The epuivalence of two definitions of homotopy sets for Kan complexes

By

Eiitirô HONMA and Tetuo KANEKO

(Received December 20, 1959)

As we remarked in §§ 1 and 2 of [1], the following proposition holds. The purpose of this paper is to give its proof. Free use will be made of the definitions and notations of [1].

PROPOSITION 1. 1°. *If* (K, L) *is a Kan pair with base point* $\varphi \in L_0$, DEFINITIONS 1.7 *and* 1.10 *in* [1] *of* $\pi_n(K, L, \varphi)$ *are equivalent for* $n \geqq 0$. 2°. *If* $(K; L, M)$ *is a Kan triad with base point* $\varphi \in (L \cap M)_0$, DEFINITIONS 2.4 *and* 2.7 *in* [1] *of* $\pi_n(K; L, M, \varphi)$ *are equivalent for* $n \geqq 2$. *I.e. the natural embedding map* $i_k: K \to S|K|$ *given in* [7] *induces one-to-one onto maps* $(i_k)_*: \pi_n(K, L, \varphi) \to \pi_n(S|K|, S|L|, i_k(\varphi))$ *and* $(i_k)_*: \pi_n(K; L, M, \varphi) \to \pi_n(S|K|; S|L|, S|M|, i_k(\varphi))$ *where* π_n *means the set defined by* DEFINITIONS 1.7 *and* 2.4 *in* [1].

Proof of 1°. The equivalence follows from THEOREM 7.3 in [1], REMARK 1 in [3, §4] and the five lemma for $n \geqq 2$, and by their definitions for $n=0$.

To show that $(i_k)_*$ is one-to-one onto for $n=1$, consider $\pi_1(K, L, \varphi)$ and $\pi_1(S|K|, S|L|, i_k(\varphi))$. In this case we may assume that K is connected, i.e. $\pi_0(K, \varphi)=0$. Then we can construct the c. s. s. group $G(K; \varphi)$ which is a loop complex of K rel. φ [2, THEOREM 9.2]. Put $U=G(K; \varphi) \times_t K$, $C=G(K; \varphi) \times_t L$ and $\psi=(e_0, \varphi) \in U_0$ where t is a twisting function defined by $t\sigma=\bar{\sigma}$, e_0 is the identity element of the group $G(K; \varphi)_0$. By LEMMA 9.3 in [2] U is contractible. Let $p: U \to K$ be given by $p(\rho, \sigma)=\sigma$ for $(\rho, \sigma) \in U$. Then p is a fibre map: $(U, C, \psi) \to (K, L, \varphi)$ and (U, C) is a Kan pair. By THEOREM 8.3-2) and PROPOSITION 8.2 in [1], $p_*: \pi_1(U, C, \psi) \to \pi_1(K, L, \varphi)$ and $(S|P|)_*: \pi_1(S|U|, S|C|, i_U(\psi)) \to \pi_1(S|K|, S|L|, i_k(\varphi))$ are one-to-one onto.

Consider the following commutative diagram:

$$\begin{array}{ccccc}
\pi_1(K, L, \varphi) & \xleftarrow{\;p_*\;} & \pi_1(U, C, \psi) & \xrightarrow{\;\delta\;} & \pi_0(C, \psi) \\
\downarrow (i_k)_* & & \downarrow (i_U)_* & & \downarrow (i_C)_* \\
\pi_1(S|K|, S|L|, i_k(\varphi)) & \xleftarrow{\;(S|p|)_*\;} & \pi_1(S|U|, S|C|, i_U(\psi)) & \xrightarrow{\;\delta'\;} & \pi_0(S|C|, i_C(\psi)),
\end{array}$$

where δ and δ' are the boundary operations induced by the 0-th face operation, $(i_C)_*$ is one-to-one onto [3, §4 REMARK 1]. Therefore to show that $(i_k)_*$ is one-to-one onto it sufficies to prove the following.

LEMMA 2. *If (U, C) is a Kan pair with base point $\phi \in C_0$ and if U is contractible, then $\delta : \pi_1(U, C, \phi) \rightarrow \pi_0(C, \phi)$ is one-to-one onto.*

(In this case, $S|U|$ is also contractible and that δ' is one-to-one onto is verified by the same method.)

Proof. Since $\pi_0(U, \phi)=0$ it is clear that δ is onto.

