A Spectral Characterization of a Class of C*-algebras

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(Received October 10, 1977)

1. Introduction

R. A. Hirschfeld and B. E. Johnson [2] studied those C*-algebras of which every self-adjoint element has a finite spectrum. T. Ogasawara and K. Yoshinaga [4] proved that a C*-algebra is dual if and only if every self-adjoint element has a spectrum without limit points other than zero. In this paper we present conditions on a C*-algebra under which every self-adjoint element has a countable spectrum.

2. Preliminaries

We state at first the definition of a dual C*-algebra.

A C*-algebra A is called dual if there is a Hilbert space H such that A is *-isomorphic to a C*-algebra of the C*-algebra of all compact operators on H.

A C*-algebra A is called liminal if for every irreducible representation π of A, $\pi(a)$ is compact for each $a \in A$.

If A is a C*-algebra, A^* denotes its conjugate space and A^{**} denotes its second conjugate space. Assuming A is in its universal representation, then the σ-weak closure of A can be identified with A^{**}.

If A and B are C*-algebras, $A \otimes_\alpha B$ denotes their spaitial C*-tensor product, $A^{**} \otimes^- B^{**}$ denotes the W*-tensor product of A^{**} and B^{**}, and $A^* \otimes_{\alpha'} B^*$ denotes the norm closure of the algebraic tensor product of A^* and B^* in $(A \otimes_\alpha B)^*$.

If X is a compact Hausdorff space, $C(X)$ denotes the C*-algebra of all continuous functions on X, and $C(X)^{*+}$ denotes the set of all positive linear functionals on $C(X)$. By the Riesz representation theorem we can identify $C(X)^*$ with the space of all bounded complex regular Borel measures on X. We recall that a pure atomic functional ϕ on $C(X)$ is of the form:

$$\phi = \sum_{i=1}^{\infty} \alpha_i \delta_{t_i},$$

where $\{\alpha_i\}$ is a sequence in the complex field with $\sum_{i=1}^{\infty} |\alpha_i| < \infty$, $\{t_i\}$ is a sequence in X, and δ_t denotes the evaluation functional of a point $t \in X$.

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The following lemma is obtained by a method similar to [3, Lemma].

LEMMA 1. *Let X be a compact Hausdorff space. Suppose that $(C(X)\otimes_{\alpha} C(X))^* = C(X)^*\otimes_{\alpha'} C(X)^*$. Then every self-adjoint element of $C(X)$ has a countable spectrum.*

PROOF. For $\phi \in C(X\times X)^*$, define $\phi_{\Delta} \in C(X\times X)^*$ by

$$\phi_{\Delta}(a) = \iint_{X\times X} \chi_{\Delta}(s, t) a(s, t) d\phi(s, t)$$

where χ_{Δ} denotes the characteristic function of the diagonal set $\Delta = \{(t, t) : t\in X\}$. Let μ, $\nu \in C(X)^*$. By the Fubini theorem, we have

$$\begin{aligned}(\mu\times\nu)_{\Delta}(a) &= \iint_{X\times X} \chi_{\Delta}(s, t) a(s, t) d\mu(s) d\nu(t)\\ &= \int_X \left(\int_X \chi_{\Delta}(s, t) a(s, t) d\mu(s) \right) d\nu(t)\\ &= \int_X a(t, t) \mu(\{t\}) d\nu(t)\,.\end{aligned}$$

Since $\{t\in X : \mu(\{t\}) \neq 0\}$ is at most countable, $(\mu\times\nu)_{\Delta}$ is purely atomic. Now, $C(X)\otimes_{\alpha} C(X)$ can be identified with $C(X\times X)$. Hence each element of $\{\phi_{\Delta} : \phi \in C(X\times X^*\}$ is purely atomic since the subspace of all purely atomic functionals on $C(X\times X)$ is closed.

Let $\Psi \in C(X)^*$. For $a\in C(X\times X)$, define $\Psi^{-} \in C(X\times X)^{*+}$ by

$$\Psi^{-}(a) = \int_X a(t, t) d\Psi(t)\,.$$

Since Δ contains the support of $\Psi^{-}{}_{\Delta} = \Psi^{-}$. Thus, Ψ^{-} is of the form:

$$\Psi^{-} = \textstyle\sum_{i=1}^{\infty} \alpha_i \delta_{(t_i,\, t_i)}\,.$$

For a positive element $a\in C(X)$, the function $a^{-} : (s, t) \longrightarrow a(s)^{1/2} a(t)^{1/2}$ is in $C(X\times X)$. Then we have

$$\Psi(a) = \Psi^{-}(a^{-}) = \textstyle\sum_{i=1}^{\infty} a(t_i)^{1/2} a(t_i)^{1/2} = (\sum_{i=1}^{\infty} \alpha_i \delta_{t_i})(a)\,.$$

Therefore, Ψ is purely atomic. It follows from [5, Theorem] that every self-adjoint element of $C(X)$ has a countable spectrum.

LEMMA 2. *Let A be a C^*-algebra. Suppose that $(A\otimes_{\alpha} B)^* = A^*\otimes_{\alpha'} B^*$ for an arbitary C^*-algebra B. Then every self-adjoint element of A has a countable spectrum.*

PROOF. Let h be a self-adjoint element of A and C be a maximal commutative self-adjoint subalgebra of A which contains h. It is easy to see that

$$(C\otimes_{\alpha} C)^* = C^*\otimes_{\alpha'} C^*.$$

By Lemma 1, h has a countable spectrum. This completes the proof.

