Nihonkai Math. J.
Vol.1(1990),209-227

A Characterization of Generalized Saddle Points for Vector-Valued Functions via Scalarization*

Tamaki TANAKA

Abstract. In this paper, we propose abstract concepts of saddle points of a vector-valued function f defined on a product $X \times Y$ of infinite-dimensional sets X and Y in locally convex topological vector spaces. Three notions of the generalized saddle points are considered, and a notions of semi-saddle points, which is also known as "Nash equilibrium points" for a two-person nonzero-sum game in game theory, is defined for a pair of scalarized functions. Various necessary conditions, sufficient conditions and existence conditions are explored for each type of the generalized saddle points. These conditions give a connection between each type of the generalized saddle points and the corresponding type of the semi-saddle points.

Key words. vector optimization, scalarization, two-person game, cone saddle points, equilibrium points, convex-concave functions.

1. Introduction

Saddle point problems are important in the areas of optimization theory and game theory. As for optimization theory, the main motivation for studying saddle points of scalar-valued functions in the past has been their connection with characterizing solutions to dual problems. Also, as for game theory, the main motivation in the past has been the determination of two-person zero-sum games based on the minimax principle. That is, saddle points of the payoff function for the game are optimal strategy pairs for players. Unfortunately, no vector-valued saddle point problems have been formulated in any application in those areas and the other areas. However, we think that they will eventually lead to worthwhile new developments in mathematics as vector optimization has been explored widely. Therefore, we will generalize a saddle point concept and investigate the generalized saddle points which are called "cone saddle points".

Since scalarization method is of great importance on characterizing and computing for vector optimization theory, we will adopt the same approach to characterize the generalized saddle points. There have recently appeared many papers connected with scalarization in

*AMS subject classifications. 90C31, 90D05, 90D35

vector optimization; see [1], [8], [9], [10], [12], and [20]. In these papers, "optimal", "weakly optimal", and "properly optimal"— these optimalities are investigated widely. There are, however, few papers devoted to scalarization of generalized saddle points for vector-valued functions except [14] and [19]. In the two papers, the existence of generalized saddle points is guaranteed by the existence of (ordinary) saddle points of appropriate scalarized functions. Also, in [19], a notion of generalized saddle points is defined by means of the concept of cone extreme points or cone efficiency (in the sense of [22]), and called "cone saddle points". Various properties of the cone extreme points can be found in [22], [17], and [18].

The aim of this paper is to characterize the generalized saddle points for a vector-valued function via scalarization. For this purpose, we will define cone saddle points and weak cone saddle points, and introduce a notion of proper cone saddle points anew. Then we will establish some sufficient conditions and necessary conditions for the three types of the cone saddle points via scalarization. Also, we will present some existence results for each type of the cone saddle points.

The organization of the paper is as follows. In Section 2, we will define three types of cone saddle points: "proper Z_+-saddle points", "Z_+-saddle points", and "weak Z_+-saddle points" for a vector-valued function. Also, we will introduce a notion of semi-saddle points for a pair of functionals, which is also known as "Nash equilibrium points" for a two-person nonzero-sum game in game theory. Moreover, we will present some sufficient conditions for each type of the cone saddle points. Next, in Section 3 we will give some necessary conditions for each type of the cone saddle points. Also, we try to make clear the relation between each type of the cone saddle points for a vector-valued function and the corresponding type of the semi-saddle points for a pair of the scalarized functions. Finally, in Section 4 we will prove some existence theorems for each type of the cone saddle points, and give their corollaries with respect to some special convex properties of vector-valued functions.

Throughout the paper, let E, F, Z be three real Hausdorff locally convex topological vector spaces (l.c.s.) possessing each base $\mathcal{B}_E$, $\mathcal{B}_F$, $\mathcal{B}_Z$ of (convex symmetric) neighborhoods of the origin, respectively. We assume that $Z_+ \subset Z$ is a pointed convex cone (i.e., $tZ_+ \subset Z_+$ for any $t \geq 0$, $Z_+ \cap (-Z_+) = \{0\}$, and Z_+ is nonempty and convex), which induces a partial ordering $\leq_{Z_+}$ in Z as follows: For vectors $z_1, z_2 \in Z$, we denote

$$z_1 \leq_{Z_+} z_2, \quad \text{whenever } z_2 - z_1 \in Z_+.$$

Also, we assume that $\mathrm{int}Z_+ \neq \emptyset$, where the set $\mathrm{int}Z_+$ denotes the interior of the set Z_+. Then, $\mathrm{int}Z_+^0 := (\mathrm{int}Z_+) \cup \{0\}$ is a nontrivial pointed convex cone and induces a partial ordering $\leq_{\mathrm{int}Z_+^0}$ weak than $\leq_{Z_+}$ in Z. An element z_0 of a subset A of Z is said to be a Z_+-extreme point of A if there is no point $z \in A$ such that $z \neq z_0$ and $z_0 - z \in Z_+$, i.e.,

$$\{z \in A \mid z \leq_{Z_+} z_0, z \neq z_0\} = \emptyset,$$

which is equivalent to

$$A \cap (z_0 - Z_+) = \{z_0\}.$$

In this paper, we will call the Z_+-extreme point of A a minimal point of A. Also, if the set $-Z_+ := \{-z \mid z \in Z_+\}$ is denoted by Z_-, the Z_--extreme point of A is called a maximal point of A. The set of all Z_+-extreme points of A is denoted by $\mathrm{Ext}[A \mid Z_+]$; see [17], [18], and [22] for details. Then, the sets $\mathrm{Ext}[A \mid Z_+]$ and $\mathrm{Ext}[A \mid Z_-]$ consist of the minimal and maximal points of A, respectively. Finally, we will denote the dual cone for the convex cone Z_+ in the dual space Z^* of all continuous linear functionals on Z by

$$Z_+^* := \{z^* \in Z^* \mid \langle z^*, z\rangle \geq 0, \ \forall z \in Z_+\}.$$

Also, we will denote the positive dual cone for the convex cone Z_+ in Z^* by

$$Z_+^{*i} := \{z^* \in Z^* \mid \langle z^*, z\rangle > 0, \ \forall z \in Z_+ \setminus \{0\}\}.$$

It is easily seen that the sets Z_+^* and $Z_+^{*i} \cup \{0\}$ are pointed convex cones but the set Z_+^{*i} is not always the topological interior of the set Z_+^* (e.g., the usual positive cone in l^2).

2. Sufficient conditions for the cone saddle points

First, we start with recalling the contingent cone $K(A; z)$ of tangents to a subset A of Z at a vector z (e.g., see [3, p.55]). A vector v in Z belongs to $K(A; z)$ if and only if, for any $U \in \mathcal{B}_Z$ and $\varepsilon > 0$, there exist a scalar $t \in (0, \varepsilon)$ and a vector $w \in v + U$ such that $z + tw \in A$ (thus $z \in \mathrm{cl}A$ necessarily). It follows that the cone $K(A; z)$ contains the sequential (Bouligand) tangent cone $T(A; z)$ defined as

$$T(A;z) := \left\{ \lim_{n\to\infty} \lambda_n(z_n - z) \mid \lambda_n \geq 0, z_n \in A \text{ for } \forall n, \text{ and } z = \lim_{n\to\infty} z_n \right\}$$

If the set A is convex, the contingent cone $K(A; z)$ at any vector $z \in \mathrm{cl}A$ is a convex cone and

$$A - z \subset K(A; z). \tag{1}$$

Many papers (e.g., [1], [9] and [10]) give a definition of proper optimality by means of the sequential tangent cone, but in this paper we shall define proper cone saddle points by means of the contingent cone.

