On Cone-Extreme Points in $\mathbf{R}^n$

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1. Introduction

Recently, the decision problems in $\boldsymbol{R}^n$ ordered by a convex cone have been investigated by many authors (cf. [1], [2], [4] and [7]). In [7], Yu used the nonpositive orthant $\boldsymbol{R}^n_-$ as a convex cone C and defined the cone extreme points. Further, he introduced the concept of acute to the convex cone C and showed that this led some properties of cone extreme points. Hartley introduced also the concept of cone compactness and showed that this is sufficient to guarantee the existence of an efficient point in [1]. Moreover, in [6], Tanino and Sawaragi introduced the concepts of $\boldsymbol{R}^p_+$–boundedness and $\boldsymbol{R}^p_+$–closedness, and gave some properties to $\boldsymbol{R}^p_+$–bounded sets and $\boldsymbol{R}^p_+$–closed sets.

In this paper we give the concepts of cone compactness, cone boundedness and cone closedness and investigate the characterization of the set of all cone extreme points of a subset A under a given cone C, denoted by Ext $[A|C]$. And we study the following:

(i) Ext $[A|C]\neq\phi$,

(ii) $A\subset$Ext $[A|C]+C$,

and

(iii) compactness or cone compactness of Ext $[A|C]$.

This paper is organized in the following way. In Section 2, we discuss the various properties of acute convex cones and Ext $[A|C]$. In Section 3, we study (i), (ii) and (iii) under C–compactness of a set A. In Section 4, we show them under C–boundedness and C–closedness of a set A. In addition, we investigate the relations among cone compactness, cone boundedness, and cone closedness.

2. Preliminaries and Cone Extreme Points

For a set $A\subset\boldsymbol{R}^n$, its closure, interior, and relative interior are denoted by cl$A$, int$A$, and ri$A$, respectively.

DEFINITION 2.1. *A cone* $C \subset \boldsymbol{R}^n$ is said to be *acute* if there exists an open half-space H such that $\mathrm{cl}\, C \subset H \cup \{0\}$. *A cone* $C \subset \boldsymbol{R}^n$ is said to be *poined* if $C \cap (-C) = \{0\}$.

Note that if C is acute then C is pointed. Moreover if $C \subset \boldsymbol{R}^n$ is a convex cone, then, by [2], the following facts are equivalent:

(i) C is acute,

(ii) $\mathrm{cl}\, C$ is pointed,

and

(iii) $\mathrm{cl}\, C$ is acute.

Throughout the paper, we will use a cone with the origin 0 as the vertex in $\boldsymbol{R}^n$.

DEFINITION 2.2. Let C be a cone in $\boldsymbol{R}^n$. A point $x_0 \in A$ is said to be a *C-extreme point* of A if there is no points $x \in A$ such that $x \neq x_0$ and $x_0 \in x + C$. We denote the *set of all C-extreme points* of A by $\mathrm{Ext}\,[A \mid C]$.

By Lemma 4.1 in [7], we have

$$\mathrm{Ext}\,[A \mid C_2] \subset \mathrm{Ext}\,[A \mid C_1] \quad \text{if } C_1 \subset C_2 \tag{1}$$

and

$$\mathrm{Ext}\,[A + C \mid C] \subset \mathrm{Ext}\,[A \mid C]. \tag{2}$$

If C contains no nontrivial subspaces, then

$$\mathrm{Ext}\,[A \mid C] \subset \mathrm{Ext}\,[A + C \mid C], \tag{3}$$

and, from (2) and (3), $\mathrm{Ext}\,[A \mid C] = \mathrm{Ext}\,[A + C \mid C]$.

PROPOSITION 2.1. *Let C be an acute convex cone in $\boldsymbol{R}^n$. Then* $\mathrm{Ext}\,[A + \mathrm{ri}\, C^0 \mid C] = \mathrm{Ext}\,[A \mid C]$ *where* $\mathrm{ri}\, C^0 = \mathrm{ri}\, C \cup \{0\}$.

