A note on SO(3)-action on CP_3

By

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Introduction

Let $SU(2)$ be the special unitary group of dimension 2 and $SO(3)$ identified with $Ad\ SU(2)$, where $Ad: SU(2) \to O(3)$ is the adjoint representation. If an $SU(2)$-action on CP_3 (= the complex projective 3-space) has $Ker\ Ad$ as its ineffective kernel, it induces an $SO(3)$-action on CP_3. We shall call the action of $SO(3)$ induced by a linear $SU(2)$-action on CP_3 linear action.

In this note we shall prove that possible orbit types of $SO(3)$ actions on CP_3 are like those of linear actions. This note also contains a correction of an argument in the paper [6] ([6], p. 5) of one of the present authors.

We shall use the following notations.

S= the standard maximal torus of $SU(2)$

$$T = Ad\ S = \left\{ \begin{bmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix} t \in R \right\}; \text{ the maximal torus of } SO(3).$$

$$a = Ad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix}, \quad b = Ad \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix}$$

$N = N(T) = T \cup aT$, the normalizer of T in $SO(3)$

$D_2 = \{e, a, b, ab\} = Z_2 + Z_2$

ϕ_r; the irreducible representation of $SU(2)$ of degree $r+1$

$[z_1, z_2, z_3, z_4]$; the homogeneous coordinate on CP_3.

1. Linear actions on CP_3

1.1. The action induced by ϕ_3.

Consider the action of $SU(2)$ on CP_3 induced by ϕ_3. Recall that $\phi_3: SU(2) \longrightarrow U(4)$ is given by

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$$\phi_3\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}=\begin{bmatrix}\alpha^3 & \sqrt{3}\alpha^2\beta & \sqrt{3}\alpha\beta^2 & \beta^2\\ \sqrt{3}\alpha^2\gamma & \alpha^2\delta+2\alpha\beta\gamma & \beta^2\gamma+2\alpha\beta\delta & \sqrt{3}\beta^2\delta\\ \sqrt{3}\alpha\gamma^2 & \beta\gamma^2+2\alpha\gamma\delta & \alpha\delta^2+2\beta\gamma\delta & \sqrt{3}\beta\delta^2\\ \gamma^3 & \sqrt{3}\gamma^2\delta & \sqrt{3}\gamma\delta^2 & \delta^3\end{bmatrix}$$

It is clear that this action induces an $SO(3)$-action on CP_3. By direct computations we can show that

(i) principal isotropy subgroups are trivial.

(ii) $F(T, CP_3)=$ {isolated four points}

(iii) $F(a, CP_3)\cap F(b, CP_3)=\emptyset$

(iv) $SO(3)_{(1,0,1,0)}=D_3$ (=dihedral subgroup).

1.2. The action induced by ϕ_2

Recall that ϕ_2: $SU(2)\longrightarrow U(3)$ is given by

$$\phi_2\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}=\begin{bmatrix}\alpha^2 & \sqrt{2}\alpha\beta & \beta^2\\ \sqrt{2}\alpha\gamma & \alpha\delta+\beta\gamma & \sqrt{2}\beta\delta\\ \gamma^2 & \sqrt{2}\gamma\delta & \delta^2\end{bmatrix}$$

The action of $SU(2)$ on CP_3 induced by $\phi_2+\phi_0$ induces an action of $SO(3)$ on CP_3. We can show that

(i) principal isotropy subgroups are trivial.

(ii) $F(T, CP_3)=$union of one 2-sphere and isolated two points.

(iii) $F(N, CP_3)=$ {isolated two points}.

(iv) $F(SO(3), CP_3)=\{[0, 0, 0, 1]\}$.

(v) There is a point whose isotropy subgroup is Z_2.

(vi) Possible isotropy subgroups are $\{e\}$, (T), (N), G, (Z_2).

1.3. The action induced by $\phi_1+\phi_1$.

Recall that ϕ_1; $SU(2)\longrightarrow U(2)$ is given by

$$\phi_1\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}=\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}$$

This action also induces an action of $SO(3)$ on CP_3. For this action, we have

(i) principal isotropy subgroups are trivial.

(ii) $F(T, CP_3)=F(b, CP_3)=$union of two 2-spheres

(iii) $F(D_2, CP_3)=\emptyset$

(iv) Possible isotropy subgroups are $\{e\}$ and (T).