Throughout the paper, let $X \subset E$ and $Y \subset F$ be nonempty, and let $f : X \times Y \longrightarrow Z$ be a vector-valued function. Also, we will use the following symbols:

$$f(X, y) := \bigcup_{x \in X} \{f(x, y)\},$$

$$f(x, Y) := \bigcup_{y \in Y} \{f(x, y)\},$$

and

$$M := f(X, Y).$$

Then, three types of the cone saddle points for a vector-valued function f are defined as follows:

DEFINITION 2.1. (a) A point (x_0, y_0) is said to be a Z_+-saddle point of f with respect to $X \times Y$, a Z_+-saddle point for short, if

$$f(x_0, y_0) \in \mathrm{Ext}[f(x_0, Y) \mid Z_-] \cap \mathrm{Ext}[f(X, y_0) \mid Z_+]. \tag{2}$$

(b) A point (x_0, y_0) is said to be a weak Z_+-saddle point of f with respect to $X \times Y$, a weak Z_+-saddle point for short, if

$$f(x_0, y_0) \in \mathrm{Ext}\big[f(x_0, Y) \mid \mathrm{int} Z_-^0\big] \cap \mathrm{Ext}\big[f(X, y_0) \mid \mathrm{int} Z_+^0\big]. \tag{3}$$

(c) A point (x_0, y_0) is said to be a proper Z_+-saddle point of f with respect to $X \times Y$, a proper Z_+-saddle point for short, if the point (x_0, y_0) is a Z_+-saddle point and

$$\begin{aligned} 0 \ \in \ & \mathrm{Ext}[\mathrm{cl}K\,(f(X, y_0) + Z_+; f(x_0, y_0)) \mid Z_+] \\ & \cap \mathrm{Ext}[\mathrm{cl}K\,(f(x_0, Y) + Z_-; f(x_0, y_0)) \mid Z_-]. \end{aligned} \tag{4}$$

As for (a) and (b), we gave the definitions in [19]. Also, the definition (c) of proper Z_+ -saddle points is found to be acceptable on refering to [1, Def.2], [9, Def.1.1], [10, Def.1.1] and [20, Def.3.1] If the sets $f(X, y_0) + Z_+$ and $f(x_0, Y) + Z_-$ are convex, then, by (1), any point (x_0, y_0) satisfing the condition (4) is a Z_+-saddle point of f, and hence a proper Z_+-saddle point. For the convenience, we will denote the set of all Z_+-saddle points (resp. weak Z_+-saddle points, proper Z_+-saddle points) by S (resp. S^w , S^p), and then the following relationship holds:

$$S^p \subset S \subset S^w.$$

Moreover, we have $S = S^w$ whenever $\mathrm{int} Z_+^0 = Z_+$.

Next, we will formulate a sufficient condition for each of the three types of the cone saddle points. For this purpose, we will review Jahn's definition [8, p.204]. It is slightly changed as follows:

DEFINITION 2.2. Let A be a nonempty subset of Z, and z_0 be a vector of A.

(a) A functional $\varphi : A \longrightarrow R$ is called monotonically increasing with respect to the lower (resp. upper) section on A at z_0 if

$$\varphi(z) \leq \varphi(z_0) \ \text{ for any } z \in (\{z_0\} + Z_-) \cap A$$

$$(\ \text{resp. } \varphi(z) \geq \varphi(z_0) \ \text{ for any } z \in (\{z_0\} + Z_+) \cap A\ ).$$

(b) A functional $\varphi : A \longrightarrow R$ is called strongly monotonically increasing with respect to the lower (resp. upper) section on A at z_0 if

$$\varphi(z) < \varphi(z_0) \ \text{ for any } z \in (\{z_0\} + Z_-) \cap A, z \neq z_0$$

$$(\ \text{resp. } \varphi(z) > \varphi(z_0) \ \text{ for any } z \in (\{z_0\} + Z_+) \cap A, z \neq z_0\ ).$$

(c) A functional $\varphi : A \longrightarrow R$ is called strictly monotonically increasing with respect to the lower (resp. upper) section on A at z_0 if

$$\varphi(z) < \varphi(z_0) \ \text{ for any } z \in (\{z_0\} + \mathrm{int} Z_-) \cap A$$

$$(\ \text{resp. } \varphi(z) > \varphi(z_0) \ \text{ for any } z \in (\{z_0\} + \mathrm{int} Z_+) \cap A\ ).$$

Also, we will define a semi-saddle point for a pair of functionals, which is also known as a Nash equilibrium point for a two-person nonzero-sum game in game theory (e.g., see [13, p.287]).

DEFINITION 2.3. Let two (real-valued) functionals g_1 and g_2 be defined on $X \times Y$.

(a) A point (x_0, y_0) is said to be a semi-saddle point of (g_1, g_2) with respect to $X \times Y$, a semi-saddle point of (g_1, g_2) for short, if

$$g_1(x_0, y_0) \leq g_1(x, y_0),$$

and

$$g_2(x_0, y_0) \geq g_2(x_0, y),$$

for any $x \in X$ and $y \in Y$.

(b) A point (x_0, y_0) is said to be a strict semi-saddle point of (g_1, g_2) with respect to $X \times Y$, a strict semi-saddle point of (g_1, g_2) for short, if

$$g_1(x_0, y_0) < g_1(x, y_0),$$

and

$$g_2(x_0, y_0) > g_2(x_0, y),$$

for any $x \in X$, $x \neq x_0$ and $y \in Y$, $y \neq y_0$.

If $g_1 = g_2$, then a semi- (resp. strict semi-) saddle point (x_0, y_0) of (g_1, g_2) is an ordinary saddle (resp. strict saddle) point of g_1.

Now, we will state a basic characterization of each type of the cone saddle points. Here, "l.s.c." is the abbreviation for lower semicontinuous, "u.s.c." for upper semicontinuous.

