# On hypersurfaces in spaces of constant curvature satisfying a certain condition on the curvature tensor

By

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## 1. Introduction

If a Riemannian manifold $M$ is locally symmetric, then its curvature tensor $R$ satifies

$$(*) \qquad R(X, Y) \cdot R = 0 \qquad \text{for all tangent vectors } X \text{ and } Y,$$

where the endomorphism $R(X, Y)$ operates on $R$ as a derivation of the tensor algebra at each point of $M$. Conversely, does this algebraic condition on the curvature tensor field $R$ imply that $M$ is locally symmetric?

We conjecture that the answer is affirmative in the case where $M$ is irreducible and complete and dim $M \geqq 3$.

Recently, K. Nomizu [4], has given an affirmative answer in the case where $M$ is a complete hypersurface in a Euclidean space.

In this paper, let $\bar{M}$ be a $(n+1)$-dimensional connected pseudo-Riemannian manifold with constant curvature c, and the main purpose is to consider the hypersurfaces of $\bar{M}$ satisfying the condition $(*)$.

Now, we give a short summary of those parts of the theory of hypersurfaces which are necessary for what follows.

Let $M$ be a real hypersurface immersed in $\bar{M}$ and $g$ be the induced pseudo-Riemannian metric from the pseudo-Riemannian metric $\tilde{g}$ of $\bar{M}$. And let $H$ be the second fundamental form with respect to this immersion and $A$ be a field of endomorphism which corresponds to $H$, that is, $H(X, Y) = g(AX, Y)$, where $X$ and $Y$ are tangent vectors to $M$.

By definition, the directions of the lines of curvature of $M$ are given by the vectors $\rho_m{}^i$ which satisfy

$$(1.1) \qquad (H_{ij} - \lambda_m g_{ij}) \rho_m{}^i = 0, \qquad \text{for } m, i, j = 1, 2, \ldots, n.$$

where $H_{ij}$ and $g_{ij}$ are the components of $H$ and $g$, respectively, and $\lambda_m$ is a principal

normal curvature and is a root of the determinant equation

(1.2)        $|H_{ij} - \lambda g_{ij}| = 0.$

If the principal normal curvature are real and none of the lines of curvature are tangent to null vectors, we call $M$ a proper hypersurface of $\bar{M}$ and term the immersion a proper immersion respectively. In this paper, we assume that an immersion means always a proper immersion.

The equation of Gauss expresses the curvature tensor $R$ of $M$ by means of $A$

(1.3)        $R(X, Y) = \varepsilon \cdot AX \wedge AY + cX \wedge Y,$

where, in general, $X \wedge Y$ denotes the endomorphism which maps $Z$ upon $g(Z, Y)X - g(Z, X)Y$, and $\varepsilon = +1$, or $-1$.

The type number $k(x)$ at $x$ is, by definition, the rank of $A$ at $x$

## 2. Reduction of condition (∗)

At a point $x \in M$, let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space $T_x(M)$ such that $Ae_i = \lambda_i e_i$, $1 \leq i \leq n$. Then the equation of Gauss implies

$$R(e_i, e_j) = (\varepsilon \lambda_i \lambda_j + c) e_i \wedge e_j.$$

By computing

$$(R(e_i, e_j) \cdot R)(e_k, e_l) = [R(e_i, e_j), R(e_k, e_l)]$$

$$-R(R(e_i, e_j)e_k, e_l) - R(e_k, R(e_i, e_j)e_l),$$

we find that it is zero except possibly in the case where $k = i$ and $l \neq i$, $j(i \neq j)$. For this case we have

(2.1)        $(R(e_i, e_j) \cdot R)(e_i, e_l) = \delta (\varepsilon \lambda_i \lambda_j + c) \lambda_l (\lambda_j - \lambda_i) e_j \wedge e_l,$

where $\delta = +1$, or $-1$.

Thus we see that condition (∗) is equivalent to

(2.2)        $(\varepsilon \lambda_i \lambda_j + c) \lambda_l (\lambda_j - \lambda_i) = 0$        for $l \neq i$, $j$, where $i \neq j$.

Suppose that the type number $k(x)$ is $\geq 2$ at a point $x \in M$ and assume that $\lambda_1, \ldots, \lambda_k$ are non-zero principal normal curvatures, that is, non-zero eigenvalues of $A$ at $x$ and $\lambda_{k+1} = \ldots = \lambda_n = 0$.

