An elementary proof of Gleason-Kahane-Zelazko's theorem for complex Banach algebra with a hermitian involution

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1. Introduction

Gleason [2], Kahane and Zelazko [3] proved independently the following;

THEOREM (Gleason-Kahane-Zelazko). *Let A be a complex unital Banach algebra and let f be a linear functional on A. Then f is multiplicative if and only if $f(a)\in Sp(a)$ $(a\in A)$.*

Their proof is based on Hadamard's factorization theorem. By Choda and Nakamura [1], an elementary proof of this theorem for B*-algebra was presented without depending on such a theorem from the theory of functions.

The purpose of our paper is to present an elementary proof of this theorem for a complex Banach algebra with a hermitian involution. Throughout this paper, we use the standard notations and terminologies from [4].

2. The main theorem

LEMMA. *Let A be a complex Banach algebra with a hermitian involution and let f be a linear functional on A.*
If $f(a)\in Sp_A(a)$ $(a\in A)$, then we have

$$f(xh)=f(x)f(h) \quad (x\in A,\ h\in A_h),$$

where A_h denotes the set of all self-adjoint elements of A.

PROOF. We shall suppose, without loss of generality, that A possesses an identity element 1.

Let $k\in A_h$ be such that $f(k)=0$, and B be a maximal commutative $*$-subalgebra of A which contains 1 and k, and Φ_B be the carrier space of B. Then we get

$$\mathrm{Sp}_A(x)=\mathrm{Sp}_B(x) \quad (x\in B).$$

Since $k^2+ik\in B$ and $f(k^2+ik)=f(k^2)$, there exists, from our assumption, an element $\Psi\in\Phi_B$ such that

$$\Psi(k^2+ik)=f(k^2),$$

and so

$$\Psi(k^2)+i\Psi(k)=f(k^2).$$

We have $\Psi(k)\in \mathrm{Sp}_B(k)=\mathrm{Sp}_A(k)\subset\mathbb{R}$ and $f(k^2)\in\mathrm{Sp}_A(k^2)\subset\mathbb{R}$.

It follows that $\Psi(k)=0$,

and consequently $f(k^2)=0$.

On the other hand, we have, from [5, Theorem 1],

$$\mathrm{Sp}_A(x^*x)\subset\mathrm{R}^+\cup(0) \quad (x\in A).$$

Therefore $f(x^*x)\geqq 0$ $(x\in A)$, namely f is positive on A.

Hence we have

$$|f(xk)|^2\leqq \mathrm{f}(xx^*)f(k^2) \quad (x\in A,\ k\in A_h).$$

Thus we have $f(xk)=0$ $(x\in A)$ for any $k\in A_h$ such that $f(k)=0$.

Now, let h be an arbitrary element of A_h. Then we have

$$f(h)1-h\in A_h$$

since $f(h)\in\mathrm{Sp}_A(h)\subset\mathbb{R}$.

Moreover, $f(f(h)1-h)=0$.

Hence we have $f(x(f(h)1-h))=0 (x\in A)$.

Consequently, $f(xh)=f(x)f(h) (x\in A,\ h\in A_h)$.

This completes the proof.

Remark. It is easy to verify the following statement:

Let A be a complex Banach algebra and f be a linear functional on A such that $f(a)\in\mathrm{Sp}_A(a)$ $(a\in A)$. Let $x\in A$ be such that $f(x)=0$ and $\|x\|<1$, then we have

$$\lambda f(x^2)\in\mathrm{Sp}_A(x) \quad (\lambda\in\mathbb{C},\ |\lambda|\leqq 1).$$

Consequently, if the above element x satisfies $\mathrm{Sp}_A(x)\subset\mathbb{R}$, then it follows that $f(x^2)=0$.

Theorem. *Let A and f be the same as in Lemma. Then f is multiplicative if and only if $f(a)\in Sp_A(a)$ $(a\in A)$.*

Proof. The "necessary" part is well known. As for the sufficiency, suppose f is a linear functional on A such that $f(a)\in\mathrm{Sp}_A(a)$ $(a\in A)$.

For any pair x and y in A, there exist hermitian elements h_1 and h_2 such that $y=h_1+ih_2$, so we have

$$f(xy)=f(x(h_1+ih_2))=f(xh_1)+if(x)f(h_2)$$
$$=f(x)f(h_1)+if(x)f(h_2)=f(x)\cdot f(y).$$

This completes the proof.

REMARK. 1. Theorem is false for real Banach algebra, e. g.

$$A = C_R([0, 1]) \text{ and } f(x) = \int_0^1 x(t)dt, \text{ see [3].}$$

REMARK 2. *Let A be a commutative Banach algebra such that, for each $x \in A$, there exist h and k with the following properties;*

$$Sp_A(h) \subset \mathbb{R},\ Sp_A(k) \subset \mathbb{R} \text{ and } x = h + ik.$$

Then, a linear functional on A is multiplicative if and only if

$$f(a) \in Sp_A(a)\ (a \in A).$$

PROOF. We shall suppose, without loss of generality, that A possesses an identity element 1. As in the proof of Theorem, we shall sketch only the proof of "if" part. Let f be a linear functional on A such that $f(a) \in \mathrm{Sp}_A(a)$ $(a \in A)$.

By the method in the proof of Lemma, we have $f(x^2)=0$ for any $x \in A$ such that $\mathrm{Sp}_A(x) \subset \mathbb{R}$ and $f(x)=0$. Therefore for any pair $x, y \in A$ such that

$$\mathrm{Sp}_A(x) \subset \mathbb{R},\ \mathrm{Sp}_A(y) \subset \mathbb{R} \text{ and } f(x)=f(y)=0,$$

we have $f((x+y)^2)=2f(xy)$.

Thus, there exists $\Psi \in \Phi_A$ such that $\Psi((x+y)^2)=2f(xy)$.

Since $\Psi((x+y)^2)=(\Psi(x)+\Psi(y))^2$, $\Psi(x) \in \mathrm{Sp}_A(x) \subset \mathbb{R}$, $\Psi(y) \in \mathrm{Sp}_A(y) \subset \mathbb{R}$, we have $f(xy) \in \mathbb{R}$.

Let $\phi \in \Phi_A$ be such that $\phi(xy+ix)=f(xy+ix)=f(xy)$.

Since $\phi(xy)=\phi(x)\phi(y)$, $\phi(x) \in \mathrm{Sp}_A(x) \subset \mathbb{R}$, $\phi(y) \in \mathrm{Sp}_A(y) \subset \mathbb{R}$, we have $\phi(x)=0$, consequently $f(xy)=0$.

Now, let $x \in A$ be any element such that $f(x)=0$.

From our assumption, there exist $h, k \in A$ such that

$$\mathrm{Sp}_A(h) \subset \mathbb{R},\ \mathrm{Sp}_A(k) \subset \mathbb{R} \text{ and } x=h+ik.$$

Therefore $f(h) \in \mathbb{R}$, $f(k) \in \mathbb{R}$ and $f(h)+if(k)=0$, hence $f(h)=f(k)=0$.

We have $f(x^2)=f(h^2)+2if(hk)-f(k^2)=0$.

For any $x \in A$, put $f(x)=\lambda$.

Since $f(x-\lambda \cdot 1)=0$, $f((x-\lambda \cdot 1)^2)=0$.

It follows that

$$0=f(x^2)-2\lambda f(x)+\lambda^2=f(x^2)-\lambda^2,$$

thus we have $f(x^2)=f(x^2)$.

Consequently for any pair $x, y \in A$, we have $f((x+y)^2)=f(x+y)^2$, hence $f(xy)=f(x)f(y)$. This completes the proof.

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