PROOF. In order to show $\mathrm{Ext}\,[A + \mathrm{ri}\, C^0 \mid C] \subset \mathrm{Ext}\,[A \mid C]$, for any $x_0 \in \mathrm{Ext}\,[A + \mathrm{ri}\, C^0 \mid C]$, suppose that $x_0 \notin A$. Then there are $\hat{x} \in A$ and $\hat{d} \in \mathrm{ri}\, C^0$ such that

$$x_0 = \hat{x} + \hat{d}. \tag{4}$$

Since $A \subset A + \mathrm{ri}\, C^0$ and $\mathrm{ri}\, C \subset C$, we have

$$\hat{x} \in A + \mathrm{ri}\, C^0 \text{ and } 0 \neq \hat{d} \in C. \tag{5}$$

But, (4) and (5) contradict the fact that $x_0 \in \mathrm{Ext}\,[A + \mathrm{ri}\, C^0 \mid C]$. Hence, we have $x_0 \in A$. Since $x_0 \in \mathrm{Ext}\,[A + \mathrm{ri}\, C^0 \mid C]$, there is no $x \in A$, $x \neq x_0$, such that $x_0 \in x + C$, that is $x_0 \in \mathrm{Ext}\,[A \mid C]$.

Conversely, take any $x_0 \notin \mathrm{Ext}\,[A + \mathrm{ri}\, C^0 \mid C]$. If $x_0 \notin A$, then $x_0 \notin \mathrm{Ext}\,[A \mid C]$. Assume that $x_0 \in$ A, then there are $\hat{x} \in A + \mathrm{ri}\, C^0$ and nonzero $\hat{d} \in C$ such that

$$x_0 = \widehat{x} + \widehat{d}. \tag{6}$$

And there is $\widetilde{x} \in A$ and $\widetilde{d} \in \mathrm{ri}\, C^0$ such that

$$\widehat{x} = \widetilde{x} + \widetilde{d}. \tag{7}$$

From (6) and (7), we have

$$x_0 = \widetilde{x} + (\widetilde{d} + \widehat{d}). \tag{8}$$

Since $\mathrm{ri}\, C^0$ is an acute convex cone and $\mathrm{ri}\, C^0 + C \subset C$, we have

$$0 \neq \widetilde{d} + \widehat{d} \in C. \tag{9}$$

From (8) and (9), we have

$$x_0 \notin \mathrm{Ext}\, [A \mid C].$$

PROPOSITION 2. 2. *Let C and D be two cones in $\boldsymbol{R}^n$. The following results hold:*

(i) $\mathrm{Ext}\, [A+B \mid C] \subset \mathrm{Ext}[A \mid C] + \mathrm{Ext}\, [B \mid C]$

and

(ii) $\mathrm{Ext}\, [A + \mathrm{Ext}\, [B \mid D] \mid C] \subset \mathrm{Ext}\, [A \mid C] + B$.

PROOF. Take any $z \notin \mathrm{Ext}\, [A \mid C] + \mathrm{Ext}\, [B \mid C]$. If $z \notin A+B$ then $z \notin \mathrm{Ext}\, [A+B \mid C]$. Suppose that $z \in A+B$, that is, there are $x \in A$ and $y \in B$ such that $z = x+y$. Then $x \notin \mathrm{Ext}\, [A \mid C]$ or $y \notin \mathrm{Ext}\, [B \mid C]$. If $x \notin \mathrm{Ext}[A \mid C]$, then there is $x' \in A$ such that $x' \neq x$ and $x \in x' + C$. This clearly implies that

$$x+y \in x'+y+C \quad \text{and} \quad x'+y \neq x+y,$$

hence,

$$z = x+y \notin \mathrm{Ext}\, [A+B \mid C].$$

Similarly, if $y \notin \mathrm{Ext}\, [B \mid C]$, then $z \notin \mathrm{Ext}\, [A+B \mid C]$. Thus $\mathrm{Ext}\, [A+B \mid C] \subset \mathrm{Ext}\, [A \mid C] + \mathrm{Ext}\, [B \mid C]$. The remaining statements follow immediately.