When $c \neq 0$, from (2.2), we can see that non-zero eigenvalues of $A$ are classified into at most two classes, that is,

$$\lambda_{i_1} = \ldots\ldots\ldots = \lambda_{i_s} \ (= \lambda),$$

$$\lambda_{j_1} = \ldots\ldots\ldots = \lambda_{j_t} (= \mu), \quad \text{where } s + t = n, \text{ and } \varepsilon \lambda \mu + c = 0.$$

Then, by the observations we made above, we can define three linear subspaces,

say, $T_0(M)$, $T_1(M)$, $T_2(M)$ of the tangent space $T_x(M)$ at $x \in M$ as follows:

$$T_0(x) = \{X \in T_x(M) ; AX = 0\}$$

$$T_1(x) = \{X \in T_x(M) ; AX = \lambda X\}$$

$$T_2(x) = \{X \in T_x(M) ; AX = \mu X\}.$$

We have $T_x(M) = T_0(M) + T_1(M) + T_2(M)$ (direct sum). For any $Z \in T_x(M)$, $Z_0$, $Z_1$, and $Z_2$ will denote the components of $Z$ in $T_0(x)$, $T_1(x)$ and $T_2(x)$, respectively. We shall only consider the following two cases in this paper.

I.   $c \neq 0$, and $k(x) \geqq 3$ at a point $x \in M$, $s \geqq 1$, $t \geqq 1$.

II.   $c = 0$, and $k(x) \geqq 3$ at a point $x \in M$.

Then we have the following for the case I.

**LEMMA 2.1.** *If $c \neq 0$ and the rank of $A$ is $\geqq 3$ at $x_0 \in M$, then there is a neighborhood $U$ of $x_0$ on which the dimension of $T_1(x)$ and the dimension of $T_2(x)$ are constant at each point $x \in U$, and the non-zero eigenvalues $\lambda(x)$ and $\mu(x)$ of $A$ are differentiable functions. Where $\varepsilon\lambda(x)\mu(x) + c = 0$ and $\lambda(x) > \mu(x)$, at $x \in U$.*

And we have also the following for the case II.

**LEMMA 2.2.** *If $c = 0$ and $k(x_0) \geqq 3$ at $x_0 \in M$, then then there is a neighborhood $U$ of $x_0$ on which $k(x) \geqq 3$ at each point $x \in U$, and the non-zero eigenvalue $k(x)$ is a differentiable function.*

### 3. Lemmas

In this paper, we shall assume that $M$ is oriented (so that a unit normal field is defined on the whole $M$) and the type number $k(x)$ is $\geqq 3$ everywhere on $M$. By the observations we made in §2, for the case I, the functions $s(x)$ and $t(x)$ are locally constant and hence are constant functions, say, $s$ and $t$, respectively, since $M$ is connected, and moreover, for the case II, the function $k(x)$ is locally constant and hence is a constant function, say, $k$.

**LEMMA 3.1.** *For the case I, $T_1$ and $T_2$ are differentiable.*

**PROOF.** For any point $x_0 \in M$, let $\{X_1, \ldots, X_s\}$ be a basis of $T_1(x_0)$ and $\{X_{s+1}, \ldots X_n\}$ be a basis of $T_2(x)$. We extend $X_i$'s to vector fields on $M$ and define vector fields

$$Y_i = (A - \mu I)X_i \qquad \text{for } 1 \leqq i \leqq s$$

and

$$Y_j = (A - \lambda I)X_j \qquad \text{for } s+1 \leqq j \leqq n,$$

where $I$ denotes the identity transformation. At $x_0$, we have $Y_i=(\lambda-\mu)X_i$ for $1\leq i\leq s$ and $Y_j=-(\lambda-\mu)X_j$ for $s+1\leq j\leq n$.

Thus $Y_1,\ldots,Y_n$ are linearly independent at $x_0$ and hence in a neighborhood $U$ of $x_0$. At each point of $U$, we have

$$(A-\lambda I)Y_i=(A-\lambda I)(A-\mu I)X_i=0 \qquad \text{for } 1\leq i\leq s,$$

$$(A-\mu I)Y_j=(A-\mu I)(A-\lambda I)X_j=0 \qquad \text{for } s+1\leq j\leq n.$$

Hence $Y_1,\ldots,Y_s$ from a basis of $T_1$ and $Y_{s+1},\ldots,Y_n$ from a basis of $T_2$.