THEOREM 2.4. *Let φ_1 and φ_2 be functionals from M into R, and a point $(x_0, y_0) \in X \times Y$ be given.*

(i) *Suppose that the functionals φ_1 and φ_2 are u.s.c. and l.s.c., respectively, linear, strongly monotonically increasing with respect to the lower section on Z_- (or the upper section on Z_+) at $0 \in Z$. If the point (x_0, y_0) is a semi-saddle point of $(\varphi_1 \circ f, \varphi_2 \circ f)$, then (x_0, y_0) is a proper Z_+-saddle point of f.*

(ii) *Suppose that the functionals φ_1 and φ_2 are monotonically increasing with respect to the lower and upper section on M at $f(x_0, y_0)$, respectively. If the point (x_0, y_0) is a strict semi-saddle point of $(\varphi_1 \circ f, \varphi_2 \circ f)$, then (x_0, y_0) is a Z_+-saddle point of f.*

(iii) *Suppose that the functionals φ_1 and φ_2 are strictly monotonically increasing with respect to the lower and upper section on M at $f(x_0, y_0)$, respectively. If the point (x_0, y_0) is a semi-saddle point of $(\varphi_1 \circ f, \varphi_2 \circ f)$, then (x_0, y_0) is a weak Z_+-saddle point of f.*

PROOF. We only prove the part (i) of the assertion. First, in order to show the condition (2) to hold, we suppose to the contrary that

$$f(x_0, y_0) \notin \text{Ext}[f(X, y_0) \mid Z_+].$$

Then, there exists some vector $\hat{x} \in X$ such that

$$f(x_0, y_0) \in f(\hat{x}, y_0) + Z_+ \quad \text{and} \quad f(\hat{x}, y_0) \neq f(x_0, y_0),$$

and hence

$$f(\hat{x}, y_0) - f(x_0, y_0) \in Z_- \setminus \{0\}.$$

This implies

$$\varphi_1(f(\hat{x}, y_0)) < \varphi_1(f(x_0, y_0)),$$

which is a contradiction. Therefore the condition (2) holds, and hence the point (x_0, y_0) is a Z_+-saddle point of f.

Next, in order to show the condition (4) to hold, we suppose to the contrary that

$$0 \notin \mathrm{Ext}[\mathrm{cl}K(f(X, y_0) + Z_+; f(x_0, y_0)) \mid Z_+].$$

Then, there exists a nonzero vector $h \in Z$ such that $0 \in h + Z_+$ and

$$h \in \mathrm{cl}K(f(X, y_0) + Z_+; f(x_0, y_0)). \tag{5}$$

Therefore,

$$(f(x_0, y_0) + h) - f(x_0, y_0) = h \in Z_- \setminus \{0\},$$

which implies

$$\varphi_1(f(x_0, y_0) + h) < \varphi_1(f(x_0, y_0)).$$

Since the functional φ_1 is u.s.c., there exists a (convex symmetric) neighborhood $U \in \mathcal{B}_Z$ such that

$$\varphi_1(f(x_0, y_0) + z) < \varphi_1(f(x_0, y_0))$$

for any $z \in h + U$. Since the functional φ_1 is linear, we have

$$\varphi_1(z) < 0 \quad \forall z \in h + U. \tag{6}$$

On the other hand, it follows that there exist a vector $\hat{z} \in h + U$ and a scalar $\hat{t} > 0$ such that

$$f(x_0, y_0) + \hat{t}\hat{z} \in f(X, y_0) + Z_+,$$

from (5). Therefore there exist a vector $\hat{x} \in X$ and a vector $\hat{z}_+ \in Z_+$ such that

$$\hat{t}\hat{z} = f(\hat{x}, y_0) - f(x_0, y_0) + \hat{z}_+.$$

Since the functional φ_1 is linear, we have

$$\hat{t}\varphi_1(\hat{z}) \geq \varphi_1(\hat{z}_+) \geq 0.$$

Hence, $\varphi_1(\hat{z}) \geq 0$, which is a contradiction to the condition (6). Thus the condition (4) holds, and then the point (x_0, y_0) is a proper Z_+-saddle point of f. □

In order to give a few corollaries to the theorem, for a vector-valued function f and each (continuous) linear functional $z^* \in Z^*$, we will consider the z^*-scalarized function which is defined by

$$\overline{f_{z^*}}(x, y) := \langle z^*, f(x, y) \rangle.$$

COROLLARY 2.5. (i) *If there exist* $z_1^*, z_2^* \in Z_+^{*i}$ *such that a point* $(x_0, y_0) \in X \times Y$ *is a semi-saddle point of* $(\overline{f_{z_1^*}}, \overline{f_{z_2^*}})$, *then the point* (x_0, y_0) *is a proper* Z_+*-saddle point of* f.

(ii) *If there exist* $z_1^*, z_2^* \in Z_+^*$ *such that a point* $(x_0, y_0) \in X \times Y$ *is a strict semi-saddle point of* $(\overline{f_{z_1^*}}, \overline{f_{z_2^*}})$ (*thus* $z_1^*, z_2^* \in Z_+^* \setminus \{0\}$ *necessarily*), *then the point* (x_0, y_0) *is a* Z_+*-saddle point of* f.

(iii) *If there exist* $z_1^*, z_2^* \in Z_+^* \setminus \{0\}$ *such that a point* $(x_0, y_0) \in X \times Y$ *is a semi-saddle point of* $(\overline{f_{z_1^*}}, \overline{f_{z_2^*}})$, *then the point* (x_0, y_0) *is a weak* Z_+*-saddle point of* f.

PROOF. It is easily proved as shown in [8, Cor.2.3] that each functional $z^* \in Z_+^{*i}$ (resp. $z^* \in Z_+^*$, $z^* \in Z_+^* \setminus \{0\}$) is strongly monotonically (resp. monotonically, strictly monotonically) increasing with respect to the lower and upper section on Z at any point of Z. The assertion is clear by Theorem 2.4. □

COROLLARY 2.6. (i) *If there exists* $z^* \in Z_+^{*i}$ *such that a point* $(x_0, y_0) \in X \times Y$ *is a saddle point of* $\overline{f_{z^*}}$, *then the point* (x_0, y_0) *is a proper* Z_+*-saddle point of* f.

(ii) *If there exists* $z^* \in Z_+^*$ *such that a point* $(x_0, y_0) \in X \times Y$ *is a strict saddle point of* $\overline{f_{z^*}}$ (*thus* $z^* \in Z_+^* \setminus \{0\}$ *necessarily*), *then the point* (x_0, y_0) *is a* Z_+*-saddle point of* f.

(iii) *If there exists* $z^* \in Z_+^* \setminus \{0\}$ *such that a point* $(x_0, y_0) \in X \times Y$ *is a saddle point of* $\overline{f_{z^*}}$, *then the point* (x_0, y_0) *is a weak* Z_+*-saddle point of* f.

PROOF. The proof is straightforward from Corollary 2.5. □

REMARK 2.7. Theorem 3.3 in [19] follows directly from (i) of Corollary 2.6. Also, we notice that every one of Corollaries 3.1-3.4 in [19] is an existence theorem for proper Z_+ -saddle points.

3. Necessary conditions for the cone saddle points

In this section, we will formulate some necessary conditions for each type of the cone saddle points. They are converse to Corollary 2.5. For this purpose, we will start with a few definitions, which are usual notions in vector optimization.

DEFINITION 3.1. Let A and D be a subset and a cone in Z, respectively. The set A is said to be D-convex if the set $A + D$ is a convex set in Z.

