A Theorem of Schur Type for Locally Symmetric Spaces

By

Nobuhiro INNAMI

(Received October 31, 1988)

Abstract. By showing hidden hypotheses in Schur's lemma on spaces of constant curvature we get a new version for locally symmetric spaces.

0. Statement

Let M be a connected Riemannian manifold with dimension $n \geq 3$. Schur proved in 1886 that M is a space of constant curvature if the sectional curvature depends only on the points (see [2], [3]). In the present note we improve the theorem and have a theorem of the same type for locally symmetric spaces.

Let ∇ be the *Riemannian connection* and let R be the *Riemannian curvature tensor* given by

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = R(X, Y) Z$$

where X, Y, Z are vector fields and $[\cdot, \cdot]$ is the Lie bracket. We say that the eigenspaces of R are *parallel* if the following condition is satisfied: For any geodesic ν and for any unit parallel vector field v along ν the eigenspaces of $R(\cdot, v)v : T_pM \longrightarrow T_pM$ are parallel along ν where p is the foot point of v. The locally symmetric spaces have this property. If the sectional curvature depends only on the points, then the condition is automatically satisfied, since the eigenspaces of $R(\cdot, v)v : T_pM \longrightarrow T_pM$ is either $v^\perp$ or T_pM which are parallel along ν, where $v^\perp$ is the space orthogonal to v.

Theorem. *Let M be a connected Riemannian manifold with dimension $n \geq 3$. Suppose there exist functions $c_1, \ldots, c_i$ on M such that (1) the distinct eigenvalues of $R(\cdot, v)v : T_pM \longrightarrow T_pM$ are $c_1(p), \ldots, c_i(p)$ for any point $p \in M$ and any unit vector $v \in T_pM$ with $c_i(p)=0$ and (2) if $c_j = \lambda_j c_1$ then λ_j are constants on M for $j=1, \ldots, i-1$ (always $\lambda_1=1$* and *$\lambda_i=0$). If the eigenspaces of R are parallel and dim Ker $R(\cdot, v)v \leq n-2$ for any unit vector v, then M is a locally symmetric space.*

Here, ***Ker*** $R(\cdot, v)v$ is by definition the kernel of $R(\cdot, v)v$: $T_pM \longrightarrow T_pM$. The

Schur theorem covers two cases, namely, $i=1$, or $i=2$ and $\dim \operatorname{Ker} R(\cdot, v)v=1$. In those cases it is not necessary to assume the condition (2) for c_j's and the parallel property of R, because they are automatically satisfied.

1. Preliminaries

Let M be a Riemannian manifold. We say that M is a *locally symmetric space* if the geodesic symmetry is isometry in some neighborhood of each point of M. The condition is equivalent to that the Riemannian curvature tensor is parallel.

A tensor field K of type (1, 3) is called a *curvature tensor* if it satisfies the following condition:

(1) $K(x, y)z=-K(y, x)z$.

(2) $K(x, y)z+K(y, z)x+K(z, x)y=0$.

(2) $\langle K(x, y)z, w\rangle=-\langle K(x, y)w, z\rangle$.

(4) $\langle K(x, y)z, w\rangle=\langle K(z, w)x, y\rangle$.

The curvature tensor K is said to satisfy the *second Bianchi identity* if

$$(\nabla_W K)(X, Y)Z+(\nabla_X K)(Y, W)Z+(\nabla_Y K)(W, X)Z=0$$

for any vector fields X, Y, Z and W on M. The Riemannian curvature tensor satisfies the second Bianchi identity.

For the proof of Theorem we provide a lemma. We write $F=F\circ\pi$.

Lemma. *Let M be a connected Riemannian manifold with dimension $n\geq 3$. Suppose there exist a function F on M and a curvature tensor K such that $R=FK$. If K satisfies the second Bianchi identity and dim Ker $K(\cdot, v)v\leq n-2$ fnr any unit vector v, then F is constant. In particular, if K is in addition parallel, then M is locally symmetric.*

Proof. From the assumption we have

$$((\nabla_W R)(X, Y))Z=(WF)K(X, Y)Z+((\nabla_W K)(X, Y))Z$$

where X, Y, Z, W are vector fields. By the second Bianchi identity we find

$$(WF)K(X, Y)Z+(XF)K(Y, W)Z+(YF)K(W\ X)Z=0. \tag{1.1}$$

Let p be a point of M. Put

$$c=\max\{|\langle K(u, v)v, u\rangle|\,;\, u, v\in T_pM,\ |v|=|u|=1,\ u\perp v\}.$$

Let x, z be orthonormal vectors in T_pM such that $|\langle K(x, z)z, x\rangle|=c$. Then x is an eigenvector of $K(\cdot, z)z$ with eigenvalue $b\neq 0$ ($b=c$ or $-c$). Since $\langle K(z, x)x, z\rangle=\langle K(x, z)z, x\rangle=b$, we find also that z is an eigenvector of $K(\cdot, x)x$ with eigenvalue b. Let y be a unit eigenvector of $K(\cdot, z)z$ with eigenvalue d such that $y\perp z$, $y\perp x$. Then, putting

$\boldsymbol{Z}=\boldsymbol{W}=\boldsymbol{z}$, $\boldsymbol{X}=\boldsymbol{x}$, $\boldsymbol{Y}=\boldsymbol{y}$ in (**1.1**), we have

$$(zF)K(x, y)z+(xF)dy-(yF)bx=0.$$

Hence,

$$(zF)\langle K(x, y)z, x\rangle+(xF)d\langle y, x\rangle-(yF)b\langle x, x\rangle=0.$$

