A Banach algebra which is an ideal in the second conjugate space II

By

Seiji WATANABE

(Received May 20, 1975)

1. Introduction

Let A be a Banach algebra, A^{**} its second conjugate space. Then A^{**} becomes a Banach algebra under the Arens multiplications. For any Banach space X, let π be the cononical embedding of X into X^{**}. When does A^{**} contain $\pi(A)$ as an ideal? In [5] we investigated the condition under which $\pi(A)$ is an ideal in A^{**}. Here we shall consider the following problem.

(1) When is $\pi(A)$ a two-sided ideal in A^{**}?

(2) When is $\pi(A)$ a block subalgebra in A^{**}?

i. e. $\pi(A)A^{**}\pi(A)\subset\pi(A)$.

If $\pi(A)$ is an ideal in A^{**}, it is a block subalgebra of A^{**}.

A Banach algebra A is called *weakly compact* if every left and right multiplication operators on A are weakly compact.

In [5] we have shown that $\pi(A)$ is an ideal in A^{**} if and only if A is weakly compact.

In §3 we shall investigate the special case, and obtain an improvement of a result in [5].

We shall use the notations and definitions given in [5] without notice.

2. General case

Let A be a Banach algebra. Denote by L_a (resp. R_a) the left (resp. right) multiplication operator on A.

Then we have

$$L_a^*(f)=f\circ a,\quad R_a^*(f)=a*f,\quad L_a^{**}(F)=\pi(a)\circ F$$

$$R_a^{**}(F)=F*\pi(a)\quad (a\in A, f\in A^*, F\in A^{**})$$

where T^* (resp. T^{**}) denote the conjugate (resp. second conjugate) operator of an operator T.

Hence we have the following two theorems from the well-known result on weakly compact operators [see 2].

THEOREM 1. *The following three statements are equivalent.*

(1) $\pi(A)$ *is a two-sided ideal in* A^{**}.

(2) A *is weakly compact.*

(3) $f \longrightarrow f \circ a$ *and* $f \longrightarrow a*f$ *are weakly compact on* A^* *for each* $a \in A$.

THEOREM 2. *The following three statements are equivalent.*

(1) $\pi(A)$ *is a block subalgebra of* A^{**}.

(2) $L_a \circ R_b$ *is weakly compact on* A *for each* $a, b \in A$.

(3) $f \longrightarrow a*f \circ b$ *is weakly compact on* A^* *for each* $a, b \in A$.

Next we have the following useful proposition.

PROPOSITION 3. *Let* I *and* B *be a closed two-sided ideal and a closed subalgebra in* A *respectively. Suppose that* $\pi(A)$ *is a two-sided ideal (resp. block subalgebra) of* A^{**}. *Then* $\pi(B)$ *is a two-sided ideal (resp. block subalgebra) of* B^{**} *and* $\pi(A/I)$ *is a two-sided ideal (resp. block subalgebra) of* $(A/I)^{**}$.

PROOF. Let $\{x_n\}$ be a bounded sequence in B and a be in B. Then there exists a subsequence $\{ax_{n'}\}$ of $\{ax_n\}$ such that a weak limit of $ax_{n'}$ exists in A. Since B is weakly closed in A, weak lim $\{ax_{n'}\}$ is in B. On the other hand, let $\{[y_n]\}$ be a bounded sequence in A/I and $[a]$ be in A/I where $[z]$ is a canonical image of $z \in A$ in A/I. Then we may assume that $\{y_n\}$ is a bounded sequence. Hence we can choose a subsequence $\{y_{n'}\}$ of $\{y_n\}$ such that a weak limit $ay_{n'}$ exists in A. Since $(A/I)^*$ is isometrically isomorphic to the polar of I in A^*, weak limit $[a][y_{n'}]$ exists in A/I.

Consequently $\pi(B)$ and $\pi(A/I)$ are left ideals in B^{**} and in $(A/I)^{**}$ respectively. For the other cases we can prove in a similar way.

3. Special Banach algebras

It is well-known that a C*-algebra A is dual if and only if $\pi(A)$ is a two-sided ideal in A^{**}. Recently P. K. Wong [7] proved that a semi-simple Banach algebra A which is a dense two-sided ideal of a semi-simple annihilator Banach algebra is a two-seded ideal in A^{**}. Particularly a semi-simple annihilator Banach algebra is a two-sided ideal in the second conjugate space. More generally, if A be a semi-simple modular annihilator Banach algebra, is $\pi(A)$ a two-sided ideal in A^{**}? The answer is negative in general. Indeed we have the following Theorem.