And we have also the following

LEMMA 3.2. *For the case II, $T_0$ and $T_1$ are differentiable.*

LEMMA 3.3. *For the case I, $T_1$ and $T_2$ are involutive.*

PROOF. We recall the Codazzi equation: $(\nabla_X A)Y=(\nabla_Y A)X$, where $\nabla$ denotes the Levi-Civita connection with respect to the pseudo-Riemannian metric $g$ of $M$.

Suppose that $X$ and $Y$ are vector fields belonging to $T_1$. Then

$$(\nabla_X A)Y=\nabla_X(AY)-A(\nabla_X Y)=(X\lambda)Y+\lambda(\nabla_X Y)-A(\nabla_X Y),$$

and

$$(\nabla_Y A)X=(Y\lambda)X+\lambda(\nabla_Y X)-A(\nabla_Y X).$$

Thus we get

$$(X\lambda)Y-(Y\lambda)X+(\lambda I-A)[X, Y]=0.$$

Since $(X\lambda)Y-(Y\lambda)X\in T_1$ and $(\lambda I-A)[X, Y]=(\lambda-\mu)[X, Y]_2$, we get

$$(X\lambda)Y-(Y\lambda)X=0 \text{ and } [X, Y]_2=0.$$

The second identity shows that $[X, Y]\in T_1$, proving that $T_1$ is involutive. Similarly, $T_2$ is involutive.

And we have also the following

LEMMA 3.4. *For the case II, $T_0$ and $T_1$ are involutive.*

## 4. Some results

For the case I, let $M_1(x)$ and $M_2(x)$ be the maximal integral manifolds of $T_1$ and $T_2$, respectively.

Then we have the following

LEMMA 4.1. *For the case I, if $s\geq 2$ and $t\geq 2$, then $\lambda(x)$ and $\mu(x)$ are constant functions on $M$.*

PROOF. Since $s\geq 2$, we may choose $X, Y\in T_1(x)$ such that $X$ and $Y$ are linearly

independent. Extending $X$ and $Y$ to vector fields belonging to $T_1(x)$, we have $(X\lambda)Y-(X\lambda)X=0$ at $x$. Thus $X\lambda=Y\lambda=0$ at $x$. Therefore, the function $\lambda(x)$ is constant on each $M_1(x)$. Similarly, since $t\geqq 2$, the function $\mu(x)$ is also constant on each $M_2(x)$. However, $\varepsilon\lambda\mu+c=0$, thus it follows that $\lambda(x)$ is a constant function on $M$, say, $\lambda$. And moreover, $\mu(x)$ is also a constant function on $M$, say, $\mu$.

From Lemma 4.1., we can easily prove the following

LEMMA 4.2. *If $s\geqq 2$ and $t\geqq 2$, then $M_1(x)$ and $M_2(x)$ are both totally geodesic and $T_1(x)$ and $T_2(x)$ are paprallel.*

Thus, by virtue of above two lemmas, we can see that $M_1(x)$ and $M_2(x)$ are spaces of constant curvatures $c_1$, $c_2$, respectively, where

$$c_1 = \varepsilon\,\lambda^2+c$$

$$c_2 = \varepsilon\,\mu^2+c.$$

Then, we see that $c_1$ and $c_2$ satisfy the following equation

(4.1) $$1/c_1+1/c_2=1/c.$$

And moreover, we can show that for a point $x\in M$, there is a coordinate system $\{x^1,\ldots\ldots, x^s, x^{s+1},\ldots\ldots, x^n\}$ with origin $x$ in a neighborhood $U$ of $x$ such that $\{\partial/\partial x^1, \ldots\ldots, \partial/\partial x^s\}$ and $\{\partial/\partial x^{s+1},\ldots\ldots, \partial/\partial x^n\}$ are local bases for $T_1$ and $T_2$, and with respect to this coordinate system the first fundamental form of $M$ is given as follows

(4.2)
$$ds^2 = \frac{\varepsilon_1(dx^1)^2+\ldots\ldots+\varepsilon_s(dx^s)^2}{\left[1+\frac{c_1}{4}(\varepsilon_1(x^1)^2+\ldots+\varepsilon_s(x^s)^2)\right]^2}$$
$$+\frac{\varepsilon_{s+1}(dx^{s+1})^2+\ldots\ldots+\varepsilon_n(dx^n)^2}{\left[1+\frac{c_2}{4}(\varepsilon_{s+1}(x^{s+1})^2+\ldots+\varepsilon_n(x^n)^2)\right]^2},$$

where each $\varepsilon$ is $+1$, or $-1$. (See for example A. Fialkow [1].)
Then $M$ may be immersed in $\bar{M}$ by means of the algebraic equations