PROPOSITION 2. 3. *Let C be a convex cone in $\boldsymbol{R}^n$. Then, for any $x \in A$,*

$$\mathrm{Ext}\, [(x - \mathrm{cl}\, C) \cap A \mid C] \subset \mathrm{Ext}\, [A \mid C].$$

PROOF. Let $x \in A$, and take any $x_0 \notin \mathrm{Ext}\, [A \mid C]$. If $x_0 \notin (x - \mathrm{cl}\, C) \cap A$, then $x_0 \notin \mathrm{Ext}\, [(x - \mathrm{cl}\, C) \cap A \mid C]$. Assume $x_0 \in (x - \mathrm{cl}\, C) \cap A$, then there is $d' \in \mathrm{cl}\, C$ such that $x_0 = x - d'$. Since there are $\widehat{x} \in A$ and $\widehat{d} \in C$ such that

$$\widehat{x} \neq x_0 \quad \text{and} \quad x_0 = \widehat{x} + \widehat{d}, \tag{10}$$

we have $\widehat{x} + \widehat{d} = x - d'$, that is, $\widehat{x} = x - (\widehat{d} + d')$. Since $\mathrm{cl}\, C$ is a convex cone, $\widehat{d} + d' \in \mathrm{clC}$, and hence $\widehat{x} \in x - \mathrm{cl}\, C$. Thus

$$\hat{x} \in (x-\mathrm{cl}C) \cap A,$$

and from (10),

$$x_0 \notin \mathrm{Ext}\ [(x-\mathrm{cl}C) \cap A \mid C].$$

This completes the proof.

3. Cone Compactness

DEFINITION 3.1. Let C be a cone in $\boldsymbol{R}^n$. A set A is said to be *C-compact* if $(x-\mathrm{cl}\,C) \cap A$ is compact for any $x \in A$.

A compact set is C-compact. However, taking C to be $\boldsymbol{R}^2_-$ in $\boldsymbol{R}^2$ and $A=\{x \in \boldsymbol{R}^2 \mid x_1 + x_2 \leq 1\}$, we can see that A is $\boldsymbol{R}^2_-$-compact but not compact (cf. p. 214 in [**1**]).

LEMMA 3.1. *Let C_1 and C_2 be two cones in $\boldsymbol{R}^n$. If A is C_2-Compact and $C_1 \subset C_2$ then A is C_1-compact.*

PROOF. For any $x \in A$, $x-\mathrm{cl}C_1$ is closed and $(x-\mathrm{cl}C_2) \cap A$ is compact. It follows that $(x-\mathrm{cl}C_1) \subset (x-\mathrm{cl}C_2)$ from $C_1 \subset C_2$. So, we have $(x-\mathrm{cl}C_1) \cap A=(x-\mathrm{cl}C_1) \cap (x-\mathrm{cl}C_2) \cap A$, which shows that $(x-\mathrm{cl}C_1) \cap A$ is compact. Therefore A is C_1-compact.

If C is a convex cone in $\boldsymbol{R}^n$, by Theorem 6.3 in [**3**], the following (i), (ii) and (iii) are equivalent:

(i) A is C-compact,

(ii) A is $\mathrm{cl}C$-compact,

and

(iii) A is $\mathrm{ri}C$-compact.

THEOREM 3.1. *Let A and C be a nonempty subset and an acute convex cone in $\boldsymbol{R}^n$,* respectively. *If A is C-compact, then* Ext $[A \mid C] \neq \phi$. *Moreover, if C is closed, then* $A \subset \mathrm{Ext}\ [A \mid C]+C$.

PROOF. For any $x_0 \in A$, $(x_0-\mathrm{cl}C) \cap A$ is compact and $\mathrm{cl}\,C$ is acute. By using Corollary 4.6 in [**7**], it follows that

$$\mathrm{Ext}\ [(x_0-\mathrm{cl}C) \cap A \mid \mathrm{cl}C] \neq \phi. \qquad (11)$$

From (1) and Proposition 2.3,

$$\mathrm{Ext}\ [(x_0-\mathrm{cl}C) \cap A \mid \mathrm{cl}C] \subset \mathrm{Ext}\ [A \mid C], \qquad (12)$$

which implies that Ext $[A \mid C] \neq \phi$.