2. $SO(3)$-action on homotopy complex projective 3-space

Let M be a homotopy complex projective 3-space and $SO(3)$ act on M. We can show the following

PROPOSITION (2. 1) *There is a point x in M such that $SO(3)_x$ is a maximal torus.*

PROOF. Assume the contrary. Then for every point $x \in M$, $H^*(SO(3)(x); Q) = H^*(pt., Q)$ or $H^*(S^3; Q)$. By Vietoris-Beagle theorem, the orbit map π: $M \to M/SO(3)$ induces an isomorphism π^*: $H^2(M/SO(3); Q) \to H^2(M; Q)$, and hence any generator of $H^*(M, Q)$ is in the image of π^*. Since dim $M/SO(3) \leqq 4$, this contradicts to the structure of $H^*(M; Q)$. This completes the proof.

From results in [2] and [4], it follows that possible types of $F(T, M)$ are

Case 1. $F(T, M)$ = union of two 2-spheres

Case 2. $F(T, M)$ = union of one 2-sphere and two isolated points

Case 3. $F(T, M)$ = union of isolated four points.

We shall consider the case 1. Denote $F(T, M) = S_1^2 \cup S_2^2$. It is clear that $F(d, M) = F(T, M)$ for every element d of T of finite order. This implies that principal isotropy subgroup are trivial. We prove the following

PROPOSITION (2. 2) $F(D_2, M) = \emptyset$

PROOF. Assume the contrary. By the same arguments as in [6] ([6], p. 5) is follows that $F(D_2, M)$ consists of just four points; put $F(D_2, M) = \{x_1, x_2, x_3, x_4\}$. Since $F(N, M) = F(a, F(T, M))$, $F(b, M) = F(T, M)$ and $F(a, M) \cap F(b, M) = F(D_2, M)$, we have $F(N, M) = F(D_2, M)$. From a result in [3] ((3. 7) in [3]) it follows that

$$G(x_1) \cap F(D_2, M) = \begin{cases} \{\text{three points}\} & \text{if } G_{x_1} = N \\ \{\text{one point}\} & \text{if } G_{x_1} = G. \end{cases}$$

Here G denotes $SO(3)$. If $F(G, M) = \emptyset$, $G(x_1) = G(x_i)$ for some $i = 2, 3, 4$ and hence $x_i = gx_1$ for some $g \in G$. Hence we have $N = gNg^{-1}$ and hence $g \in N$, which implies $x_i = x_1$. Thus we have $F(G, M) \neq \emptyset$. By similar arguments we can prove that it is not the case in which $\emptyset \neq F(G, M) \subsetneqq F(N, M)$. Thus we have proved that $F(G, M) = F(N, M)$. We may assume that $\{x_1, x_2\} \subseteq S_1^2$ and $\{x_3, x_4\} \subset S_2^2$. Let π: $S^7 \to M$ be a principal S^1-bundle. By a result in [4] the action of $SU(2)$ on M define by Ad can be lifted on S^7. Then $F(S, S^7)$ is connected. Since $\pi^{-1}F(G, M) \subset F(SU(2), S^7) \subset F(S, S^7)$, we have $F(G, M) \subset \pi F(SU(2), S^7) \subset \pi F(S, S) \subset F(T, M)$. Thus $F(G, M)$ is contained in $\pi F(S, S^7)$ which is a component of $F(T, M)$. This is a contradiction. This completes the proof of the proposition.

It is easy to conclude that possible isotropy subgroup types are $\{e\}$ and $\{T\}$.

Next we shall consider the case 2. Denote $F(T, M) = S_1^2 \cup \{x_1, x_2\}$. Assume that dim $F(b, M) = 4$. We may assume $F(b, M) = F \cup \{x\}$, where F is a manifold of dimension 4 (see [2] Chap. VII). Since $F(b, M)$ is invariant under a, x lies in $F(a, F(b, M)) = F(D_2, M)$. Since x is an isolated point of $F(b, M)$ $F(a, M)$ and $F(ab, M)$, the Borel formula at x ([1]) leads a contradiction. Denote $F(b, M) = S_1^2 \cup S_2^2$ and assume $\{x_1, x_2\} \subset S_2^2$. By a similar argument in [6], we can show that $F(D_2, M)$ consists of just four points; put

$F(D_2, M)=\{y_1, y_2, y_3, y_4\}$ and assume $\{y_1, y_2\}\subset S_1^2$ and $\{y_3, y_4\}\subset S_2^2$. Since $F(N, M)=(a, F(T, M)=F(a, S_1^2)\cup F(a, \{x_1, x_2\})$, we have $F(N, M)=\{y_1, y_2\}$ or $F(N, M)=\{y_1, y_2, y_3, y_4\}$.