(4.3) $$e_1(z^1)^2+\ldots\ldots+e_{s+2}(z^{s+1})^2=1/c_1,$$

$$e_{s+2}(z^{s+2})^2+\ldots\ldots+e_{n+2}(z^{n+2})^2=1/c_2,$$

where the $e$'s are $+1$, or $-1$ and the $\bar{M}$ is defined by

(4.4) $$e_1(z^1)^2+\ldots\ldots+e_{n+2}(z^{n+2})^2=1/c.$$

The $z$'s are the point coordinates of Weierstrass. Since $n\geqq 4$ and $\lambda\mu\neq 0$, the $M$ is indeformable in $\bar{M}$.

The equations (4.3) show that $M$ may also be considered as the intersection of two hypersylinders in a flat space of $n+2$ dimensions.

Thus we have the following

THEOREM 4.3. *Let $\bar{M}$ be complete, connected and of constant curvature $c \neq 0$ and defined by (4.4). If $M$ is connected complete proper hypersurface of $\bar{M}$ of the case I and $s \geq 2$, $t \geq 2$, then $M$ may be considered as the intersection of two hypersylinders in a flat space of $n+2$ dimensions.*

As a result, M is, of course, symmetric.

Next, for the case II, let $M_0(x)$ and $M_1(x)$ be the maximal integral manifolds of $T_0$ and $T_1$ respectively.

LEMMA 4.4. *If $X$ belongs to $T_1(x)$, then $X\lambda = 0$.*

From this lemma, we see that the function $\lambda$ is constant on each maximal integral manifold of $T_1$, that is, on each $M_1(x)$.

We now let $X \in T_1$, $Y \in T_0$ and compute the both sides of the Codazzi equation:

$$(\nabla_X A)Y = \nabla_X(AY) - A(\nabla_X Y) = -A(\nabla_X Y) = -\lambda(\nabla_X Y)_1,$$

$$(\nabla_Y A)X = \nabla_Y(AX) - A(\nabla_Y X) = \nabla_Y(\lambda X) - A(\nabla_Y X)$$

$$= Y\lambda \cdot X + \lambda(\nabla_Y X) - A(\nabla_Y X)$$

$$= Y\lambda \cdot X + \lambda(\nabla_Y X)_0.$$

Therefore we have

$$(\nabla_Y X)_0 = 0, \qquad \text{that is,} \quad \nabla_Y X \in T_1$$

and

$$(Y\lambda)X = -\lambda(\nabla_X Y)_1 = -A(\nabla_X Y).$$

We have hence

LEMMA 4.5. *If $X \in T_1$, $Y \in T_0$, then $A(\nabla_X Y) = -(Y\lambda)X$.*

LEMMA 4.6.
(i)  *If $Y \in T_0$, then $\nabla_Y(T_1) \subset T_1$.*
(ii)  *If $Y \in T_0$, then $\nabla_Y(T_0) \subset T_0$.*
(iii)  *If $Y \in T_0$, $X \in T_1$ and $[X, Y] = 0$, then $\nabla_X Y \in T_1$.*

By virtue of Lemma 4.5. and Lemma 4.6., we have

(4.5)          $$\nabla_Y X = -\frac{Y\lambda}{\lambda} \cdot X. \qquad \text{for } X \in T_1, \ Y \in T_0, \ [X, Y] = 0.$$

By virtue of Lemma 3.4., for a point $x \in M$, there is a coordinate system $\{x^1, ..., x^n\}$ in a neighborhood $U$ of $x$ such that $\{\partial/\partial x^1, ......, \alpha/\partial x^k\}$ and $\{\partial/\partial x^{k+1}, ......, \partial/\partial x^n\}$ are local bases for $T_1$ and $T_0$, respectively.

