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TOTALLY REAL SUBMANIFOLDS OF S^6

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ABSTRACT: We obtain an estimate for the index of a 3-dimensional compact totally real submanifold of the nearly Kaehler six dimensional sphere S^6.

A six dimensional unit sphere S^6 has an almost complex structure J defined by the vector cross product in the space of purely imaginary cayley numbers. This almost complex structure is not integrable and satisfies $(\bar{\nabla}_{\bar{X}} J)(\bar{X}) = 0$ for any vector field $\bar{X}$ on S^6, where $\bar{\nabla}$ is the Riemannian connection on S^6 (and hence S^6 is a nearly Kaehler manifold) (cf.[2]). It is also known that there does not exist a 4-dimensional almost complex submanifold of S^6 (cf.[2]). However there are 3-dimensional totally real submanifolds of S^6 (cf.[1]). 3-dimensional totally real submanifolds of S^6 are minimal and orientable ([1]).

Let M be a 3-dimensional totally real submanifold of S^6 with the tangent bundle TM and the normal bundle ν. We denote by the same letter g the Riemannian metric on S^6 as well as that induced on M. The Riemannian connection $\bar{\nabla}$ induces the Riemannian connection ∇ on M and the connection $\nabla^{\perp}$ in the normal bundle ν and we have the following Gauss and Weingarten formulae

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$$(1) \quad \bar{\nabla}_X Y = \nabla_X Y + h(X,Y)$$

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \quad , \quad X,Y \in \chi(M),\ N \in \nu,$$

where h is the second fundamental form and the tangential component $A_N X$ of $-\bar{\nabla}_X N$ is related to the second fundamental form h by $g(h(X,Y),N) = g(A_N X, Y)$, $X,Y \in \chi(M)$, $\chi(M)$ is the Lie-algebra of smooth vector fields on M. Since M is totally real, for each normal vector field N there exists $X \in \chi(M)$ such that $JX = N$. Define a tensor field G of type (1,2) on S^6 by $G(X,Y) = (\bar{\nabla}_{\bar{X}} J)(\bar{Y})$, $\bar{X},\bar{Y} \in \chi(S^6)$. Then using (1) and $(\bar{\nabla}_{\bar{X}} J)(\bar{X}) = 0$, it is easy to get the following (cf. [1]):

(2) $h(X,Y) = JA_{JY}X$, $\nabla^{\perp}_X JY = J\nabla_X Y + G(X,Y)$, and $G(X,Y) \in \nu$, for $X,Y \in \chi(M)$.

It is also observed that if $\{e_i\} = \{e_1, e_2, e_3\}$ is an orthonormal basis of the tangent space T_pM of M at $p \in M$, then

$$JG(e_2,e_2) = e_3,\ JG(e_2,e_3) = e_1,\ JG(e_3,e_1) = e_2.$$

In the present note, we assume that 3-dimensional totally real submanifold M of S^6 is without boundary and $f: M \to S^6$ is the totally real immersion. Let f_t be the normal variation of f induced from a normal vector field $N \in \nu$. Then, since M is minimal, we have the following second variation formula (cf. [3]).

$$\alpha''(N) = \int_M \{\|\nabla^{\perp} N\|^2 - \bar{R}(N) - \|A_N\|^2\} dV,$$

where $\bar{R}(N) = \sum_{i=1}^{3} \bar{R}(N,e_i\,;\, e_i,N)$, $\bar{R}$ being the curvature tensor of S^6 and $\{e_i\} = \{e_1,e_2,e_3\}$ is any oriented orthonormal basis of T_pM at any $p\in M$. It is well-known that every compact minimal submanifold without boundary in a sphere is not stable. In fact, J.Simons has proved the following

PROPOSITION: Let M be a compact closed m-dimensional minimal submanifold immersed in S^n. Then the index of M is greater than or equal to (n−m) and equality holds only when M is S^m. The nullity of M is greater than or equal to (m+1)(n−m), and equality holds only when M is S^m.

The object of the present note is to improve the above result in the special case where M is a 3-dimensional compact totally real submanifold of S^6. Namely, we shall prove the following

THEOREM. Let M be a 3-dimensional compact totally real submanifold of S^6. Then the index of M is greater than or equal to $3+b_1$, where b_1 is the first Betti number of M.