Next, let C be closed then $\mathrm{cl}C=C$, and take any $x_0 \in A$. By (11), there is $y \in (x_0-C) \cap A$ such that

$$y \in \mathrm{Ext}\,[(x_0-C)\cap A\,|\,C].$$

From this it follows that $x_0 \in y+C \subset \mathrm{Ext}\,[(x_0-C)\cap A\,|\,C]+C$.
Further, from (12), $x_0 \in \mathrm{Ext}\,[A\,|\,C]+C$.

THEOREM 3.2. *Let C be an acute convex cone in $\boldsymbol{R}^n$. If A is C-compact, and*

$$(C\setminus\{0\})+\mathrm{cl}\,C\subset C, \tag{13}$$

then $A\subset \mathrm{Ext}\,[A\,|\,C]+C$.

PROOF. Suppose that there is $x_0 \in A\setminus(\mathrm{Ext}\,[A\,|\,C]+C)$. If $x_0 \in \mathrm{Ext}\,[A\,|\,C]$, then $x_0=x_0+0 \in \mathrm{Ext}\,[A\,|\,C]+C$ since $0\in C$. This is a contradiction. Therefore, there exist $x_1 \in A$ and nonzero $d_1 \in C$ such that $x_0=x_1+d_1$, and so $x_1=x_0-d_1 \in x_0-C$. Thus, $x_1 \in (x_0-C)\cap A$. If $x_1 \in \mathrm{Ext}\,[A\,|\,C]$, then

$$x_0=x_1+d_1 \in \mathrm{Ext}\,[A\,|\,C]+C.$$

This is a contradiction. Thus, $x_1 \notin \mathrm{Ext}\,[A\,|\,C]$. Since $(x_0-\mathrm{cl}C)\cap A$ is $\mathrm{cl}C$-compact, from Theorem 3.1, Proposition 2.3 and (1), it follows that

$$\begin{aligned} x_1 &\in (x_0-C)\cap A \\ &\subset \mathrm{Ext}[(x_0-\mathrm{cl}C)\cap A\,|\,\mathrm{cl}\,C]+\mathrm{cl}C \\ &\subset \mathrm{Ext}[A\,|\,C]+\mathrm{cl}\,C. \end{aligned}$$

Thus, there exist $\hat{x} \in \mathrm{Ext}\,[A\,|\,C]$ and $\hat{d} \in \mathrm{cl}C$ such that $x_1=\hat{x}+\hat{d}$, and hence $x_0=x_1+d_1 = \hat{x}+\hat{d}+d_1$. Since $\hat{d} \in \mathrm{cl}C$ and $0\neq d_1 \in C$, we have $\hat{d}+d_1 \in C$ by the assumption (13). Therefore,

$$x_0 \in \mathrm{Ext}\,[A\,|\,C]+C.$$

This contradicts the fact that $x_0 \notin \mathrm{Ext}\,[A\,|\,C]+C$.

In general, it is known that $\mathrm{ri}\,C+\mathrm{cl}\,C=\mathrm{ri}\,C$ for any convex cone C (cf. [**3**]). But it is not necessary that

$$(C\setminus\{0\})+\mathrm{cl}\,C\subset C. \tag{13}$$

However, closed or open convex cones satisfy this property (13). Moreover, if a cone C can be expressed as an intersection of arbitrary number of closed or open convex cones, then C also satisfies (13), (cf. p. 112 in [**2**]).

COROLLARY 3.1. *Let C be an acute convex cone with the property* (13) *in $\boldsymbol{R}^n$. If A is C-compact then*

$$A+C=\mathrm{Ext}\,[A\,|\,C]+C \tag{14}$$

PROOF. Clearly, $A+C\supset \mathrm{Ext}\,[A\,|\,C]+C$. On the other hand, from Theorem 3.2, it follows that $A+C\subset \mathrm{Ext}\,[A\,|\,C]+C$ by $C+C=C$. Thus (14) holds.

COROLLARY 3.2. *Let C_1 and C_2 be two acute convex cones in $\boldsymbol{R}^n$. Assume that*

(i) A is C_2-*compact,*

(ii) $(C_i\backslash\{0\})+\mathrm{cl}\,C_i\subset C_i, \quad i=1, 2$

and

(iii) $C_1\subset C_2$.

