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Cartan hypersurfaces and reflections

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Abstract

One gives a characterization of the Cartan hypersurfaces in spheres by means of volume-preserving local reflections.

1. Introduction and statement of the results

In this short note we will treat some geometrical properties of a special class of minimal hypersurfaces M embedded in a sphere $S^{n+1}(c)$ of curvature c. We always suppose M to be connected and compact.

We start with the definition of this class.

Definition. A *Cartan hypersurface* in a sphere $S^{n+1}(c)$ is a compact hypersurface with principal curvatures $-(3c)^{1/2}$, 0, $(3c)^{1/2}$ with the same multiplicity.

These hypersurfaces were discovered by E.Cartan in his work about isoparametric hypersurfaces in real space forms [2], [3]. First, he discovered the socalled classical Cartan hypersurface in $S^4(1)$. It is the only complete hypersurface, up to congruence, with three distinct constant principal curvatures. Further, it is an "algebraic" manifold defined by a polynomial of order three. It is minimally embedded and moreover, it is a homogeneous space $SO(3)/\mathbb{Z}_2 \times \mathbb{Z}_2$ which may be viewed as a tube of radius $\pi/2$ about a Veronese surface. (See also [6] for a description.) Next, E.Cartan also proved that these hypersurfaces only exist when $n = 3, 6, 12, 24$ and that the compact ones are always homogeneous.

Many authors studied *isoparametric hypersurfaces*, i.e. hypersurfaces with constant principal curvatures, in real space forms. Every family of isoparametric hypersurfaces contains a unique minimal one and the Cartan hypersurfaces are the compact ones where there are exactly three distinct principal curvatures. In the reference list we give

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a number of references about the general theory of isoparametric families and also about the characterization of the Cartan hypersurfaces using one or other special property of their geometry. In particular, see [1]-[3], [6], [8], [9], [12], [14]-[19], [27].

Here we shall concentrate on another aspect of their rich geometry. Let (M, g) be a Riemannian manifold and let P be an embedded submanifold with compact closure. Then we may define the socalled *local reflection* φ_P with respect to P. In [4], [5], [7], [20]-[22] we studied the properties of these reflections in relation with the properties of the curvature of the ambient space (M, g) and the extrinsic and intrinsic geometry of the submanifold P. See [24], [25], [26] for a survey and for more references. In this paper we shall use the local reflections with respect to embedded compact hypersurfaces to give a characterisation of the Cartan hypersurfaces by means of ***volume-preserving*** reflections. This is a quite natural direction since volume-preserving reflections φ_P lead at once to ***minimal*** submanifolds P [20]. Note that an ***isometric*** reflection φ_P leads to a ***totally geodesic*** submanifold P, as is well-known (see for example [20]).

More precisely, we shall prove the following theorems.

Theorem 1. *The Cartan hypersurface in $S^4(c)$ is, up to congruence, determined by the following property: it is the only compact hypersurface P in $S^4(c)$ with constant scalar curvature and such that the local reflection φ_P is a non-isometric volume-preserving diffeomorphism.*

Theorem 2. *Let P be a compact hypersurface in $S^{n+1}(c)$ with constant scalar curvature and such that the shape operator has exactly three different eigenvalues. Then P is a Cartan hypersurface if and only if φ_P is a volume-preserving non-isometric local reflection.*

Note that we cannot delete the constant scalar curvature condition. See [1] for an example of a hypersurface P in $S^4(1)$ with nonconstant scalar curvature and volume-preserving reflection φ_P.

2. Preliminaries

Before giving the proofs of these theorems we recall some facts about the geometry in a tubular neighborhood of a submanifold. We refer to [10], [11], [26] for more details.

Let (M, g) be an n-dimensional Riemannian manifold and let P be a q-dimensional (connected) embedded submanifold which is relatively compact. Further, let $\mathcal{U}_P(r)$ denote a tubular neighborhood about P, i.e.

$$\mathcal{U}_P(r) = \{m \in M \mid \text{there exists a geodesic } \gamma \text{ with length } L(\gamma) \le r \text{ from } m \text{ to } P \text{ meeting } P \text{ orthogonally } \}.$$

We always suppose that r is smaller than the distance from P to its nearest focal point.

To describe this neighborhood we use Fermi coordinates. Therefore, let $m \in P$ and let $\{E_1, \ldots, E_n\}$ be a local orthonormal frame field of (M, g) defined along P in a neighborhood of m. We specialize the moving frame such that $E_1, \ldots, E_q$ are tangent vector fields and $E_{q+1}, \ldots, E_n$ normal vector fields of P. Further, let $(y_1, \ldots, y_q)$ be a system of coordinates in a neighborhood of m in P such that $\frac{\partial}{\partial y_i}(m) = E_i(m)$, $i = 1, \ldots, q$. Then the Fermi coordinates $(x_1, \ldots, x_n)$ with respect to m, $(y_1, \ldots, y_q)$ and $E_{q+1}, \ldots, E_n$ are defined in an open neighborhood $\mathcal{U}_m$ of m in M by

$$x_i(\exp_\nu \sum_{q+1}^{n} t_\beta E_\beta) = y_i, \quad i = 1, \ldots, q,$$

$$x_a(\exp_\nu \sum_{q+1}^{n} t_\beta E_\beta) = t_a, \quad a = q+1, \ldots, n$$

where $\exp_\nu$ is the restriction of the exponential map to the normal bundle $\nu = T^\perp P$ of P.