Now we shall prove the following

PROPOSITION (2. 3) *$F(N, M)=\{y_1, y_2\}$ and $\emptyset\neq F(G, M)\subsetneqq F(N, M)$.*

PROOF. By completely similar arguments as in [6], we can prove that $F(N, M)=F(D_2, M)$. If $F(G, M)=\emptyset$, $G(y_i)\cap F(D_2, M)$ consists of just three points for $i=1, 2$ and hence $G(y_1)=G(y_2)$ which leads a contradiction. Thus $F(G, M)\neq\emptyset$. Assume $F(G, M)=F(N, M)$. Then $G(y_3)$ is 3-dimensional. From a result in [4], it follows that $G(y_3)\cap F(D_2, M)$ consists of just six points or three points, which is clearly impossible. This completes the proof of the proposition.

Consider the local representation at the unique fixed point and the action of $SO(3)$ on S^5 induced by the representation. Since the action has S^2 as an orbit, a result in [7] shows that principal isotropy subgroups are trivial.

Moreover we can prove the following

PROPOSITION (2. 4) *There exists a point whose isotropy subgroup is a cyclic group of order 2.*

PROOF. Consider $M=GS_2^2$. Put $F(G, M)=\{y_1\}$. Notice that there is an element $g\in G$ such that $gy_2=y_3$ or y_4. In fact one can choose an element g such that $a=gbg^{-1}$ and $g^2=e$. Then $gy_2\in F(D_2, M)$. Assume $gy_2=y_2$. Then $G_{y_2}=gG_{y_2}g^{-1}$, i. e. $N=gNg^{-1}$, which implies $g\in N$. Thus $gy_2=y_3$ or y_4. We may assume $gy_2=y_3$. We can show that $F(T, M)=\{x_1, x_2, y_2\}$. If $hz(h\in G, z\in S_2^2)$ is in $GS_2^2\cap F(T, M)$, then $hz\in S_1^2\cup\{x_1, x_2\}$. Assume $hz\in S_1^2$ and hence $h^{-1}Th\subset G_z$. Since S_2^2 is N-invariant and $hz\in S_1^2$, $h\in N$. It follows from the fact that dim $G_z=1$ that $G_z=h^{-1}Th$ or $h^{-1}Nh$. Consider the case in which $G_z=h^{-1}Th$. Since $b\in G_z$, $b=h^{-1}th(t\in T)$. Since $t^2=1$, we have $t=b$, and hence $h\in N$, which contradicts to the assumption. Thus we have $G_z=h^{-1}Nh$ and $b=h^{-1}nh$. Since a is the unique element in N of order 2, $b=h^{-1}ah$ and hence $h^2=e$. Thus we have $a\in G_z$ and $z\in F(D_2, M)$. Since $ha=ah$, we have $ahz=haz=hz$ and hence $hz\in F(D_2, M)$ which implies $hz=y_2$. It is now easy to prove the proposition from the following

PROPOSITION (2. 5) *Principal isotropy subgroups of G-action on M_0 are cyclic of order 2.*

PROOF. Let H be principal isotropy subgroup of the action of G on M_0. Clearly H is cyclic. Consider the slice representation at y_2, $\rho_{y_2}: N\rightarrow 0(2)$. Note that dim $M_0=4$. It is clear that $\rho_{y_2}|T: T\rightarrow 0(2)$ is given by

$$(\rho_{y_2}\mid T)\begin{bmatrix}\cos t & \sin t & 0\\ -\sin t & \cos t & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}\cos nt & \sin nt\\ -\sin nt & \cos nt\end{bmatrix}.$$

Let $\rho_{y_2}(a)=(x_{ij})\in O(2)$. It follows from $at=t'a$ (t' is transpose of t) that $x_{11}^2+x_{12}^2$

$=1$, $x_{12}=x_{21}$ and $x_{11}=-x_{22}$. Thus principal isotropy subgroup of ρ_{y_2} is the subgroup generated by a and $\begin{bmatrix} \cos 2\pi/n & \sin 2\pi/n \\ -\sin 2\pi/n & \cos 2\pi/n \end{bmatrix}$ which is a dihedral group D_n. Since must be cyclic, we have $n=1$, which prove the proposition.

It is easy to conclude that possible isotropy subgroup types are $\{e\}$, (Z_2), (T), (N), G and (odd dihedral group).

Remark. We can not determine whether (odd dihedral group) appears as isotropy subgroup actually.

We shall consider the last case 3. Denote $F(T, M)=\{x_1, x_2, x_3, x_4\}$. We shall prove the following.