Then

$$A\subset \mathrm{Ext}\,[A\,|\,C_1]+C_1\subset \mathrm{Ext}\,[A\,|\,C_2]+C_2. \tag{15}$$

PROOF. The proof is straightforward from Corollary 3.1.

COROLLARY 3.3. *Let C be an acute convex cone with the property* (13) *in $\boldsymbol{R}^n$. If A is C-compact then*

$$\mathrm{Ext}\,[A\,|\,\mathrm{ri}\,C^0]+\mathrm{ri}\,C^0\subset \mathrm{Ext}[A\,|\,C]+C\subset \mathrm{Ext}[A\,|\,\mathrm{cl}\,C]+\mathrm{cl}\,C. \tag{16}$$

Further, if $\mathrm{int}\,C\neq\phi$, *then*

$$\mathrm{Ext}\,[A\,|\,\mathrm{int}\,C^0]+\mathrm{int}\,C^0\subset \mathrm{Ext}\,[A\,|\,C]+C\subset \mathrm{Ext}\,[A\,|\,\mathrm{cl}\,C]+\mathrm{cl}\,C, \tag{17}$$

where $\mathrm{int}\,C^0=\mathrm{int}\,C\cup\{0\}$.

PROOF. Clearly A is both $\mathrm{cl}\,C$-compact and $\mathrm{ri}\,C$-compact. Since C, $\mathrm{cl}\,C$ and $\mathrm{ri}\,C$ satisfy the property (13), by Corollary 3.1,

$$A+C=\mathrm{Ext}\,[A\,|\,C]+C,$$

$$A+\mathrm{cl}\,C=\mathrm{Ext}\,[A\,|\,\mathrm{cl}\,C]+\mathrm{cl}\,C$$

and

$$A+\mathrm{ri}\,C^0=\mathrm{Ext}\,[A\,|\,\mathrm{ri}\,C^0]+\mathrm{ri}\,C^0.$$

Thus (16) holds. Further, if $\mathrm{int}\,C\neq\phi$ then $\mathrm{ri}\,C=\mathrm{int}\,C$, and hence (17) holds.

Next we study compactness and cone compactness of the set of all cone extreme points.

THEOREM 3.3. *Let A be a nonempty compact set in $\boldsymbol{R}^n$ and let C be a cone in $\boldsymbol{R}^n$ such that $C\backslash\{0\}$ is open. Then* $\mathrm{Ext}\,[A\,|\,C]$ *is compact. Further, if C is acute then* $\mathrm{Ext}\,[A\,|\,C]\neq\phi$.

PROOF. By compactness of A and $A\supset\mathrm{Ext}\,[A\,|\,C]$, it suffices to show that $\mathrm{Ext}[A\,|\,C]$ is closed. Suppose that there is a sequence $\{x_n\}\subset\mathrm{Ext}\,[A\,|\,C]$ converging to $\bar{x}\notin\mathrm{Ext}[A\,|\,C]$. Since $\bar{x}\in A$ and $\bar{x}\notin\mathrm{Ext}\,[A\,|\,C]$, there exists $\bar{y}\in A$ and nonzero $\bar{c}\in C$ such that $\bar{x}=\bar{y}+\bar{c}$. Since $C\backslash\{0\}$ is open, there exists an open neighborhood U of $\bar{c}$ such that

$$U \subset C \text{ and } 0 \notin U. \tag{18}$$

Consequently $\bar{y}+U$ is a neighborhood of $\bar{x}$ and hence there is a number $N>0$ such that $x_k \in \bar{y}+U$ for any $k>N$. Therefore, for any $k>N$, there exists $c_k \in U$ such that $x_k = \bar{y}+c_k$. From (18), $0 \neq c_k \in C$ and $\bar{y} \in A$, and from this it follows that $x_k \notin \text{Ext}\ [A \mid C]$. This is a contradiction. Thus Ext $[A \mid C]$ is closed. Furthermore, if C is acute then Ext $[A \mid C] \neq \phi$ by Theorem 3.1.

If we drop the assumption that $C \setminus \{0\}$ is open then Ext $[A \mid C]$ is not always compact.