In order to prove the above Theorem, we shall prepare a second variation formula for a 3-dimensional compact totally real submanifold. Let M be a 3-dimensional compact totally real submanifold of S^6. If R is the curvature tensor of M, from Gauss equation we have

$$R(X,Y;Z,W) = g(Y,Z)g(X,W) - g(X,Z)g(Y,W) + g\big(h(Y,Z),h(X,W)\big) - g\big(h(X,Z),h(Y,W)\big),$$

from which using the minimality of M, we obtain the following expression for the Ricci tensor of M

$$\mathrm{Ric}\,(X,X) = 2\|X\|^2 - \sum_{i=1}^{3} \|h(X,e_i)\|^2,$$

at each point $p\varepsilon M$, where $\{e_i\} = \{e_1,e_2,e_3\}$ is an oriented orthonormal basis of T_pM. Using (2) in the above equation, we get for any $X\varepsilon\chi(M)$, that

$$(3) \qquad -\|A_N\|^2 = \mathrm{Ric}\,(X,X) - 2\|X\|^2, \; N = JX.$$

Now let $\{X_i\} = \{X_1,X_2,X_3\}$ be the local orthonormal frame field on a neighborhood U of p such that $X_i=e_i$, i=1,2,3 and $\nabla_{e_i}X_j = 0$ (i,j=1,2,3) at p. For any $X\varepsilon\chi(M)$, we denote by η the dual 1-form to X. We may write $X = \sum_{i=1}^{3} a_iX_i$ on U. Then taking account of (1) and the definition of the tensor field G, we get

$$\begin{aligned}(4) \quad \nabla^{\perp}_{e_j} N &= \nabla^{\perp}_{e_j} (JX) \\ &= \sum_{i=1}^{3} \nabla^{\perp}_{e_j}(a_iJX_i) \\ &= \sum_{i=1}^{3} (e_j\cdot a_i)JX_i + G(e_j,X), \; (j=1,2,3)\end{aligned}$$

at p. By the direct calculation, we get

$$(5)\quad 2\sum_{i,j=i}^{3} g\big((e_j a_i)Je_i\ ,\ JG(e_j,X)\big)$$

$$= -2\sum_{j=1}^{3} g(\nabla_{e_j}X,\ JG(e_j,X))$$

$$= -2\sum_{j,k=1}^{3} g(\nabla_{e_j}X,e_k)\ g(JG(e_j,X),\ e_k)$$

$$= -\sum_{j,k=1}^{3} d\eta(e_j,e_k)\ g(G(e_j,e_k),\ JX)$$

at p. We define a smooth function F_X on M by

$$(6)\quad F_X(p) = \sum_{j,k=1}^{3} d\eta(e_j,e_k)g(G(e_j,e_k),\ JX),\ p\in M.$$

Then by (4), (5), (6), we get

$$(7)\quad \|\nabla^{\perp}N\|^2 = \|\nabla^{\perp}JX\|^2 = \|\nabla X\|^2 + 2\|X\|^2 - F_X.$$

Using the expression for the curvature tensor $\bar{R}$ of S^6, we get $\bar{R}(N) = 3\|X\|^2$. Hence, using (3) and (7), the formula for the second variation becomes

$$(8)\quad \alpha''(X) = \alpha''(JX) = \int_M \{\|\nabla X\|^2 + Ric(X,X) - F_X - 3\|X\|^2\}dV.$$

On one hand, computing the divergence of the vector field $W = \nabla_X X + (divX)X$ (where divX is the divergence of X), and using $\int_M (divW)dV = 0$, we obtain (cf. [4])

$$(9)\quad \int_M \{\|\nabla X\|^2 + Ric(X,X) - \frac{1}{2}\|d\eta\|^2 - (\delta\eta)^2\}\, dV = 0.$$

From (8) and (9), we then obtain

$$(10)\quad \alpha''(X) = \alpha''(JX) = \int_M \{\frac{1}{2}\|d\eta\|^2 + (\delta\eta)^2 - F_X - 3\|X\|^2\} dV,$$

for any $X \in \chi(M)$, where η is the 1-form dual to X.