With a nonempty subset A of Z, we associate the cone generated by the set A which is defined by

$$\text{cone}[A] := \{\lambda a \mid \lambda \geq 0 \text{ and } a \in A\}.$$

Also, for a point $(x_0, y_0) \in X \times Y$, we define the following cones of Z:

$$K_1 := \text{cone}\,[f(X, y_0) + Z_+ - f(x_0, y_0)]\,,$$

$$K_2 := \text{cone}\,[f(x_0, Y) + Z_- - f(x_0, y_0)]\,.$$

THEOREM 3.2. *Let a point $(x_0, y_0) \in X \times Y$ be given.*

(i) *Suppose that the pointed convex cone Z_+ has a compact base; i.e.,*

$$Z_+ = \bigcup_{\lambda \geq 0} \lambda B$$

where the set B is compact and convex, and $0 \notin B$. If the point (x_0, y_0) is a proper Z_+-saddle point of f such that the sets $f(X, y_0)$ and $f(x_0, Y)$ are Z_+-convex and Z_--convex, respectively, then there exist $z_1^*, z_2^* \in Z_+^{*i}$ such that the point (x_0, y_0) is a semi-saddle point of $(\overline{f_{z_1^*}}, \overline{f_{z_2^*}})$.

(ii) *Suppose that the two cones $K_1 \setminus \{0\}$ and $K_2 \setminus \{0\}$ are open. If the point (x_0, y_0) is a Z_+-saddle point of f such that the sets $f(X, y_0)$ and $f(x_0, Y)$ are Z_+-convex and Z_--convex, respectively, then there exist $z_1^*, z_2^* \in Z_+^* \setminus \{0\}$ such that the point (x_0, y_0) is a strict semi-saddle point of $(\overline{f_{z_1^*}}, \overline{f_{z_2^*}})$.*

(iii) *If the point (x_0, y_0) is a weak Z_+-saddle point of f such that the sets $f(X, y_0)$ and $f(x_0, Y)$ are Z_+-convex and Z_--convex, respectively, then there exist $z_1^*, z_2^* \in Z_+^* \setminus \{0\}$ such that the point (x_0, y_0) is a semi-saddle point of $(\overline{f_{z_1^*}}, \overline{f_{z_2^*}})$.*

PROOF. (i) We assume that the point (x_0, y_0) is a proper Z_+-saddle point of f, and that the sets $f(X, y_0)$ and $f(x_0, Y)$ are Z_+-convex and Z_--convex, respectively. It suffices to prove that there exist $z_1^*, z_2^* \in Z_+^{*i}$ such that

$$\overline{f_{z_1^*}}(x_0, y_0) \leq \overline{f_{z_1^*}}(x, y_0) \quad \forall x \in X, \tag{7}$$

$$\overline{f_{z_2^*}}(x_0, y_0) \geq \overline{f_{z_2^*}}(x_0, y) \quad \forall y \in Y. \tag{8}$$

We only prove the existence of $z_2^* \in Z_+^{*i}$ satisfying the condition (8). Let

$$N := \mathrm{cl}K(f(x_0, Y) + Z_-; f(x_0, y_0)),$$

then the set N is a closed convex cone and it follows that

$$f(x_0, Y) + Z_- - f(x_0, y_0) \subset N,$$

from the condition (1). Since the point (x_0, y_0) is a proper Z_+-saddle point, we have

$$N \cap Z_+ = \{0\}.$$

By [1, Prop.3], there is some $z_2^* \in Z_+^{*i}$ satisfying the following condition:

$$\langle -z_2^*, z \rangle \geq 0 \quad \forall z \in N,$$

which implies that the condition (8) holds.

(ii) We assume that the point (x_0, y_0) is a Z_+-saddle point of f, and that the sets $f(X, y_0)$ and $f(x_0, Y)$ are Z_+-convex and Z_--convex, respectively. We only prove the existence of $z_1^* \in Z_+^* \setminus \{0\}$ satisfying the following condition:

$$\overline{f_{z_1^*}}(x_0, y_0) < \overline{f_{z_1^*}}(x, y_0) \tag{9}$$

for any $x \in X$, $x \neq x_0$. By assumption, we have

$$(f(X, y_0) + Z_+ - f(x_0, y_0)) \cap Z_- = \{0\},$$

and so $(K_1 \setminus \{0\}) \cap Z_- = \emptyset$. Since the set $K_1 \setminus \{0\}$ is an open convex cone and the set Z_- is a convex cone, from Dubovitskii-Milyutin's theorem (e.g., see [6, p.37] or [7, p.116]), it follows that there exists some $z_1^* \in Z_+^* \setminus \{0\}$ satisfying the following condition:

$$\langle z_1^*, z\rangle > 0 \quad \forall z \in K_1 \setminus \{0\}.$$

This implies that the condition (9) holds for any $x \in X$, $x \neq x_0$.

(iii) We assume that the point (x_0, y_0) is a weak Z_+-saddle point of f, and that the sets $f(X, y_0)$ and $f(x_0, Y)$ are Z_+-convex and Z_--convex, respectively. We only prove the existence of $z_1^* \in Z_+^* \setminus \{0\}$ satisfying the condition (7). Let

$$A := f(X, y_0) + Z_+ - f(x_0, y_0),$$

then the set A is a convex set and we have

$$A \cap \mathrm{int} Z_- = \emptyset.$$

By a weak separation theorem [15, Thm. 3.3.3], there is some $z_1^* \in Z^* \setminus \{0\}$ satisfying the following condition:

$$\sup_{z \in \mathrm{int} Z_-} \langle z_1^*, z\rangle \leq \inf_{z \in A} \langle z_1^*, z\rangle.$$

From $0 \in A \cap Z_-$, it follows that

$$\sup_{z \in \mathrm{int} Z_-} \langle z_1^*, z\rangle = \inf_{z \in A} \langle z_1^*, z\rangle = 0,$$

which implies that $z_1^* \in Z_+^* \setminus \{0\}$ and

$$\langle z_1^*, z\rangle \geq 0 \;\; \forall z \in A.$$

This completes the proof. □

REMARK 3.3. In part (i) of the theorem, the pointed convex cone Z_+ is locally compact if and only if it has a compact base, in which case the cone Z_+ is necessarily closed (e.g., see Lemma 1 in [7, p.76]). As for part (ii) of the theorem, we have an another result: Suppose that two cones K_1, K_2 are closed, and the pointed convex cone Z_+ has a compact base. If the point (x_0, y_0) is a Z_+-saddle point of f such that the sets $f(X, y_0)$ and $f(x_0, Y)$ are Z_+-convex, Z_--convex, respectively, then there exist $z_1^*, z_2^* \in Z_+^{*i}$ such that the point (x_0, y_0) is a semi-saddle point of $(\overline{f_{z_1^*}}, \overline{f_{z_2^*}})$. The proof is similar to that of part (i) of the theorem.