Therefore,

$$yF=0,$$

since

$$\langle K(x, y)z, x\rangle=\langle K(z, x)x, y\rangle=b\langle z, y\rangle=0.$$

Let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ be orthonormal eigenvectors of $\boldsymbol{K}(\cdot, \boldsymbol{y})\boldsymbol{y}$ with $\boldsymbol{e}_1 \perp \boldsymbol{y}$, $\boldsymbol{e}_2 \perp \boldsymbol{y}$. Again, putting $\boldsymbol{W}=\boldsymbol{Z}=\boldsymbol{y}$, $\boldsymbol{X}=\boldsymbol{e}_1$, $\boldsymbol{Y}=\boldsymbol{e}_2$ in (**1.1**), we have

$$(e_1F)b_2e_2-(e_2F)b_1e_1=0,$$

where $\boldsymbol{b}_1$, $\boldsymbol{b}_2$ are eigenvalues of $\boldsymbol{K}(\cdot, \boldsymbol{y})\boldsymbol{y}$ for $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, resp. Since we may assume that $\boldsymbol{b}_1\neq \boldsymbol{0}$ because of $\boldsymbol{dim\ Ker}(\cdot, \boldsymbol{y})\boldsymbol{y}\leq \boldsymbol{n}-\boldsymbol{2}$, we see that $\boldsymbol{e}_2\boldsymbol{F}=\boldsymbol{0}$. Further, by the same reasoning, we can find a unit vector $\boldsymbol{e}_1' \perp \boldsymbol{e}_1$ which is an eigenvector of $\boldsymbol{K}(\cdot, \boldsymbol{y})\boldsymbol{y}$ with eigenvalue $\boldsymbol{b}_1'\neq \boldsymbol{0}$. Hence, $\boldsymbol{e}_1\boldsymbol{F}=\boldsymbol{0}$ also. Since $\boldsymbol{y}$ and the eigenvectors of $\boldsymbol{R}(\cdot, \boldsymbol{y})\boldsymbol{y}$ span $\boldsymbol{T}_p\boldsymbol{M}$, the derivative of $\boldsymbol{F}$ is zero. This implies that $\boldsymbol{F}$ is constant on $\boldsymbol{M}$.

We can get the same result even if $\boldsymbol{R}$ is not the Riemannian curvature tensor but any curvature tensor satisfying the second Bianchi identity.

2. Proof of Theorem

We prove the theorem here. Let ν be a geodesic in $\boldsymbol{M}$ with $\nu(\boldsymbol{0})=\boldsymbol{v}$, $|\boldsymbol{v}|=1$ and let $\boldsymbol{w}$ be a unit parallel vector field along ν. Let $\boldsymbol{E}_j \subset \boldsymbol{w}^\perp$ be the eigenspace of $\boldsymbol{R}(\cdot, \boldsymbol{w})\boldsymbol{w}$ with eigenvalue $\boldsymbol{c}_j$ for each $\boldsymbol{j}=\boldsymbol{1},\dots, \boldsymbol{i}$. If a parallel vector field $\boldsymbol{e}$ along ν is given by $\boldsymbol{e}=\boldsymbol{e}_1+\dots+\boldsymbol{e}_i$, $\boldsymbol{e}_j \in \boldsymbol{E}_j$, then $\boldsymbol{e}_j$ is parallel along ν, since so is $\boldsymbol{E}_j$ for all $\boldsymbol{j}$. Let $\boldsymbol{K}$ be a tensor field on $\boldsymbol{M}$ of type (**1, 3**) given by $\boldsymbol{K}=(1/\boldsymbol{c}_1)\boldsymbol{R}$. Then ,we have

$$K(e, w)w=\Sigma_{j=1}^{i}\frac{1}{c_1}R(e, w)w=\Sigma_{j=1}^{i}\lambda_j e_j.$$

Hence,

$$(\nabla_v K)(e, w)w=0.$$

Therefore, it follows that

$$\langle(\nabla_v K)(x, y)x, y\rangle=0$$

for any point $\boldsymbol{p} \in \boldsymbol{M}$ and any vectors $\boldsymbol{v}, \boldsymbol{x}, \boldsymbol{y} \in T_p\boldsymbol{M}$. By the identity (**1.10**) in [1] and the definition of $\boldsymbol{K}$, we know that $\langle \boldsymbol{K}(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{z}, \boldsymbol{w}\rangle$ is a sum of terms of the form $\pm\langle \boldsymbol{K}(*, \cdot) *, \cdot\rangle$. From this we have that $\boldsymbol{K}$ is parallel on $\boldsymbol{M}$. Lemma implies that $\boldsymbol{c}_1$ is constant on $\boldsymbol{M}$ and therefore the Riemannian curvature tensor is parallel on $\boldsymbol{M}$. This completes the proof.

References

[1] J. Cheeger and D. Ebin: Comparison Theorems in Riemannian Geometry. North-Holland, Amsterdam, 1975.

[2] B.-y. Chen: Geometry of Submanifolds. Marcel Dekker, New York, 1973.

[3] F. Schur: *Ueber den Zusammenhang der Räume constanten Riemann'schen Krümmungsmasses mit den projectiven Räumen.* Math. Ann., 27 (1886), 537–567.

Nobuhiro Innami
Department of Mathematics
Faculty of Science
Niigata University
Niigata, 950-21
JAPAN