EXAMPLE 3.1. Let

$$\begin{aligned} A = &\{(x, y) \in \boldsymbol{R}^2 \mid -3 \leq y \leq (x+2)^2-3,\ -2 \leq x \leq -1\} \\ &\cup \{(x, y) \in \boldsymbol{R}^2 \mid -3 \leq y \leq 2x,\ -1 \leq x \leq 0\} \\ &\cup \{(x, y) \in \boldsymbol{R}^2 \mid -3 \leq y \leq -2x,\ 0 \leq x \leq 1\} \\ &\cup \{(x, y) \in \boldsymbol{R}^2 \mid -3 \leq y \leq (x-2)^2-3,\ 1 \leq x \leq 2\}, \end{aligned}$$

and

$$C = \{(x, y) \in \boldsymbol{R}^2 \mid y \leq -2|x|\}.$$

It is seen that Ext $[A \mid C]$ is not compact;

$$\begin{aligned} \text{Ext}\ [A \mid C] = &\{(0, 0)\} \\ &\cup \{(x, y) \in \boldsymbol{R}^2 \mid y = (x+2)^2-3,\ -2 \leq x < -1\} \\ &\cup \{(x, y) \in \boldsymbol{R}^2 \mid y = (x-2)^2-3,\ 1 < x \leq 2\}. \end{aligned}$$

THEOREM 3.4. *Let A and C_+ be a compact subset and a cone in $\boldsymbol{R}^n$, respectively, and $C_- = -C_+$. We assume that*

$C_+ \setminus \{0\}$ is open, or else

C_+ is closed.

Then Ext $[A \mid C_-]$ *is C_+-compact.*

PROOF. In case $C_+ \setminus \{0\}$ is open, the conclusion is immediately obtained by Theorem 3.3. We assume that C_+ is closed, and hence so is C_-. If Ext $[A \mid C_-] = \phi$, the conclusion is clear, and assume that Ext $[A \mid C_-] \neq \phi$. We show that $(x_0 - \text{cl}\, C_+) \cap \text{Ext}\ [A \mid C_-]$ is compact for every $x_0 \in \text{Ext}\ [A \mid C_-]$. Suppose that there is $\widehat{x} \neq x_0$ such that $\widehat{x} \in (x_0 - \text{cl}\, C_+) \cap \text{Ext}\ [A \mid C_-]$. Then there exists nonzero $c' \in C_-$ such that $\widehat{x} = x_0 + c'$. This follows that $\widehat{x} \notin \text{Ext}\ [A \mid C_-]$, which is a contradiction. Thus $(x_0 - \text{cl}\, C_+) \cap \text{Ext}\ [A \mid C_-] = \{x_0\}$, which is compact in $\boldsymbol{R}^n$, and hence C_+-compact.

In Example 3.1, Ext $[A \mid C]$ is $(-C)$-compact. The following example shows that

Ext $[A|C_-]$ is not compact even if A is convex. However, it is C_+-compact.

EXAMPLE 3.2. Let

$$A=\left\{(x, y, z) \in \boldsymbol{R}^3 | x^2+y^2 \leq \left(\frac{1}{2} z\right)^2, x\leq 0, -4 \leq x \leq 0\right\}$$

$$\cup\left\{(x, y, z) \in \boldsymbol{R}^3 | \frac{x^2}{4}+y^2 \leq \left(\frac{1}{2} z\right)^2, x\geq 0, -4 \leq z \leq 0\right\}$$

and

$$C_-=\left\{(x, y, z)\in \boldsymbol{R}^3 | x^2+y^2\leq \left(\frac{1}{2} z\right)^2, z \leq 0\right\}.$$

It is seen that Ext $[A|C_-]$ is not compact;

$$\operatorname{Ext}[A|C_-]=\left\{ (x, y, z) \in \boldsymbol{R}^3 | \frac{x^2}{4}+y^2=\left(\frac{1}{2} z\right)^2, x>0, -4 \leq z \leq 0\right\}.$$

In general, it is known that $A+B$ is compact if A and B are two compact sets in a topological vector space. However, even if both A is compact and B is C-compact in $\boldsymbol{R}^n$, $A+B$ is necessarily neither compact nor C-compact in $\boldsymbol{R}^n$.