By setting $g_{\alpha\beta}(x^1, ......, x^n) = g(\partial/\partial x^\alpha, \partial/\partial x^\beta)$ for $k+1 \leq \alpha$, $\beta \leq n$, we have

$$\partial g_{\alpha\beta}/\partial x^r = g(\nabla_{\partial/\partial x^r}(\partial/\partial x^\alpha),\ \partial/\partial x^\beta) + g(\partial/\partial x^\alpha,\ \nabla_{\partial/\partial x^r}(\partial/\partial x^\beta)).$$

But since Lemma 4.6 (iii), implies $\nabla_{\partial/\partial x^r}(\partial/\partial x^\alpha) \in T_1$ for $1 \le r \le k$, we have

$$g(\nabla_{\partial/\partial x^r}(\partial/\partial x^\alpha),\ \partial/\partial x^\beta) = 0$$

and, similarly, $g(\partial/\partial x^\alpha,\ \nabla_{\partial/\partial x^r}(\partial/\partial x)) = 0$. We have thus $\partial g_{\alpha\beta}/\partial x^r = 0$. Moreover, by setting $g_{pq}(x^1, \ldots, x^n) = g(\partial/\partial x^p, \partial/\partial x^q)$ for $1 \le p,\ q \le k$,
we have

$$\partial g_{pq}/\partial x^\alpha = g(\nabla_{\partial/\partial x^\alpha}(\partial/\partial x^p),\ \partial/\partial x^q) + g(\partial/\partial x^p,\ \nabla_{\partial/\partial x^\alpha}(\partial/\partial x_p)).$$

But (4.5) implies $\nabla_{\partial/\partial x^\alpha}(\partial/\partial x^p) = \nabla_{\partial/\partial x^p}(\partial/\partial x^\alpha) = -\dfrac{\partial\lambda}{\partial x^\alpha}\partial/\partial x^p$ for $k+1 \le \alpha \le n$.
Hence

(4.6) $\qquad \partial g_{pq}/\partial x^\alpha = -(2\partial\lambda/\partial x^\alpha)g_{pq}.$

Thus we have the following

**THEOREM 4.7.** *For the case II, there is a coordinate system in a neighborhood $U$ of $x$ such that with respect to this coordinate system the metric tensor of $M$ is given as follows*

(4.7)
$$g_{ij}(x) = \begin{pmatrix} g_{pq}(x^1, \ldots, x^n) & 0 \\ 0 & g_{\alpha\beta}(x^{k+1}, \ldots, x^n) \end{pmatrix}$$

*where $g_{pq}(x^1, \ldots, x^n)$ must satisfy (4.6), and $g_{\alpha\beta}$ are constant on $U$.*

Lastly, let us give an example of hypersurface, of the case I, in the hyperbolic space of dimension $n+1$, say, $H^{n+1}$. Then $H^{n+1}$ is isometric to the Riemannian half space

$$ds^2 = (1/z^1)^2((dz^1)^1 + \ldots + (dz^{n+1})^2),\ z \in R^{n+1},\ z^1 > 0.$$

Let $M$ be a hypersurface of $H^{n+1}$, defined by the equations:

$$z^1 = e^u,\qquad z^i = e^u w^i(x^1, \ldots, x^{n-1}),\qquad i = 2, \ldots, n+1,\qquad u \in R,\qquad n \ge 3,$$

where $e^u$ denotes the exponential function of $u$.
Then the Riemannian metric of $M$ is expressed as follows

$$g_{pq} = \sum_{i=2}^{n+1}(\partial w^i/\partial x^p)(\partial w^i/\partial x^q),\qquad g_{**} = 1 + \sum_{i=2}^{n+1}(w^i)^2,\qquad p,\ q = 1, \ldots, n-1,$$

where the first fundamental from of $M$ is written in the form, $ds^2 = g_{pq}dx^p dx^q + g_{**}dudu.$
Then, if $M$ satisfies the condition $(*)$, then, $s$ or $t = 1$ and from the similar arguments as before, we have

$$\sum_{i=2}^{n+1} (w^i) = r^2, \qquad \text{where } r \text{ is a constant.}$$

Therefore, $M$ is homothetic to the direct product Riemannian space of $(n-1)$-dimensional sphere with radius $|r|$ and a real line $R$.

*Remarks*, (i) If $c \neq 0$ and $k(x) \geq 3$ at some point $x \in M$, then we can show that the condition (*) is equivalent to the condition, $R(X, Y) \cdot S = 0$, at $x$. Where $S$ denotes the Ricci tensor of $M$.

(ii) The assumption, $k(x) \geq 3$ everywhere on $M$, can be replaced by the assumption, $k(x) \geq 3$ at some point $x \in M$.

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## References

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