COROLLARY 3.4. *Suppose that two sets $f(X, y)$ and $f(x, Y)$ are Z_+-convex and Z_--convex, for any $y \in Y$ and $x \in X$, respectively. Then*

$$S^w = \bigcup_{z_1^*, z_2^* \in Z_+^* \setminus \{0\}} \left\{ \text{ semi–saddle points of } (\overline{f_{z_1^*}}, \overline{f_{z_2^*}}) \right\}.$$

Moreover, if the pointed convex cone Z_+ has a compact base, then

$$S^p = \bigcup_{z_1^*, z_2^* \in Z_+^{*i}} \left\{ \text{ semi–saddle points of } (\overline{f_{z_1^*}}, \overline{f_{z_2^*}}) \right\}.$$

PROOF. The proof is straightforward from Corollary 2.5 and Theorem 3.2. □

In order to present an another corollary, we will review the following definition (e.g., see [19, Def.3.2]).

DEFINITION 3.5. Let X and Y be two convex sets.

(a) A vector-valued function $f : X \times Y \longrightarrow Z$ is said to be Z_+-(convex-concave) in $X \times Y$ if the function f satisfies the following conditions: for each $x \in X$ and $y \in Y$,

(i) $\lambda f(x_1, y) + (1-\lambda) f(x_2, y) \in f(\lambda x_1 + (1-\lambda)x_2, y) + Z_+$ for every $x_1, x_2 \in X$, $\lambda \in [0,1]$, and

(ii) $\mu f(x, y_1) + (1-\mu) f(x, y_2) \in f(x, \mu y_1 + (1-\mu)y_2) + Z_-$ for every $y_1, y_2 \in Y$, $\mu \in [0,1]$.

(b) A vector-valued function $f : X \times Y \longrightarrow Z$ is said to be strictly Z_+-(convex-concave) in $X \times Y$ if the function f satisfies the following conditions: for each $x \in X$ and $y \in Y$,

(i) $\lambda f(x_1, y) + (1-\lambda) f(x_2, y) \in f(\lambda x_1 + (1-\lambda)x_2, y) + \mathrm{int} Z_+$ for every $x_1, x_2 \in X$, $x_1 \neq x_2$, $\lambda \in [0,1]$, and

(ii) $\mu f(x, y_1) + (1-\mu) f(x, y_2) \in f(x, \mu y_1 + (1-\mu)y_2) + \mathrm{int} Z_-$ for every $y_1, y_2 \in Y$, $y_1 \neq y_2$, $\mu \in [0,1]$.

A strictly Z_+-(convex-concave) function is also a Z_+-(convex-concave) function.

COROLLARY 3.6. *If the vector-valued function f is Z_+-(convex-concave), then*

$$S^w = \bigcup_{z_1^*, z_2^* \in Z_+^* \setminus \{0\}} \left\{ \text{semi–saddle points of } (\overline{f_{z_1^*}}, \overline{f_{z_2^*}}) \right\}.$$

Moreover, if the pointed convex cone Z_+ has a compact base, then

$$S^p = \bigcup_{z_1^*, z_2^* \in Z_+^{*i}} \left\{ \text{semi–saddle points of } (\overline{f_{z_1^*}}, \overline{f_{z_2^*}}) \right\}.$$

PROOF. The proof follows directly from Corollary 3.4. □

4. Existence theorems for the cone saddle points

In this section, we will prove some existence theorems for each type of the cone saddle points. To begin with, we define the following set-valued maps T_φ and U_φ for a functional φ from M into R:

$$T_\varphi(y) := \left\{ x_0 \in X \mid \varphi(f(x_0, y)) = \min_{x \in X} \varphi(f(x, y)) \right\}, \quad y \in Y;$$

$$U_\varphi(x) := \left\{ y_0 \in Y \mid \varphi(f(x, y_0)) = \max_{y \in Y} \varphi(f(x, y)) \right\}, \quad x \in X.$$

If, for all $y \in Y$, $T_\varphi(y)$ is a nonempty closed convex subset of X, then the map T_φ is said to be a nonempty closed convex map. Also, the set-valued map T_φ is said to be u.s.c. from Y to X if, for each neighborhood V of the origin $0 \in X$ and each vector $y_0 \in Y$, there exists a neighborhood U of the origin $0 \in Y$ such that $T_\varphi(y) \subset T_\varphi(y_0) + V$ for all $y \in y_0 + U$; see [2, Def.2]. The map U_φ is analogously defined.

First, we give some existence results for proper Z_+-saddle points.

THEOREM 4.1. *Let X and Y be compact convex sets. Suppose that there exist functionals φ_1 and φ_2 from M into R satisfying the following conditions:*

(i) *The set-valued maps T_{φ_1} and U_{φ_2} are nonempty closed convex u.s.c. maps.*

(ii) *The functionals φ_1 and φ_2 are u.s.c. linear and l.s.c. linear, respectively.*

(iii) *The functionals φ_1 and φ_2 are strongly monotonically increasing with respect to the lower section on Z_- (or the upper section on Z_+) at the origin $0 \in Z$.*

Then the vector-valued function f has at least one proper Z_+-saddle point.

PROOF. The proof is based on Browder's coincidence theorem (e.g., see [2] and [16]). By (i), we can use [16, Thm.2.5] and so there exists some point $(x_0, y_0) \in X \times Y$ such that $x_0 \in T_{\varphi_1}(y_0)$ and $y_0 \in U_{\varphi_2}(x_0)$. Consequently, we have

$$\varphi_1(f(x_0, y_0)) \leq \varphi_1(f(x, y_0)) \quad \forall x \in X,$$

and

$$\varphi_2(f(x_0, y_0)) \geq \varphi_2(f(x_0, y)) \quad \forall y \in Y.$$

This implies that the point (x_0, y_0) is a semi-saddle point of $(\varphi_1 \circ f, \varphi_2 \circ f)$. From (i) of Theorem 2.4, it follows directly that the point (x_0, y_0) is a proper Z_+-saddle point of f. □

COROLLARY 4.2. *Let X and Y be compact convex sets, and f a continuous function. Suppose that there exist functionals $z_1^*, z_2^* \in Z^*$ satisfying the following conditions:*

(i) $\forall y \in Y, \quad x \longmapsto \overline{f_{z_1^*}}(x, y)$ *is convex, and*
$\forall x \in X, \quad y \longmapsto \overline{f_{z_2^*}}(x, y)$ *is concave.*

(ii) *The functionals z_1^* and z_2^* are strongly monotonically increasing with respect to the lower section on Z_- (or the upper section on Z_+) at the origin $0 \in Z$.*

Then the vector-valued function f has at least one proper Z_+-saddle point.