LEMMA 3. *Let A be a C^*-algebra. Suppose that A contains a minimal projection. Then A contains a non-zero dual closed two-sided ideal.*

PROOF. Let J be the closed two-sided ideal generated by a minimal projection. Then

J is a minimal ideal. Since J is simple and contains a minimal projection, the image of an irreducible representation of J is *-isomorphic to the C^*-algebra of all compact operators on the representation space. Hence J is dual.

3. Theorem

We are in the position to state and prove our theorem.

THOEREM. *Let A be a C^*-algebra. Then the following statements are equivalent:*

(1) *Every self-adjoint element of A has a countabe spectrum.*

(2) *A has a composition series $(I_\rho)_{0 \leqq \rho \leqq \alpha}$ such that $I_{\rho+1}/I_\rho$ is dual.*

(3) *The second conjugate space of A is atomic, that is, it is a sum of factors of type I.*

PROOF. If A has no identity element, A_1 denotes the C*-algebra obtained by adjoining an identity element to A. It is easy to see that each of the three statements holds if it holds with A_1 in place of A. Thus there is no loss of generality in assuming that A has an identity element.

(1)$\longrightarrow$(2). The proof is a modification of the method used by J. Tomiyama [7]. We first show that there is a minimal projection in A. Let C be a maximal commutative self-adjoint subalgebra of A and let X be the carrier space of C. Since C has an identity element, X is compact. Then the Gelfand transformation: $x \longrightarrow x^\wedge$ is a *-isomorphism of C onto $C(X)$. Let x' be a real function of $C(X)$. Then we have a unique element x of C such that $x^\wedge = x'$. The range of x' is the spectrum of x, which is countable. By [5, Theorom], X has no perfect set. Therefore, there is at least one isolated point x_0 in X, and so the characteristic function p of $\{x_0\}$ is a minimal projection in $C(X)$.

Identifying C with $C(X)$, we may assume that p belongs to C. Let q be a non-zero positive element of A such that $q \leqq p$. For each $a \in C$, we have $ap = pa = \gamma p$ for some complex number γ. Then $aq = apq = \gamma q = qa$. Since C is a maximal commutative self-adjoint subalgebra of A, q belongs to C and $q = \delta p$ for some positive real number δ. Thus there is a minimal projection in A. It follows from Lemma 3 that A contains a non-zero dual closed two-sided ideal.

Finally, let J be an arbitrary closed two-sided ideal in A. Then every self-adjoint element of A/J has a countable spectrum. By the above argument, A/J has a non-zero dual closed two-sided ideal. By transfinite induction there is a composition series $(I_\rho)_{0 \leqq \rho \leqq \alpha}$ such that $I_{\rho+1}/I_\rho$ is dual.

(2)$\longrightarrow$(3). Let I_ρ^- be the σ-weak closure of I_ρ in A^{**}. Since I_ρ^- is a σ-weakly closed two-sided ideal in A^{**}, there is a central projection z_ρ such that $I_\rho^- = A^{**} z_\rho$. Then $A^{**}(z_{\rho+1} - z_\rho)$ is the σ-weak closure of the image of representation of $I_{\rho+1}/I_\rho$. Since the second conjugate space of a dual C*-algebra is atomic, so is $A^{**}(z_{\rho+1} - z_\rho)$. Then we have that $A^{**} = \Sigma_\rho A^{**}(z_{\rho+1} - z_\rho)$. Thus, A^{**} is also atomic.

(3)$\longrightarrow$(1). Let f be a state on $A \otimes_\alpha B$, and let π be the representation of $A \otimes_\alpha B$ corresponding to f. By [1, Proposition 1] there are representations π_A and π_B of A and B, respectively, such that $\pi(x\otimes y)=\pi_A(x)\pi_B(y)=\pi_B(y)\pi_A(x)$ for $x\in A$ and $y\in B$. Since A^{**} is atomic, the σ-weak closure of $\pi_A(A)$ is atomic. Hence, the map: $\pi_A(x)\otimes\pi_B(y)\longrightarrow\pi(x\otimes y)$ extends to a normal homomorphism of $\pi_A(A)^-\otimes\pi_B(B)^-$. Thus, $f\in A^*\otimes_{\alpha'}B^*$, and so $(A\otimes_\alpha B)^*=A^*\otimes_{\alpha'}B^*$. Then Lemma 2 implies (1). This completes the proof

REMARK 1. P. Wojtaszczyk [8] considered other conditions in the separable case.

REMARK 2. Let A and B be C*-algebras. By [9, Theoreme 1] there is a canonical *-isomorphism π of $A\otimes_\alpha B$ into $A^{**}\otimes^- B^{**}$. Then π has a unique normal extension π^- to $(A\otimes_\alpha B)^{**}$; π^- is a *-homomorphism of $(A\otimes_\alpha B)^{**}$ onto $A^{**}\otimes^- B^{**}$. If π^- is a *-isomorphism, we shall say that $(A\otimes_\alpha B)^{**}$ is canonically *-isomorphic to $A^{**}\otimes^- B^{**}$. It is easy to see that $(A\otimes_\alpha B)^{**}$ is canonically *-isomorphic to $A^{**}\otimes^- B^{**}$ if and only if $(A\otimes_\alpha B)^*=A^*\otimes_{\alpha'}B^*$ (see [6, pp. 66–67]). Thus, each of the three statements of the Theorem is equivalent to the following:

(4) For an arbitrary C*-algebra B, $(A\otimes_\alpha B)^{**}$ is canonically *-isomorphic to $A^{**}\otimes^- B^{**}$.

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