On some cases where n-positivity coincides with complete positivity*

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The difference between the notions of positivity and complete positivity of various types of maps associated to operator algebras is discussed in [6]. It is shown that the commutativity of one of the associated algebras makes no difference between these two notions and conversely when this is the case, it implies the commutativity of one of the associated algebras. In this paper we investigate some typical cases where the notion of n-positivity coincides with that of complete positivity for those maps not necessarily defined between C^*-algebras. The case where maps are defined between C^*-algebras has been treated in Choi [1]. Let E and F be C^*-algebras, the duals of C^*-algebras, or the preduals of von Neumann algebras. Then our result states that every n-positive linear map of the $n\times n$ matrix space $M_n(E)$ to the space F is completely positive provided the associated algebra of E is abelian and is also the case for a n-positive linear map of E to $M_n(F)$ provided the associated algebra of F is abelian.

Let A be a C^*-algebra and M a von Neumann algebra. We can equip the $n\times n$ matrix space over the dual space A^*,

$$M_n(A^*)=\{f=[f_{ij}]\,|\,f_{ij}\in A^*\},$$

with the order of the dual space of the matrix C^*-algebra $M_n(A)=\{a=[a_{ij}]\mid a_{ij}\in A\}$. Let M_* be the predual of a von Neumann algebra M, then we can also equip the space $M_n(M_*)$ with the relative order in $M_n(M^*)$. Let E and F be C^*-algebras, the duals of C^*-algebras, or the preduals of von Neumann algebras. A linear map τ of E to F is said to be n-positive if the map

$$\tau_n : [c_{ij}]\in M_n(E)\longmapsto [\tau(c_{ij})]\in M_n(F)$$

is positive and τ is said to be completely positive if τ is n-positive for every positive integer n.

The following lemma is an immediate consequence of [2, Lemmas 4. 1 and 4. 3].

LEMMA 1. *Let E be a C^*-algebra, the dual of a C^*-algebra, or the predual of a von Neumann algebra. Then every n-positive linear map of the $n\times n$ complex matrix algebra $M_n(\mathbf{C})$ to E is completely positive and moreover every n-positive linear map of E to $M_n(\mathbf{C})$ is completely positive.*

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The following Theorems 2 and 3 give us typical cases where the notion of n-positivity coincides with that of complete positivity.

THEOREM 2. *Let E and F be C^*-algebras, the duals of C^*-algebras, or the preduals of von Neumann algebras. If the associated algebra of F is abelian, then every n-positive linear map of E to $M_n(F)$ is completely positive.*

THEOREM 3. *Let E and F be C^*-algebras, the duals of C^*-algebras, or the preduals of von Neumann algebras. If the associated algebra of E is abelian, then every n-positive linear map of $M_n(E)$ to F is completely positive.*

By [6, Proposition 1. 1], we may consider the transpose of a given n-positive linear map of $M_n(E)$ to F. Hence Theorem 3 can be reduced to Theorem 2. Therefore it suffices to prove only Theorem 2. We split our proof into the following lemmas.

LEMMA 4. *Let E be a C^*-algebra, the dual of a C^*-algebra, or the predual of a von Neumann algebra and let B be a C^*-algebra. If B is abelian, then every n-positive linear map τ of E to $M_n(B)$ is completely positive.*

PROOF. Suppose that B is abelian. Then we may assume that $B=C_0(\Omega)$, the C^*-algebra of all continuous functions on a locally compact Hausdorff space Ω vanishing at infinity. Take a positive element $[c_{ij}]$ in the $m\times m$ matrix space $M_m(E)$ and $\tau(c_{ij})=[b_{k,l}^{i,j}]_{k,l}\in M_n(B)$. For every element $t\in\Omega$, we define a linear map σ_t of $M_n(B)$ to $M_n(C)$ by

$$\sigma_t[b_{k,l}]=[b_{k,l}(t)].$$

Then by the definition, the map σ_t is completely positive for every $t\in\Omega$. Hence the composed linear map $\sigma_t\circ\tau$ of E to $M_n(C)$ is n-positive, and so by Lemma 1 the map $\sigma_t\circ\tau$ is completely positive for every $t\in\Omega$. Hence the matrix

$$[[b_{k,l}^{i,j}(t)]_{k,l}]_{i,j}=(\sigma_t\circ\tau)_m[c_{ij}]$$

is positive on $M_m(M_n(C))$ for every $t\in\Omega$. Hence the matrix $[\tau(c_{ij})]$ is positive on $M_m(M_n(B))$. Therefore τ is completely positive and the proof is completed.