Now, consider the local diffeomorphism

$$\varphi_P : p \mapsto \varphi_P(p), \; p = \exp_m(ru) \mapsto \varphi_P(p) = \exp_m(-ru)$$

for $u \in T_m^\perp P$, $\|u\| = 1$. φ_P is called the *local reflection with respect to the submanifold* P. Using the previously defined Fermi coordinates, φ_P is locally given by

$$\varphi_P : (x_1, \ldots, x_q, x_{q+1}, \ldots, x_n) \mapsto (x_1, \ldots, x_q, -x_{q+1}, \ldots, -x_n).$$

Next, let θ_P denote the volume density function defined by

$$\theta_P = (\det(g_{ij}))^{1/2}, \quad i, j = 1, \ldots, n.$$

Then we have

Lemma 3. *The local reflection φ_P is volume-preserving up to sign if and only if*

$$\theta_P(\exp_m(ru)) = \theta_P(\exp_m(-ru))$$

for all normal unit vecors $u \in T_m^\perp P$, all $m \in P$ and all sufficiently small r.

The determination of θ_P is closely related to the study of the Jacobi differential equation. A nice formula for θ_P is obtained in [13] (see also [23]) using the notion of the Wronskian of two solutions of the Jacobi equation. The remarkable feature of this formula is that θ_P is expressed completely by quantities relating only to the extrinsic geometry of P and the geodesic spheres of the ambient space which are tangent to P. In fact we have

Lemma 4.

$$\theta_P(p) = r^q \theta_m(p) \det(T(u) + B_u(r))$$

where θ_m is the volume density function of $\exp_m$ in (M, g), $T(u)$ the shape operator of P with respect to the unit vector u and

$$B_u(r)_{ij} = g(T_p(m)E_i, E_j)(m), \quad i, j = 1, \ldots, q.$$

Here $T_p(m)$ is the shape operator at m of the geodesic sphere $G_p(r)$ with center p and radius r.

To prove our theorems we will need the explicit expression when $M = S^{n+1}(c)$ and $q = n$. Then, the solution of the Jacobi equation becomes extremely easy and we get [10], [11], [26]

Lemma 5. *Let P be a hypersurface in $S^{n+1}(c)$. Then we have*

$$\theta_P(p) = \left(\frac{\sin\sqrt{c}r}{\sqrt{c}}\right)^n \det(T(u) + \sqrt{c}\cot\sqrt{c}rI).$$

From this we get the basic

Proposition 6. *The local reflection φ_P with respect to the hypersurface P in $S^{n+1}(c)$ is volume-preserving if and only if all the odd elementary symmetric functions of the principal curvatures vanish.*

This proposition is a special case of a more general theorem [20] for a class of submanifolds in locally symmetric spaces. We note again that when φ_P is an isometry, then $T(u) = 0$, i.e. P is totally geodesic.

3. Proof of the results

To prove our results we shall need two properties for isoparametric hypersurfaces in $S^{n+1}(c)$ which we recall first.

Proposition 7 [3], [14]. *Let P be an isoparametric hypersurface in $S^{n+1}(c)$ with exactly three different principal curvatures. Then these curvatures have the same multiplicity.*

Proposition 8 [16]. *Let P be a minimal isoparametric hypersurface in $S^{n+1}(c)$ with p distinct principal curvatures. Then the square $\|T\|^2$ of the length of the second fundamental form satisfies*

$$\|T\|^2 = (p-1)nc.$$

Now we are ready to give the proofs.

Proof of Theorem 1.

First, let P be a compact hypersurface in $S^4(c)$ with constant scalar curvature and suppose that φ_P is volume-preserving but not isometric. Then Proposition 6 implies that P is minimal and the principal curvatures are $-\lambda, 0, \lambda$ with $\lambda \neq 0$. Further, the scalar curvature τ of P is related to λ by the formula

$$\tau = 6c - 2\lambda^2.$$

This is an easy consequence of the Gauss equation. It shows that λ is constant and then Proposition 8 implies $\lambda^2 = 3c$ (i.e. $\tau = 0$ [17]) and the result follows.

Conversely, let P be a Cartan hypersurface in $S^4(c)$. Then it follows from the definition that the conditions in Proposition 6 are fulfilled and so, φ_P is volume-preserving.

Proof of Theorem 2.

Let P be a Cartan hypersurface in $S^{n+1}(c)$. Then the principal curvatures are $-(3c)^{1/2}$, 0, $(3c)^{1/2}$ with multiplicity $n/3$. Hence, the results follow again from Proposition 6.

Conversely, let P be a hypersurface in $S^{n+1}(c)$ with only three distinct principal curvatures and suppose that φ_P is volume-preserving but not isometric. Since the odd elementary symmetric functions of the principal curvatures vanish, we can only have $-\lambda, 0, \lambda$ with $\lambda \neq 0$. Further, the constancy of the scalar curvature implies again that λ is constant. Then Proposition 7 yields that they have the same multiplicity and finally, from Proposition 8 we get $\lambda^2 = 3c$, which proves the required result.

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