PROOF. By assumption, it is easily proved that the corresponding set-valued maps $T_{z_1^*}$ and $U_{z_2^*}$ are nonempty closed convex maps. In order to show that the map $T_{z_1^*}$ is u.s.c., we suppose to the contrary that the map $T_{z_1^*}$ is not u.s.c. at some vector $y_0 \in Y$. Then there is some neighborhood $V \in \mathcal{B}_E$ such that

$$T_{z_1^*}(y_0 + U) \not\subset T_{z_1^*}(y_0) + V \tag{10}$$

for any neighborhood $U \in \mathcal{B}_F$. Let

$$N := (T_{z_1^*}(y_0) + V)^c \cap X.$$

Since the set N is compact and the scalarized function $\overline{f_{z_1^*}}$ is continuous, we have

$$\min_{x\in X}\overline{f_{z_1^*}}(x,y_0) < \min_{x\in N}\overline{f_{z_1^*}}(x,y_0).$$

Also, let

$$\varepsilon := \min_{x\in N}\overline{f_{z_1^*}}(x,y_0) - \min_{x\in X}\overline{f_{z_1^*}}(x,y_0)$$

then $\varepsilon > 0$, and then there is some neighborhood $U_1 \in \mathcal{B}_F$ such that

$$\min_{x\in X}\overline{f_{z_1^*}}(x,y) < \min_{x\in X}\overline{f_{z_1^*}}(x,y_0) + \varepsilon/3 \tag{11}$$

for any $y \in y_0 + U_1$. On the other hand, there is some neighborhood $U_2 \in \mathcal{B}_F$ such that

$$\overline{f_{z_1^*}}(x',y) > \min_{x\in N}\overline{f_{z_1^*}}(x,y_0) - \varepsilon/3 \tag{12}$$

for any $x' \in N$ and $y \in y_0 + U_2$. Here let $U_0 := U_1 \cap U_2$ then, by the condition (10), there exist vectors $\hat{x} \in N$ and $\hat{y} \in y_0 + U_0$ such that

$$\hat{x} \in T_{z_1^*}(\hat{y}). \tag{13}$$

Consequently, by the conditions (11), (12), and (13), we get

$$\overline{f_{z_1^*}}(\hat{x},\hat{y}) = \min_{x\in X}\overline{f_{z_1^*}}(x,\hat{y}) < \overline{f_{z_1^*}}(\hat{x},\hat{y}),$$

which is a contradiction. Similarly, the map $U_{z_2^*}$ is also verified to be u.s.c. Therefore, the assertion follows directly from Theorem 4.1. □

COROLLARY 4.3. *Let X and Y be compact convex sets, and f a continuous function. Suppose that there exist functionals $z_1^*, z_2^* \in Z_+^{*i}$ such that*

$$\forall y \in Y, \quad x \longmapsto \overline{f_{z_1^*}}(x,y) \quad \textit{is convex},$$

and

$$\forall x \in X, \quad y \longmapsto \overline{f_{z_2^*}}(x,y) \quad \textit{is concave}.$$

Then the vector-valued function f has at least one proper Z_+-saddle point.

PROOF. The proof is straightforward from Corollary 4.2. □

In order to present an another corollary, we will review the following definition (e.g., see [18, Def.4.3]).

DEFINITION 4.4. Let X and Y be two convex sets.

(a) A vector-valued function $f : X \times Y \longrightarrow Z$ is said to be properly Z_+-quasi-(convex-concave) in $X \times Y$ if the function f satisfies the following conditions: for each $x \in X$ and $y \in Y$,

(i) $f(x_1,y) \in f(\lambda x_1 + (1-\lambda)x_2, y) + Z_+$ or $f(x_2,y) \in f(\lambda x_1 + (1-\lambda)x_2, y) + Z_+$ for every $x_1, x_2 \in X$, $\lambda \in [0,1]$; and

(ii) $f(x,y_1) \in f(x,\mu y_1+(1-\mu)y_2)+Z_-$ or $f(x,y_2) \in f(x,\mu y_1+(1-\mu)y_2)+Z_-$ for every $y_1, y_2 \in Y$, $\mu \in [0,1]$.

(b) A vector-valued function $f : X \times Y \longrightarrow Z$ is said to be strict-properly Z_+-quasi-(convex-concave) in $X \times Y$ if the function f satisfies the following conditions: for each $x \in X$ and $y \in Y$,

(i) $f(x_1,y) \in f(\lambda x_1+(1-\lambda)x_2,y)+\mathrm{int}Z_+$ or $f(x_2,y) \in f(\lambda x_1+(1-\lambda)x_2,y)+\mathrm{int}Z_+$ for every $x_1, x_2 \in X$, $x_1 \neq x_2$, $\lambda \in (0,1)$; and

(ii) $f(x,y_1) \in f(x,\mu y_1+(1-\mu)y_2)+\mathrm{int}Z_-$ or $f(x,y_2) \in f(x,\mu y_1+(1-\mu)y_2)+\mathrm{int}Z_-$ for every $y_1, y_2 \in Y$, $y_1 \neq y_2$, $\mu \in (0,1)$.

A strict-properly Z_+-quasi-(convex-concave) function is also a properly Z_+-quasi-(convex-concave) function. Also, the conditions expressed in Definitions 3.5 and 4.4 are mutually independent (also, see [4, Prop.4.2]).

COROLLARY 4.5. *Let X and Y be compact convex sets. If the vector-valued function f is continuous and either*

(i) *Z_+-(convex-concave) in $X \times Y$; or*

(ii) *properly Z_+-quasi-(convex-concave) in $X \times Y$,*

then the function f has at least one proper Z_+-saddle point whenever $Z_+^{*i} \neq \emptyset$.

PROOF. The case of (i), whose proof is similar to [19, Cor.3.4], is easily verified by Corollary 4.3. In the case of (ii) it suffices to prove that the corresponding set-valued maps T_{z^*} and U_{z^*} are two convex maps for some $z^* \in Z_+^{*i}$, because it is easily proved that the maps T_{z^*} and U_{z^*} are nonempty closed maps, and that they are u.s.c., in a similar way as the proof of Corollary 4.2. Now, for each vector $y \in Y$, let $x_1, x_2 \in T_{z^*}(y)$, $\lambda \in [0,1]$, then we have $\lambda x_1 + (1-\lambda)x_2 \in X$ and either

$$\overline{f_{z^*}}(x_1,y) - \overline{f_{z^*}}(\lambda x_1+(1-\lambda)x_2,y) \geq 0$$

or

$$\overline{f_{z^*}}(x_2,y) - \overline{f_{z^*}}(\lambda x_1+(1-\lambda)x_2,y) \geq 0.$$

Consequently,

$$\overline{f_{z^*}}(\lambda x_1+(1-\lambda)x_2,y) = \min_{x \in X} \overline{f_{z^*}}(x,y),$$

which implies that the set $T_{z^*}(y)$ is a convex set. Similarly, the set $U_{z^*}(x)$ is also verified to be a convex set for every vector $x \in X$. This completes the proof. □

Next, we present an another existence theorem for proper Z_+-saddle points.

THEOREM 4.6. *Let X be a compact convex set. Suppose that there exist functionals φ_1 and φ_2 from M into R satisfying the following conditions:*

(i) *The set-valued maps T_{φ_1} and U_{φ_2} are nonempty singleton continuous maps.*

(ii) *The functionals* φ_1 *and* φ_2 *are u.s.c. linear and l.s.c. linear, respectively.*

(iii) *The functionals* φ_1 *and* φ_2 *are strongly monotonically increasing with respect to the lower section on* Z_- *(or the upper section on* Z_+*) at the origin* $0 \in Z$.