EXAMPLE 3.3. Consider Ext $[A|C]$ of Example 3.1 as B, that is,

$$B=\{(0, 0)\}$$
$$\cup \{(x, y) \in \boldsymbol{R}^2 | y=(x+2)^2-3, -2\leq x<-1\}$$
$$\cup \{(x, y)\in \boldsymbol{R}^2 | y=(x-2)^2-3, 1<x\leq 2\}.$$

Let

$$A=\left\{(x, y)\in \boldsymbol{R}^2 | y=\frac{1}{2} x, 0\leq x\leq 2\right\}$$

and

$$C=\{(x, y)\in \boldsymbol{R}^2 | y\leq -2|x|\}.$$

It is seen that $A+B$ is not C-compact.

4. Cone Boundedness and Cone Closedness

DEFINITION 4.1. Let C be a cone in $\boldsymbol{R}^n$. A set A is said to be *C-bounded* if there is $a_0\in \boldsymbol{R}^n$ such that $A\subset a_0+C$. And a set A is said to be *C-closed* if $A+\operatorname{cl} C$ is closed.

The following proposition may be easily proved (cf. [**4**] and [**6**]).

PROPOSITION 4.1. *Let A and C be a nonempty subset and an acute convex cone in* $\boldsymbol{R}^n$, *respectively. If A is C-bounded and C-closed, then* $A+\operatorname{cl} C$ *is C-compact and* Ext $[A|C]\neq\phi$.

Then we have the following result.

THEOREM 4.1. *Let C be an acute convex cone with the property* (13) *in $\boldsymbol{R}^n$. If A is C-bounded and C-closed then*

$$A \subset \text{Ext}\,[A \mid \text{cl}\,C] + \text{cl}\,C \subset \text{Ext}\,[A \mid C] + \text{cl}\,C$$

and

$$A \cap \{A + (C \setminus \{0\})\} \subset \text{Ext}\,[A \mid C] + C.$$

PROOF. Since $A + \text{cl}\,C$ is C-compact by Proposition 4.1, we have

$$A + \text{cl}\,C \subset \text{Ext}\,[A + \text{cl}\,C \mid \text{cl}\,C] + \text{cl}\,C,$$

by Theorem 3.2 and Corollary 3.3. By using (1) and (2),

$$A \subset \text{Ext}\,[A \mid C] + \text{cl}\,C.$$

Moreover, since $(C \setminus \{0\}) + \text{cl}\,C \subset C$,

$$A + (C \setminus \{0\}) \subset \text{Ext}\,[A \mid C] + \text{cl}\,C + (C \setminus \{0\}) \subset \text{Ext}\,[A \mid C] + C.$$

This completes the proof.

COROLLARY 4.1. *Let C be an acute closed convex cone in $\boldsymbol{R}^n$. If A is C-bounded and C-closed, then $A \subset \text{Ext}\,[A \mid C] + C$.*

PROOF. The proof is a direct consequence of Theorem 4.1.

COROLLARY 4.2. *Let C be an acute convex cone with the property* (13) *in $\boldsymbol{R}^n$. If A is C-bounded and C-closed then*

$$A + \text{cl}\,C = \text{Ext}[A \mid \text{cl}\,C] + \text{cl}\,C.$$

Further, if C is closed then

$$A + C = \text{Ext}\,[A \mid C] + C.$$

PROOF. Clearly, $A + \text{cl}\,C \supset \text{Ext}\,[A \mid \text{cl}\,C] + \text{cl}\,C$. Conversely, by Theorem 4.1, $A \subset \text{Ext}\,[A \mid \text{cl}\,C] + \text{cl}\,C$. And hence $A + \text{cl}\,C \subset \text{Ext}\,[A \mid \text{cl}\,C] + \text{cl}\,C$. Further, if C is closed, the conclusion is clear.

Theorem 4.1, Corollary 4.1 and Corollary 4.2 correspond to Theorem 3.2, Theorem 3.1 and Corollary 3.1, respectively, but there is no relation of inclusion between each of these.