THEOREM 4. *Let* $F(X)$ *be the uniform closure of all finite rank operators on a complex Banach space* X. *Then the following four statements are equivalent.*

(1) X *is reflexive.*

(2) $\pi(F(X))$ *is a two-sided ideal in* $F(X)^{**}$.

(3) *(resp.* (4)) $\pi(F(X))$ *is a left (resp. right) ideal in* $F(X)^{**}$.

PROOF. Let X^* denote the conjugate space of X. If $a \in X$ and $f \in X^*$, we denote by

$a\otimes f$ the relation $a\otimes f(x)=f(x)a$ $(x\in X)$. Suppose that X be reflexive. For any $H\in F(X)^{**}$ there exists a net $\{T_\alpha\}$ of elements of $F(X)$ such that $\|T_\alpha\|\leq\|H\|$ and weak*-limit π $(T_\alpha)=H$. Then for any $a\in X$, $f\in X^*$ and $\varphi\in F(X)^*$, we have

$$\begin{aligned} H\circ\pi(a\otimes f)(\varphi) &= \lim_\alpha \pi(T_\alpha)\circ\pi(a\otimes f)(\varphi) \\ &= \lim_\alpha \pi((T_\alpha a)\otimes f)(\varphi) \\ &= \lim_\alpha \varphi((T_\alpha a)\otimes f). \end{aligned}$$

Now we can choose a subnet $\{T_{\alpha'}(a)\}$ of the net $\{T_\alpha(a)\}$ in X such that a weak limit $T_{\alpha'}(a)$ $(\equiv b)$ exists.
Thus $H\circ\pi(a\otimes f)(\varphi)=\varphi(b\otimes f)=\pi(b\otimes f)(\varphi)$.
Consequently for any $T\in F(X)$ and $H\in F(X)^{**}$, $H\circ\pi(T)\in\pi(F(X))$ because the set of all linear combinations of elements of $\{a\otimes f;\ a\in X,\ f\in X^*\}$ is dense in $F(X)$.
Since the reflexivity of X implies the reflexivity of X^*, $\pi(T)\circ H\in\pi(F(X))$ for any $H\in F(X)^{**}$, and $T\in F(X)$. Thus $\pi(F(X))$ is a two-sided ideal in $F(X)^{**}$.

Now take any element $f_0\in X^*$ and $a\in X$ such that $f_0(a)=1$, and fix it. We shall show that if $\pi(F(X))$ be a left ideal in $F(X)^{**}$, X is reflexive. Suppose that $\pi(F(X))$ be a left ideal in $F(X)^{**}$. For each $f\in F(X)^*$ and $G\in X^{**}$, let $\tilde{f}$ and $\tilde{G}$ be the bounded linear functionals on X and on $F(X)^*$, respectively, defined by $\tilde{f}(x)=f(x\otimes f_0)$ $(x\in X)$ and $\tilde{G}(f)=G(\tilde{f})$ $(f\in F(X)^*)$.
Then there exists $b\in X$ such that $\tilde{G}\circ\pi(a\otimes f_0)=\pi(b\otimes f_0)$. Now for any $f\in X^*$, we define a bounded linear functional F on a closed linear subspace $Z=\{x\otimes f_0;\ x\in X\}$ of $F(X)$ by the relation $F(x\otimes f_0)=f(x)$ $(x\in X)$. Then by the Hahn-Banach Theorem we have a bounded linear functional $\tilde{F}$ on $F(X)$ such that $\tilde{F}|Z=F$. On the other hand, we have

$$\begin{aligned} &\widetilde{a\otimes f_0*\tilde{F}}(x) \quad (x\in X) \\ &=\tilde{F}((x\otimes f_0)(a\otimes f_0))=F(x\otimes f_0)=f(x). \end{aligned}$$

Hence we have,

$$\pi(b\otimes f_0)(\tilde{F})=F(b\otimes f)=\pi(b)(f),$$

and

$$\tilde{G}\circ\pi(a\otimes f_0)(\tilde{F})=G(\widetilde{a\otimes f_0*\tilde{F}})=G(f).$$

Consequently X is reflexive.