Then the vector-valued function f *has at least one proper* Z_+*-saddle point.*

PROOF. The proof is based on Tychonoff's fixed point theorem [21]. By the condition (i), the composite mapping $T_{\varphi_1} \circ U_{\varphi_2} : X \longrightarrow X$ is continuous, and hence there exists some vector $x_0 \in X$ such that $x_0 = T_{\varphi_1} \circ U_{\varphi_2}(x_0)$. If we put $y_0 := U_{\varphi_2}(x_0)$, then $x_0 = T_{\varphi_1}(y_0)$. Consequently, we have

$$\varphi_1(f(x_0, y_0)) < \varphi_1(f(x, y_0)),$$

$$\varphi_2(f(x_0, y_0)) > \varphi_2(f(x_0, y)),$$

for any $x \in X$, $x \neq x_0$, and $y \in Y$, $y \neq y_0$. This implies that the point (x_0, y_0) is also a semi-saddle point of $(\varphi_1 \circ f, \varphi_2 \circ f)$. From (i) of Theorem 2.4, it follows directly that the point (x_0, y_0) is a proper Z_+-saddle point of f. □

COROLLARY 4.7. *Let* X *and* Y *be compact convex sets, and* f *a continuous function. Suppose that there exist functionals* $z_1^*, z_2^* \in Z^*$ *satisfying the following conditions:*

(i) $\forall y \in Y, \quad x \longmapsto \overline{f_{z_1^*}}(x, y)$ *is strictly convex, and*
$\forall x \in X, \quad y \longmapsto \overline{f_{z_2^*}}(x, y)$ *is strictly concave.*

(ii) *The functionals* z_1^* *and* z_2^* *are strongly monotonically increasing with respect to the lower section on* Z_- *(or the upper section on* Z_+*) at the origin* $0 \in Z$.

Then the vector-valued function f *has at least one proper* Z_+*-saddle point.*

The proof, which follows directly from Corollary 4.2, is also verified by Theorem 4.6. Secondly, we state some existence results for Z_+-saddle points.

THEOREM 4.8. *Let* X *be a compact convex set. Suppose that there exist functionals* φ_1 *and* φ_2 *from* M *into* R *satisfying the following conditions:*

(i) *The set-valued maps* T_{φ_1} *and* U_{φ_2} *are nonempty singleton continuous maps.*

(ii) *The functionals* φ_1 *and* φ_2 *are monotonically increasing with respect to the lower and upper section on* M *at each point* $z \in M$, *respectively.*

Then the vector-valued function f *has at least one* Z_+*-saddle point.*

PROOF. In a similar way as the proof of Theorem 4.6, it is shown that there exists a strict semi-saddle point (x_0, y_0) of $(\varphi_1 \circ f, \varphi_2 \circ f)$. From (ii) of Theorem 2.4, it follows directly that the point (x_0, y_0) is a Z_+-saddle point of f. □

COROLLARY 4.9. *Let* X *and* Y *be compact convex sets, and* f *a continuous function. Suppose that there exist continuous functionals* φ_1 *and* φ_2 *from* M *into* R *satisfying the following conditions:*

(i) $\forall y \in Y, \quad x \longmapsto \varphi_1 \circ f(x,y)$ *is strictly convex, and*
$\forall x \in X, \quad y \longmapsto \varphi_2 \circ f(x,y)$ *is strictly concave.*

(ii) *The functionals φ_1 and φ_2 are monotonically increasing with respect to the lower and upper section on M at each point $z \in M$, respectively.*

Then the vector-valued function f has at least one Z_+-saddle point.

PROOF. The proof is straightforward from Theorem 4.8. □

COROLLARY 4.10. *Let X and Y be compact convex sets, and f a continuous function. Suppose that there exist nonzero functionals $z_1^*, z_2^* \in Z_+^* \setminus \{0\}$ such that*

$$\forall y \in Y, \quad x \longmapsto \overline{f_{z_1^*}}(x,y) \quad \text{is strictly convex,}$$

and

$$\forall x \in X, \quad y \longmapsto \overline{f_{z_2^*}}(x,y) \quad \text{is strictly concave.}$$

Then the vector-valued function f has at least one Z_+-saddle point.

PROOF. The proof is straightforward from Corollary 4.9. □

COROLLARY 4.11. *Let X and Y be compact convex sets. If the vector-valued function f is continuous and either*

(i) *strictly Z_+-(convex-concave) in $X \times Y$; or*

(ii) *strict-properly Z_+-quasi-(convex-concave) in $X \times Y$,*

then the function f has at least one Z_+-saddle point.

PROOF. Since $\mathrm{int}Z_+ \neq \emptyset$, we have $Z_+^* \setminus \{0\} \neq \emptyset$ (e.g., see [15, Thm.3.3.3]). Therefore, the assertion of the case of (i) follows from Corollary 4.10. In the case of (ii), it suffices to prove that the corresponding set-valued maps T_{z^*} and U_{z^*} are singleton maps for some $z^* \in Z_+^*$. Now, for each $y \in Y$, let $x_1, x_2 \in T_{z^*}(y)$, $x_1 \neq x_2$, $\lambda \in (0,1)$, then we have $\lambda x_1 + (1-\lambda)x_2 \in X$ and either

$$\overline{f_{z^*}}(x_1,y) - \overline{f_{z^*}}(\lambda x_1 + (1-\lambda)x_2, y) > 0$$

or

$$\overline{f_{z^*}}(x_2,y) - \overline{f_{z^*}}(\lambda x_1 + (1-\lambda)x_2, y) > 0.$$

Consequently,

$$\overline{f_{z^*}}(\lambda x_1 + (1-\lambda)x_2, y) < \min_{x \in X} \overline{f_{z^*}}(x,y),$$

which is a contradiction. Similarly, the map U_{z^*} is also verified to be a singleton map. This completes the proof. □

REMARK 4.12. Corollary 4.11 is a generalization of Corollary 4.1 in [18].

Thirdly, we present some existence results for weak Z_+-saddle points.

THEOREM 4.13. *Let X and Y be compact convex sets. Suppose that there exist functionals φ_1 and φ_2 from M into R satisfying the following conditions:*

(i) *The set-valued maps T_{φ_1} and U_{φ_2} are nonempty closed convex u.s.c. maps.*

(ii) *The functionals φ_1 and φ_2 are strictly monotonically increasing with respect to the lower and upper section on M at each point $z \in M$, respectively.*

Then the vector-valued function f has at least one weak Z_+-saddle point.

PROOF. In a similar way as the proof of Theorem 4.1, it is shown that there exists a semi-saddle point (x_0, y_0) of $(\varphi_1 \circ f, \varphi_2 \circ f)$. From (iii) of Theorem 2.4, it follows directly that the point (x_0, y_0) is a weak Z_+-saddle point of f. □

The following corollaries are easy to prove; review the corresponding previous corollaries if necessary.