Now we are in a crucial position of the proof of the Theorem. We recall the proof of the proposition by J.Simons. We adopt the same notational convention as in the proof. We denote by Z the induced vector field on S^6 induced by a constant vector field on a 7-dimensional Euclidean space R^7. Restricting Z to M, and projecting onto normal and tangential components, gives cross sections Z^N and Z^T in ν and TM respectively. Then we get (cf.[3], p.85)

$$(11)\quad \nabla^{\perp}_X Z^N = -h(X, Z^T),$$

for any tangent vector $X \in T_pM$, $p \in M$. Then we have

$$\alpha''(Z^N) = -3 \int_M \|Z^N\|^2 dv$$

(cf. [3], Lemma 5.1.4, p.86). Hence we see that the index of M is greater than or equal to 6-3=3. It is easily observed that the 1-form dual to the vector field $JZ^N \in \chi(M)$ is harmonic if and only if

$$(12)\quad \sum_{i=1}^{3} g(\nabla_{e_i}(JZ^N), e_i) = 0, \ (i=1,2,3),$$

and

$$(13)\quad g(\nabla_{e_i}(JZ^N), e_j) - g(\nabla_{e_j}(JZ^N), e_i) = 0$$

$(i,j=1,2,3)$ at each point $p \varepsilon M$.

First we show that (12) holds. Since M is minimal, by (2) and (11), we get

$$\sum_{i=1}^{3} g(\nabla_{e_i}(JZ^N),\ e_i)$$

$$= \sum_{i=1}^{3} g(\bar{\nabla}_{e_i}(JZ^N), e_i)$$

$$= \sum_{i=1}^{3} g(G(e_i, Z^N) + J\bar{\nabla}_{e_i} Z^N,\ e_i)$$

$$= - \sum_{i=1}^{3} g(\bar{\nabla}_{ei} Z^N,\ Je_i)$$

$$= \sum_{i=1}^{3} g(h(e_i,\ Z^T),\ Je_i)$$

$$= \sum_{i=1}^{3} g(JA_{JZ^T} e_i,\ Je_i)$$

$$= \sum_{i=1}^{3} g(A_{JZ^T} e_i,\ e_i)$$

$$= \sum_{i=1}^{3} g(h(e_i, e_i),\ JZ^T) = 0$$

at $p \in M$. Thus it follows that (12) holds. Next by (2) and (11), we get

$$g(\nabla_{e_i}(JZ^N), e_j) - g(\nabla_{e_j}(JZ^N), e_i)$$

$$= g(\bar{\nabla}_{e_i}(JZ^N), e_j) - g(\bar{\nabla}_{e_j}(JZ^N), e_i)$$

$$= -2g(G(e_i,e_j),Z^N) + g(J\bar{\nabla}_{e_i} Z^N,e_j) - g(J\bar{\nabla}_{e_j} Z^N,e_i)$$

$$=-2g(G(e_i,e_j), Z^N) + g(h(e_i,Z^T),Je_j)-g(h(e_j,Z^T),Je_i)$$

$$= -2g(G(e_i,e_j), Z^N) + g(JA_{JZ^T} e_i,Je_j) -g(JA_{JZ^T}e_j, Je_i)$$

$$= -2g(G(e_i,e_j), Z^N) + g(h(e_i,e_j),JZ^T) -g(h(e_i,e_j), JZ^T)$$

$$= -2g(G(e_i,e_j), Z^N), \ (i,j=1,2,3), \text{ at } p\epsilon M.$$

Thus (13) holds if and only if $Z^N = 0$. Hence the 1-form dual to the vector field $JZ^N \epsilon \chi(M)$ is harmonic if and only if $Z^N=0$ along M. We assume that the first Betti number b_1 of M is greater than or equal to 1. Let η' be a non-zero harmonic 1-form on M and $X' \epsilon \chi(M)$ be the vector field dual to the 1-form η'. Then since $d\eta'=0$, $\delta\eta'=0$, for $N'=JX'$, from (10), we have

$$\alpha''(N') = \alpha''(JX') = -3\int_M \|X'\|^2 dV < 0.$$

Summing up the above arguments, we have the Theorem.

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