LEMMA 5. *Let E be a C^*-algebra, the dual of a C^*-algebra, or the predual of a von Neumann algebra and let B be a C^*-algebra. If B is abelian, then every n-positive linear map τ of E to $M_n(B^*)$ is completely positive.*

PROOF. Suppose that B is abelian and write $B=C_0(\Omega)$ as in Lemma 4. Then B^* is the Banach space of all finite Radon measures on Ω. Take a positive element $[c_{ij}]$ in the $m\times m$ matrix space $M_m(E)$ and put $\tau(c_{ij})=[\mu_{k,l}^{i,j}]_{k,l}\in M_n(B^*)$. Since every positive element of $M_m(M_n(B))$ is a sum of elements of the form

$$[\,[\,b_k^{i*}b_l^j\,]_{k,l}]_{i,j}\,,$$

it is enough to show that

$$\sum_{i,j=1}^{m}\sum_{k,l=1}^{n}\left\langle b_k^{i*}b_l^j,\quad \mu_{k,l}^{i,j}\right\rangle \quad \geqq 0.$$

Let $\mu=\sum_{i,j=1}^{m}\sum_{k,l=1}^{n}|\mu_{k,l}^{i,j}|$, where $|\mu_{k,l}^{i,j}|$ means the absolute value of the measure $\mu_{k,l}^{i,j}$. By Radon-Nikodym theorem, there exists a function $x_{k,l}^{i,j}\in L^1(\Omega,\mu)$ for each (i,j,k,l) such that

$$<b,\mu_{k,l}^{i,j}>=\int_\Omega x_{k,l}^{i,j}(t)b(t)\,d\mu(t) \qquad b\in B.$$

For every $b\in B$, we define a linear map ρ_b of $M_n(B^*)$ to $M_n(C)$ by

$$\rho_b\,[\varphi_{k,l}]=[\varphi_{k,l}\,(b^*b)\,].$$

By the definition ρ_b is completely positive for every $b\in B$. Hence the composed linear map $\rho_b\circ\tau$ of E to $M_n(C)$ is n-positive, and so by Lemma 1 the map $\rho_b\circ\tau$ is completely positive for every $b\in B$. Hence

$$[\,[\,\mu_{k,l}^{i,j}(b^*b)\,]_{k,l}]_{i,j}=(\rho_b\circ\tau)_m\,[c_{ij}]$$

is positive on $M_m(M_n(C))$ for every $b\in B$. Therefore for any $n\times m$ tuple $\{\lambda_k^i\,|1\leqq i\leqq m,$ $1\leqq k\leqq n\}$ of complex numbers, $\sum_{i,j=1}^{m}\sum_{k,l=1}^{n}\overline{\lambda_k^i}\lambda_l^j\,\mu_{k,l}^{i,j}$ is a positive measure on Ω, and so

$$\sum_{i,j=1}^{m}\sum_{k,l=1}^{n}\overline{\lambda_k^i}\lambda_l^j\;x_{k,l}^{i,j}(t)\geqq 0 \qquad \text{a.e } \mu.$$

Hence there exists a fixed null set such that

$$\sum_{i,j=1}^{m}\sum_{k,l=1}^{n}\overline{\lambda_k^i}\lambda_l^j\,x_{k,l}^{i,j}(t)\geqq 0 \qquad \text{a.e } \mu$$

for every $n\times m$ tuple $\{\lambda_k^i\,|\,1\leqq i\leqq m,\ 1\leqq k\leqq n\}$ of complex numbers. Therefore we have

$$\sum_{i,j=1}^{m}\sum_{k,l=1}^{n}<b_k^{i*}\,b_l^j\ ,\ \mu_{k,l}^{i,j}>$$

$$=\sum_{i,j=1}^{m}\sum_{k,l=1}^{n}\int_\Omega \overline{b_k^i(t)}\,b_l^j(t)\;x_{k,l}^{i,j}(t)\,d\mu(t)$$

$$=\int_\Omega\sum_{i,j=1}^{m}\sum_{k,l=1}^{n}\overline{b_k^i(t)}\,b_l^j(t)x_{k,l}^{i,j}(t)\,d\mu(t)\geqq 0.$$

This completes the proof.

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