Finally we shall show the implication (4)$\Longrightarrow$(1). Suppose that $\pi(F(X))$ is a right ideal in $F(X)^{**}$. For each $\varphi\in F(X)^*$ and $H\in X^{***}$, let $\tilde{\varphi}$ and $\tilde{H}$ be the bounded linear functionals on X^* and on $F(X)^*$ respectively, defined by $\tilde{\varphi}(f)=\varphi(a\otimes f)$ $(f\in X^*)$ and

$\widetilde{H}(\varphi)=H(\widetilde{\varphi})$ $(\varphi\in F(X)^*)$.
Then there exists $g\in X^*$ such that $\pi(a\otimes f_0)\circ H=\pi(a\otimes g)$.
Now for any $G\in X^{**}$, we define a bounded linear functional K on a closed linear subspace $Y=\{a\otimes f; f\in X^*\}$ of $F(X)$ by the relation $K(a\otimes f)=G(f)$ $(f\in X^*)$. Then by the Hahn-Banach Theorem we have a bounded linear functional $\widetilde{K}$ on $F(X)$ such that $\widetilde{K}|Y=G$.

Then $\widetilde{\widetilde{K}\circ a\otimes f_0}=G$. Hence we have

$$\pi(a\otimes g)(\widetilde{K})=\pi(g)(G)$$

and

$$\pi(a\otimes f_0)\circ\widetilde{H}(\widetilde{K})=H(G).$$

Thus X^* is reflexive, and so X is.
Consequently all implications are proved.

REMARK. For any complex Banach space X, $F(X)$ is a semi-simple modular annihilator Banach algebra.

However it is open whether the above problem is true or not for modular annihilator A*-algebras. This problem was posed by B. D. Malviya in [3].

Next we shall investigate the other special case.
Let G be a locally compact topological group, and μ be the left-invariant Haar measure on G. Moreover let $L^1(G)=L^1(G,\mu)$ be the group algebra of G and $M(G)$ be the measure algebra of G. When are these algebras ideals in its second conjugate space. For any compact group G, $C(G)$ (the algebra of all complex valued continuous functions with supremum norm and convolution multiplication) is always a two-sided ideal in $C(G)^{**}$. Indeed all left and right multiplication operators on $C(G)$ are strongly compact.

THEOREM 5. *The following four statments are equivalent.*

(1) *G is finite group.*
(2) *$\pi(M(G))$ is a two-sided ideal of the second dual space.*
(3) *$\pi(M(G))$ is a one-sided ideal of the second dual space.*
(4) *$\pi(M(G))$ is a block subalgebra of the second dual space.*

PROOF. If G is finite, $M(G)$ is finite dimensional, and so $M(G)$ is reflexive.
Thus the implications (1)$\Longrightarrow$(2)$\Longrightarrow$(3)$\Longrightarrow$(4) are clear.
Next suppose that $\pi(M(G)$ is a block sub-algebra of $M(G)^{**}$. Then $M(G)$ is reflexive, and so finite-dimmensional. Thus G is finite.

THEOREM 6. *The follwing four statements are equivalent.*

(1) *G is compact.*
(2) *$\pi(L^1(G))$ is a closed two-sided ideal in the second dual space.*
(3) *$\pi(L^1(G))$ is a closed one-sided ideal in the second dual space.*

(4) *$\pi(L^1(G))$ is a block subalgebra in the second dual space.*

PROOF. We denote the convolution product by $*$. (1)$\Rightarrow$(2). It is proved in [6]. (2)$\Rightarrow$(3), (3)$\Rightarrow$(4) are clear. (4)$\Rightarrow$(1). Suppose that $L^1(G)$ is a block subalgebra in $L^1(G)^{**}$. By Theorem 2 the mapping $f \longrightarrow h*f*g(g, h \in L^1(G))$ is a weakly compact operator on $L^1(G)$. Hence the mapping $f \longrightarrow h*l*f*k*g(h, l, k, g \in L^1(G))$ is (strongly) compact from [2]. Since $L^1(G)$ has a bounded approximate identity, the mapping $f \longrightarrow h*f*g(g, h\ L^1(G))$ is compact from the well-known factorization theorem. Thus, it is sufficient to prove that G is compact when $L^1(G)$ is a compact Banach algebra in the sense of Alexander [1]. Thus we may assume that the mapping $f \longrightarrow g*f*g(g, \in L^1(G))$ is compact on $L^1(G)$. Suppose that G is non-compact. Then there exists a compact subset of G such that $\mu(K)>1$. We construct inductively infinite sequence $\{a_n\}$ of elements of G such that $Ka_nK \cap Ka_mK = \phi(n \neq m)$.