COROLLARY 4.14. *Let X and Y be compact convex sets, and f a continuous function. Suppose that there exist continuous functionals φ_1 and φ_2 from M into R satisfying the following conditions:*

(i) $\forall y \in Y, \quad x \longmapsto \varphi_1 \circ f(x,y)$ *is convex, and*
$\forall x \in X, \quad y \longmapsto \varphi_2 \circ f(x,y)$ *is concave.*

(ii) *The functionals φ_1 and φ_2 are strictly monotonically increasing with respect to the lower and upper section on M at each point $z \in M$, respectively.*

Then the vector-valued function f has at least one weak Z_+-saddle point.

COROLLARY 4.15. *Let X and Y be compact convex sets, and f a continuous function. Suppose that there exist functionals $z_1^*, z_2^* \in Z_+^* \setminus \{0\}$ such that*

$$\forall y \in Y, \quad x \longmapsto \overline{f_{z_1^*}}(x,y) \quad \textit{is convex},$$

and

$$\forall x \in X, \quad y \longmapsto \overline{f_{z_2^*}}(x,y) \quad \textit{is concave}.$$

Then the vector-valued function f has at least one weak Z_+-saddle point.

COROLLARY 4.16. *Let X and Y be compact convex sets. If the vector-valued function f is continuous and either*

(i) *Z_+-(convex-concave) in $X \times Y$; or*

(ii) *properly Z_+-quasi-(convex-concave) in $X \times Y$,*

then the functon f has at least one weak Z_+-saddle point.

REMARK 4.17. Corollaries 4.3, 4.10, and 4.15 are specifications of Corollary 3.2 in [19]. Also, Corollaries 4.5, 4.11, and 4.16 are specifications of Corollary 3.4 in [19].

Finally, we have the following theorem which is a specification of Theorem 3.2 in [19].

THEOREM 4.18. *Let X and Y be compact sets. Suppose that f is a separated continuous vector-valued function of the type*

$$f(x,y) = u(x) + v(y)$$

where $u : X \longrightarrow Z$ and $v : Y \longrightarrow Z$. Then the function f has at least one weak Z_+-saddle point. Moreover, if $Z_+^{*i} \neq \emptyset$, then the function f has at least one proper Z_+-saddle point.

PROOF. Since $Z_+^* \setminus \{0\} \neq \emptyset$ (e.g., see the proof of Corollary 4.11), we have some $z^* \in Z_+^* \setminus \{0\}$. By assumption there exists some point $(x_0, y_0) \in X \times Y$ such that

$$\langle z^*, u(x_0)\rangle = \min_{x\in X}\langle z^*, u(x)\rangle,$$

$$\langle z^*, v(y_0)\rangle = \max_{y\in Y}\langle z^*, v(y)\rangle.$$

This implies that

$$\overline{f_{z^*}}(x_0, y) \leq \overline{f_{z^*}}(x_0, y_0) \leq \overline{f_{z^*}}(x, y_0).$$

From (iii) of Corollary 2.6, it follows directly that the point (x_0, y_0) is a weak Z_+-saddle point of f. Similarly, the remainder of the proof follows directly from (i) of Corollary 2.6. □

5. Conclusions

In this paper, we have introduced a notion of proper cone saddle points in addition to notions of cone saddle points and weak cone saddle points for a vector-valued function treated in [18] and [19]. By adopting the same setting as in [19], we have characterized each type of the cone saddle points via scalarization. The characterization consists of three parts: sufficient conditions, necessary conditions, and several existence results for each type of the cone saddle points. In general, scalarization means the replacement of a vector optimization problem by a suitable scalar optimization problem. From this point of view, we have introduced a suitable notion for the generalized saddle points. For scalar-valued functions, there is a notion of ordinary saddle points. For the generalization, we have defined a notion of semi-saddle points for a pair of scalar-valued functions, which is also known as "Nash equilibrium points" for a two-person nonzero-sum game in game theory. Through the characterization, we have got some connection between each type of the cone saddle points of a vector-valued function and the corresponding type of the semi-saddle points of a pair of the scalarized functions. Also, we have successfully generalized and specified several results in [19].

References

[1] J. Borwein, *Proper efficient points for maximizations with respect to cones*, SIAM J. Control Optim., 15(1977), 57-63.

[2] F. E. Browder, *Coincidence theorems, minimax theorems, and variational inequalities*, Contemp. Math., 26(1984), 67-80.

[3] F. H. Clarke, *Optimization and nonsmooth analysis*, John Wiley, New York, 1983.

[4] F. Ferro, *Minimax type theorems for n-valued functions*, Ann. Mat. Pura Appl.(4), 32(1982), 113-130.

[5] J. R. Giles, *Convex analysis with application in differentiation of convex functions*, Research Notes in Mathematics, No.58, Pitman, London, 1982.

[6] I. V. Girsanov, *Lectures on mathematical theory of extreme problems*, Lecture Notes in Economics and Mathematical Systems, No.67, Springer-Verlag, New York, 1972.

[7] R. B. Holmes, *Geometric functional analysis and its applications*, Springer-Verlag, New York, 1975.

[8] J. Jahn, *Scalarization in vector optimization*, Math. Programming, 29(1984), 203-218.

[9] J. Jahn, *A characterization of properly minimal elements of a set*, SIAM J. Control Optim., 23(1985), 649-656.

[10] J. Jahn, *Existence theorems in vector optimization*, J. Optim. Theory Appl., 50(1986), 397-406.

[11] D. G. Luenberger, *Optimization by vector space methods*, John Wiley, New York, 1969.

[12] H. Nakayama, *Geometric consideration of duality in vector optimization*, J. Optim. Theory Appl., 44(1984), 625-655.

[13] J. Nash, *Non-cooperative games*, Ann. of Math., 54(1951), 286-295.

[14] J. W. Nieuwenhuis, *Some minimax theorems in vector-valued functions*, J. Optim. Theory Appl., 40(1983), 463-475.

[15] J. Ponstein, *Approaches to the theory of optimization*, Cambridge University Press, Cambridge, 1980.

[16] S. Simons, *Cyclical coincidences of multivalued maps*, J. Math. Soc. Japan, 38(1986), 515-525.

[17] T. Tanaka, *On cone-extreme points in R^n*, Science Reports of Niigata University, 23(1987), 13-24.

[18] T. Tanaka, *Some minimax problems of vector-valued functions*, J. Optim. Theory Appl., 59(1988), 505-524.

[19] T. Tanaka, *Existence theorems for cone saddle points of vector-valued functions in infinite-dimensional spaces*, J. Optim. Theory Appl., 62(1989), 127-138.

[20] T. Tanino and Y. Sawaragi, *Duality theory in multiobjective programming*, J. Optim. Theory Appl., 27(1979), 509-529.

[21] A. Tychonoff, *Ein fixpunktsatz*, Math. Ann., 111(1935), 767-776.

[22] P. L. Yu, *Cone convexity, cone extreme points, and nondominated solutions in decision problems with multiobjectives*, J. Optim. Theory Appl., 14(1974), 319-377.

Tamaki Tanaka
Department of Information Science
Faculty of Science
Hirosaki University
3, Bunkyo-cho, Hirosaki Aomori 036
Japan

Received May 11, 1990