COROLLARY 4.3. *Let C_1 and C_2 be two acute convex cones in $\boldsymbol{R}^n$ such that $C_1 \subset C_2$. If A is C_2-bounded and C_2-closed, then*

$$A \subset \text{Ext}\,[A+\text{cl}\,C_2 | \text{cl}\,C_1]+\text{cl}\,C_1$$
$$\subset \text{Ext}\,[A+\text{cl}\,C_2 | \text{cl}\,C_2]+\text{cl}\,C_2$$
$$\subset \text{Ext}\,[A | \text{cl}\,C_2]+\text{cl}\,C_2$$
$$\subset \text{Ext}\,[A | C_2]+\text{cl}\,C_2$$
$$\subset \text{Ext}\,[A | C_1]+\text{cl}\,C_2.$$

PROOF. Since the acute convex cones $\text{cl}\,C_1$ and $\text{cl}\,C_2$ satisfy the conditions (i), (ii) and (iii) of Corollary 3.2,

$$A \subset \text{Ext}\,[A+\text{cl}\,C_2 | \text{cl}\,C_1]+\text{cl}\,C_1 \subset \text{Ext}\,[A+\text{cl}\,C_2 | \text{cl}\,C_2]+\text{cl}\,C_2.$$

The remaining statements are clear from (1) and (2).

If a subset A is compact, it is also C-closed for every cone C, but it is not necessarily C-bounded. We assume that

(a) A is compact and there is $a_0 \in \boldsymbol{R}^n$ such that

$$A \subset [C]+a_0,$$

where $[C]$ denotes the subspace generated by C, or else

(b) A is compact and $\text{int}\,C \neq \phi$.

Then A is C-bounded.

EXAMPLE 4.1. Let

$$A=\{(x, y) \in \boldsymbol{R}^2 | (x-2)^2+(y-2)^2 \leq 1\},$$

and

$$C=\{(x, y) \in \boldsymbol{R}^n | y=x,\ x \leq 0\}.$$

It is seen that A is compact and convex but not C-bounded.

Moreover, even if A is C-compact, it is necessarily neither C-bounded nor C-closed. Conversely, even if A is C-bounded and C-closed, it is not necessarily C-comapct. The following example shows that there exist cases satisfying precisely one of the following three properties: C-compact, C-bounded, and C-closed.

EXAMPLE 4.2. Let

$$C=\{(x, y) \in \boldsymbol{R}^2 | y \geq 2|x|\}$$

and three convex sets

$$A_1=\{(x, y) \in \boldsymbol{R}^2 | 1 \leq y < -x+1\},$$

$$A_2 = \{(x, y) \in \boldsymbol{R}^2 \mid x > 0,\ y > 0\} \cup \{(x, 0) \in \boldsymbol{R}^2 \mid x \geq 0\}$$

and

$$A_3 = \{(x, y) \in \boldsymbol{R}^2 \mid y > 2|x| + 1\}.$$

It is seen that A_1 is C-compact but neither C-bounded nor C-closed, A_2 is C-closed but neither C-compact nor C-bounded, and that A_3 is C-bounded but neither C-compact nor C-closed.

And if C is an acute closed convex cone, there exists a nonempty convex set $A \subset \boldsymbol{R}^n$ such that

(i) $A \subset \mathrm{Ext}\,[A \mid C] + C$,

(ii) A is not C-compact,

(iii) A is not C-bounded

and

(iv) A is not C-closed.

Example 4.3. Let

$$A = \{(x, y) \in \boldsymbol{R}^2 \mid x > 0,\ y \geq 0\}$$

and

$$C = \{(x, y) \in \boldsymbol{R}^2 \mid y \geq 2|x|\}.$$

Then it is clear that A satisfies the desired properties.

Further, even if A is a nonempty compact convex set and B is C-compact, C-bounded and C-closed where C is an acute closed convex cone in $\boldsymbol{R}^n$, $A+B$ is not necessarily C-compact.

Example 4.4. Let

$$A = \{(x, y) \in \boldsymbol{R}^2 \mid y = 3x,\ 0 \leq x \leq 1\},$$

$$B = \{(x, y) \in \boldsymbol{R}^2 \mid y = -(x-3)^2 + 2,\ 2 \leq x < 4\} \cup \left\{\left(\frac{9}{2},\ 0\right)\right\},$$

and

$$C = \{(x, y) \in \boldsymbol{R}^2 \mid y \geq 2|x|\}.$$

It is seen that $A+B$ is not C-compact.

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