We select an element a_1 of G and fix it. Next suppose $a_1, a_2, \ldots, a_n$ were chosen. Then $\bigcup_{i=1}^{n} K^{-1}Ka_iKK^{-1}$ is a compact set. We may choose a_{n+1} from the complement of this set.

$$\text{Then}\quad \chi_K * a_n\chi_K(t) = \int_G \chi_K(h)\chi_K(a_n^{-1}h^{-1}t)\,d\mu(h)$$
$$= \int_K \chi_K(a_n^{-1}h^{-1}t)\,d\mu(h)$$
$$= 0 \qquad \text{for all } t \notin Ka_nK$$

and

$$\|\chi_K * a_n\chi_K\|_1 = \int_G \left| \int_G \chi_K(h)\chi_K(a_n^{-1}h^{-1}t)\,d\mu(h) \right| d\mu(t)$$
$$= \int_G \chi_K(h) \left\{ \int_G \chi_K(a_n^{-1}h^{-1}t)\,d\mu(t) \right\} d\mu(h)$$
$$= \{\mu(K)\}^2 > 1.$$

where $a\chi_K(t) = \chi_K(a^{-1}t)$ and χ_K is the characteristic function of K. Now let S be the unit ball of $L^1(G)$. Then $\chi_K * S * \chi_K$ is relatively compact.

Let $\{V_\alpha\}$ be a fundamental family of compact neighborhoods at a point a of G and let $\{f_\alpha\}$ be a family of continuous positive functions on G such that the support of f_α is contained in V_α and $\int_G f_\alpha(t)d\mu(t)=1$, then $\chi_K * f_\alpha * \chi_K$ converges to $\chi_K * a\chi_K$ in L^1-norm [4]. Thus $\{\chi_K * a\chi_K;\ a \in G\}$ is relatively compact set in $L^1(G)$.
On the other hand,

$$\|\chi_K * a_n\chi_K - \chi_K * a_m * \chi_K\|_1 \qquad (n \neq m)$$
$$= \int_G \left| \chi_K * a_n\chi_K(t) - \chi_K * a_m\chi_K(t) \right| d\mu(t)$$

$$=\int_{Ka_nK}\left|\chi_K*a_n\chi_K(t)-\chi_K*a_m\chi_K(t)\right|d\mu(t)$$

$$+\int_{Ka_mK}\left|\chi_K*a_n\chi_K(t)-\chi_K*a_m\chi_K(t)\right|d\mu(t)$$

$$+\int_{G-(Ka_nK\cup Ka_mK)}\left|\chi_K*a_n\chi_K(t)-\chi_K*a_m\chi_K(t)\right|d\mu(t)$$

$$=\int_{Ka_nK}\left|\chi_K*a_n\chi_K(t)\right|d\mu(t)+\int_{Ka_mK}\left|\chi_K*a_m\chi_K(t)\right|d\mu(t)$$

$$=\|\chi_K*a_n\chi_K\|_1+\|\chi_K*a_m\chi_K\|_1$$

$$=2\{\mu(K)\}^2>2.$$

This contradicts to the relative compactness of $\{\chi_K*a\chi_K;\ a\in G\}$. Therefore G is compact. Thus all implications are proved.

NIIGATA UNIVERSITY

References

[1] J. C. ALEXANDER; *Compact Banach algebras.* Proc. London Math. Soc., (3) 18 (1968), 1–18.

[2] N. DUNFORD and J. SCHWARTZ; Linear operators, Part I, General theory, Interscience, New York, 1958.

[3] B. D. MALVIYA; *A problem concerning weakly completely continuous A*-algebras,* Amer. Math. Monthly, 81 (3) (1974), 267–268.

[4] S. SAKAI; *Weakly compact operators on operator algebras.* Pacific J. Math., 14 (1964), 659–664.

[5] S. WATANABE; *A Banach algebra which is an ideal in the second dual space.* Sci. Rep. Niigata Univ., Ser. A, No. 11 (1974), 95–101.

[6] P. K. WONG; *On the Arens product and annihilator algebras.* Proc. Amer. Math. Soc., 30 (1971), 79–83.

[7] P. K. WONG; *On the Arens products and certain Banach algebras,* Trans, A. M. S., 180 (1973), 437–448.