Proposition (2. 6) *$F(b, M)$ is 2-dimensional and $F(D_2, M)=\emptyset$.*

Proof. Assume $F(b, M)$ is 4-dimensional. Put $F(b, M)=F_0\cup x_1$. Since $F(b, M)$ is invariant under a, $x_1\in F(a, M)$ and hence $F(D_2, M)\neq\emptyset$. But the same arguments as in case 2 show a contradiction. Thus we may put $F(b, M)=S_1^2\cup S_2^2$ and assume $\{x_1, x_2\}\subset S_1^2$. Assume $F(D_2, M)\neq\emptyset$. Then $F(D_2, M)$ consists of four points y_1, y_2, y_3 and y_4. We assume that $y_1, y_2\in S_1^2$ and $y_3, y_4\in S_2^2$. If $F(N, M)=\emptyset$, then $G(y_i)$ is 3-dimensional and hence $G(y_i)\cap F(D_2, M)$ consists of six points, which is impossible. Thus we have $F(N, M)\neq\emptyset$. Since $F(N, M)=F(a, F(T, M))$, we have $F(N, M)=\{x_1, x_2\}$ or $\{x_1, x_2, x_3, x_4\}$. If $F(N, M)=F(T, M)$, there exists no point whose isotropy subgroup is T, which contradicts to the proposition (2. 1). Thus we have $F(N, M)=\{x_1, x_2\}$. Assume $F(G, M)\neq\emptyset$. Consider the local represemtation at a point in $F(G, M)$. This representation define an action of $SO(3)$ on S^5 whose principal isotropy subgroups are icosahedral, which is clearly impossible (see [7]). The assumption $F(G, M)=\emptyset$ also concludes a contradiction. This proves the proposition.

Moreover we can prove the following

Proposition (2. 7) *Principal isotropy subgroups are trivial.*

Proof. Let $H=Z_m$ be a principal isotropy subgroup. We may assume $H\subset T$. It is clear that m is odd. Assume that m is greater than 1 and p is a prime factor of m. Then $F(Z_p, M)$ is 4-dimensional and hence $F(Z_p, M)=F\cup x$, where $F\underset{Z_p}{\sim}CP_2$. Since $F(Z_p, M)$ is T-invariant, $x\in F(T, M)$. Consider slice representation ρ_x at x, $\rho_x\colon T\to O(4)$. Then $\rho_x=t^u+t^v$ in $R(T)=Z[t, t^{-1}]$, where u and v are integers. It is clear that $m=(u, v)$ and $F(Z_p, M)$ is 4-dimensional at x, which is a contradiction. This completes the proof of the proposition.

Thus in the case 3 possible orbit types are $\{e\}$ (Z_2), (odd dihedral group), and (T).

3. Remarks

An argument in [6] ([6], p. 5) which states that the case in which $F(T, M)=S_1^2\cup$

$\{x_1, x_2\}$ and principal isotropy subgroups are finite does not occur is incorrect. In this case we shall prove the following

PROPOSITION. *Let $SO(3)$ act on homotopy complex projective 3-space M. Assume $F(T, M)$ is union of one 2-sphere and two isolated points. Then M is diffeomorphic to the standard complex projective 3-space CP_3.*

PROOF. We have proved in this case that

(i) $F(b, M) = S_1^2 \cup S_2^2$

(ii) $F(N, M) = \{y_1, y_2\} \subset S_1^2$ and $F(G, M) = \{y_1\}$.

(iii) $M_0 = GS_2^2$, $F(T, M_0) = \{x_1, x_2, y_2\}$

(iv) principal isotropy subgroup of G-action on M_0 is cyclic of order 2.

(v) $F(N, M) = \{y_2\}$.

Consider the $SU(2)$-action on CP_2 induced by $\varphi_2: SU(2) \to U(3)$. This action induces an $SO(3)$ action on CP_2. It is easy to show that this action has the same orbit structure as the action on $SO(3)$ on M_0. Since $M_0/SO(3)$ and $CP_2(3)$ are both [0, 1], we can construct a homeomorphism $h: M_0/SO(3) \to CP_2/SO(3)$ preserving orbit structure. From a result in [2] ([2], Chap. II), it follows that M_0 is diffeomorphic to CP_2. Since inclusions $S_2^2 \to M_0$ and $S_2^2 \to M$ induce isomorphisms $H^2(M_0: Q) \to H^2(S_2^2; Q)$ and $H^2(M; Q) \to H^2(S_2^2; Q)$, we have that the inclusion $M_0 \to M$ induces an isomorphism $H^2(M; Q) \simeq H^2(M_0; Q)$. From a result in [6] (Proposition 1) it follows that M is diffeomorphic to CP_3. This completes the proof of the proposition.

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