have the Ricci tensor $K_{\alpha\beta}$ and the scalar curvature κ of (P,g)

$$\begin{aligned} K_{ij} &= R_{ij} - F^{-1}F_{i,j} - \frac{1}{2}F^2 B_{ij}, \\ K_{0i} &= \frac{1}{2}F\sum_s A_{si,s} - \frac{3}{2}\sum_s F_sA_{si}, \\ \kappa &= \sigma - \frac{1}{4}F^2 C - \frac{2}{F}\sum_s F_{s,s} \end{aligned} \tag{41}$$

and the Weyl conformal curvature tensor;

$$\begin{aligned} W_{0i0j} &= \frac{1}{2}F^2B_{ij} - \frac{1}{6}F^2C\delta_{ij} - \frac{1}{2}T_{ij} - \frac{1}{2F}(F_{i,j} - \frac{1}{3}\sum_s F_{s,s}\delta_{ij}), \\ W_{0ijk} &= \frac{1}{2}FA_{jk,i} + \frac{F}{4}(\sum_s A_{sj,s}\delta_{ik} - \sum_s A_{sk,s}\delta_{ij}) \\ &+ A_{jk}F_i - \frac{1}{2}(A_{ki}F_j + A_{ij}F_k) + \frac{3}{4}(\delta_{ik}\sum_s A_{sj}F_s - \delta_{ij}\sum_s A_{sk}F_s). \end{aligned} \tag{42}$$

LEMMA 5.1 The components of W^- are given as

$$
\begin{aligned}
W^-_{ii} &= -\frac{1}{2}FA_{jk,i} + \frac{1}{2}F^2(B_{ii} - \frac{1}{3}C) - \frac{1}{2}T_{ii} + U_{ii}, \\
W^-_{ij} &= -\frac{1}{2}FA_{jk,j} + \frac{1}{2}F^2B_{ij} - \frac{1}{2}T_{ij} + \frac{1}{4}F\sum_s A_{sk,s} + U_{ij},
\end{aligned} \tag{43}
$$

where U_{ii} and U_{ij} are given by

$$
\begin{aligned}
U_{ii} &= -\frac{1}{2F}(F_{i,i} - \frac{1}{3}\sum_s F_{s,s}) - F_iA_{jk} + \frac{1}{2}(F_jA_{ki} + F_kA_{ij}), \\
U_{ij} &= -\frac{1}{2F}F_{i,j} - \frac{3}{4}(A_{ki}F_i + A_{jk}F_j).
\end{aligned} \tag{44}
$$

Now define a 1-form $\mu = \sum_s \mu_s\theta^s$ by

$$
\mu_i = \frac{F_i}{F} + FA_{jk}, \quad i = 1,2,3. \tag{45}
$$

We take covariant derivative of μ_i;

$$
\mu_{i,\ell} = (\frac{F_i}{F} + FA_{jk})_{,\ell} \tag{46}
$$

and substitute (45) to this to get

$$
\begin{aligned}
\mu_{i,\ell} &= \{\frac{F_{i,\ell}}{F} + FA_{jk,\ell} - 2F^2A_{jk}A_{mn}\} \\
&- \mu_i\mu_\ell + F(\mu_iA_{mn} + 2\mu_\ell A_{jk}),
\end{aligned} \tag{47}
$$

where $\{i, j, k\}$ and $\{\ell, m, n\}$ are even permutations of $\{1,2,3\}$.
So,

$$
\begin{aligned}
\mu_{i,i} + \mu_i^2 &= \frac{1}{F}F_{i,i} + FA_{jk,i} + 3F\mu_iA_{jk} - 2F^2A_{jk}^2 \\
&= \frac{1}{F}F_{i,i} + FA_{jk,i} + 3F_iA_{jk} + F^2A_{jk}^2
\end{aligned} \tag{48}
$$

and summing up this

$$
\sum_s(\mu_{s,s} + \mu_s^2) = \frac{1}{F}\sum_s F_{s,s} + 3\sum F_iA_{jk} + \frac{1}{2}F^2C. \tag{49}
$$

Here we used the Bianchi identity for Γ. The summation $\sum F_i A_{jk}$ is taken over the cyclic even permutations of $\{1,2,3\}$.

So

$$\begin{aligned}(\mu_i^2+\mu_{i,i})-\frac{1}{3}\sum_s(\mu_s^2+\mu_{s,s}) &= \frac{1}{F}(F_{i,i}-\frac{1}{3}\sum_s F_{s,s})+FA_{jk,i} \\ &+ 2F_iA_{jk}-F_jA_{ki}-F_kA_{ij}-\frac{1}{6}F^2C+F^2A_{jk}^2.\end{aligned} \tag{50}$$

Thus from (46) we have

$$U_{ii}+\frac{1}{2}\{(\mu_i^2+\mu_{i,i})-\frac{1}{3}\sum_s(\mu_s^2+\mu_{s,s})\}=\frac{1}{2}FA_{jk,i}-\frac{1}{12}F^2C+\frac{1}{2}F^2A_{jk}^2 \tag{51}$$

Since $B_{ii}=A_{ij}^2+A_{ki}^2=\frac{C}{2}-A_{jk}^2$, the RHS of this turns into

$$\frac{1}{2}FA_{jk,i}-\frac{1}{2}F^2B_{ii}+\frac{1}{6}F^2C. \tag{52}$$

So we have verified the half of the following

LEMMA 5.2. If one defines $\mu_i=\frac{F_i}{F}+FA_{jk}$, then the components of W^- are written

$$\begin{aligned}W_{ii}^- &= -\frac{1}{2}T_{ii}-\frac{1}{2}\{(\mu_i)^2+\mu_{i,i})-\frac{1}{3}\sum_s((\mu_s)^2+\mu_{s,s})\},\ i=1,2,3 \\ W_{ij}^- &= -\frac{1}{2}T_{ij}-\frac{1}{4}(\mu_{i,j}+\mu_{j,i})-\frac{1}{2}\mu_i\mu_j,\ i\neq j\end{aligned} \tag{53}$$

To verify the rest part is similar, so we omit.

Therefore, from Lemma 5.2 all the components of W^- vanish if and only if

$$\mu_i=\frac{F_i}{F}+FA_{jk},\ i=1,2,3 \tag{54}$$

together with

$$\begin{aligned}T_{ii} &+ \mu_i^2+\mu_{i,i}-\frac{1}{3}\sum_s(\mu_s^2+\mu_{s,s})=0, \\ T_{ij} &+ \frac{1}{2}(\mu_{i,j}+\mu_{j,i})+\mu_i\mu_j=0,\end{aligned} \tag{55}$$

in other words,

$$R_{ij} - \frac{1}{3}\sigma\delta_{ij} + \frac{1}{2}(\mu_{i,j} + \mu_{j,i} + 2\mu_i\mu_j) - \frac{1}{3}\sum_s(\mu_s^2 + \mu_{s,s})\delta_{ij} = 0, \qquad (56)$$

or in coordinate free expressions

$$Ric_h + \frac{1}{2}(\nabla^s\mu + 2\mu\otimes\mu) = \Lambda h, \qquad (57)$$

where

$$\Lambda = \frac{1}{3}(\sigma + |\mu|^2 + \sum_s \mu_{s,s}) \qquad (58)$$

which is exactly the Einstein-Weyl equation. So we get Theorem 2.

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