Now consider two simplices σ and $\tau \in \Gamma_1(U, C, \phi)$ such that $\sigma\varepsilon^0 \sim \tau\varepsilon^0$, i.e. there exists $\gamma \in C_1$ with $\gamma\varepsilon^0=\sigma\varepsilon^0$ and $\gamma\varepsilon^1=\tau\varepsilon^0$. Let $\omega_1 \in U_2$ be a solvent of

$$[\overset{(0)}{\gamma}, \overset{(1)}{\sigma}, \overset{(2)}{\square}]$$

and let $\sigma'=\omega_1\varepsilon^2$. Let $\omega_2 \in U_2$ be a solvent of

$$[\overset{(0)}{\sigma'}, \overset{(1)}{\tau}, \overset{(2)}{\square}]$$

and let $\theta=\omega_2\varepsilon^2$. we have $\theta\varepsilon^0=\phi$ and $\theta\varepsilon^1=\phi$. Since $\pi_1(U, \phi)=0=\{\phi\eta^0\}$, there exists $\omega_3 \in U_2$ such that $\omega_3\varepsilon^0=\omega_3\varepsilon^1=\phi\eta^0$, $\omega_3\varepsilon^2=\theta$. Let $\omega_4 \in U_2$ be a solution of

$$[\overset{(0)}{\square}, \overset{(1)}{\tau\eta^0}, \overset{(2)}{\omega_2}, \overset{(3)}{\omega_3}]$$

and $\rho \in U_2$ be a solution of

$$[\overset{(0)}{\square}, \overset{(1)}{\omega_1}, \overset{(2)}{\sigma\eta^0}, \overset{(3)}{\omega_4}].$$

Then we have $\rho\varepsilon^0=\gamma \in C_1$, $\rho\varepsilon^1=\sigma$, $\rho\varepsilon^2=\tau$ and therefore $\rho : \sigma \sim \tau$ lsd. C.

Proof of 2°. The equivalence follows from THEOREM 7.1 in [1], THEOREM 1-1° and the five lemma for $n \geqq 3$.

To show that $(i_K)_*$ is one-to-one onto for $n=2$, consider $\pi_2(K; L, M, \varphi)$ and $\pi_2(S|K|; S|L|, S|M|, i_K(\varphi))$ where we may assume that K is connected. Let U, C, ϕ, be those given in the proof of 1° and moreover let $D=G(K; \varphi)\times_t M$. Then $(U; C, D)$ is a Kan triad with base point ϕ, and the following diagram is commutative:

$$\begin{array}{ccc} \pi_2(K; L, M, \varphi) & \xrightarrow{(i_K)_*} & \pi_2(S|K|; S|L|, S|M|, i_K(\varphi)) \\ \uparrow p_* & & \uparrow (S|p|)_* \\ \pi_2(U; C, D, \phi) & \xrightarrow{(i_U)_*} & \pi_2(S|U|; S|C|, S|D|, i_U(\phi)), \end{array}$$

where $p : U \rightarrow K$ is the fibre map given in the proof of 1° and p_* and $(S|p|)_*$ are one-to-one onto (THEOREM 8.3-1 in [1]). Therefore to show that $(i_K)_*$ is one-to-one onto it sufficies to prove that $(i_U)_*$ is so.

To prove that $(i_U)_*$ is onto, consider an arbitrary simplex $f \in \Gamma_2(S|U|; S|C|, S|D|, i_U(\phi))$. f is a continuous map from Δ_2 into $|U|$ where Δ_2 means the unit simplex in euclidean space R^3. Put $e=f\varepsilon^0\varepsilon^0(\varDelta_0)\in|C|\cap|D|$ for the sake of notational simplicity. It is clear that there exist a 1-simplex $g\in(S|C|\cap S|D|)_1$ and a 0-simplex $\theta\in(C\cap D)_0$ such that $g\varepsilon^0(\varDelta_0)=e$ and $g\varepsilon^1=i(\theta)$. Since $S|C|$ and $S|D|$ are Kan complexes, there exist solvents $h_0\in(S|C|)_2$ and $h_1\in(S|D|)_2$ of the following equations

$$[\overset{(0)}{g},\ \overset{(1)}{f\varepsilon^0},\ \overset{(2)}{\square}] \text{ and } [\overset{(0)}{g},\ \overset{(1)}{f\varepsilon^1},\ \overset{(2)}{\square}] \text{ respectively.}$$

We see that $h_0\varepsilon^2\in\Gamma_1(S|C|,\ S|\phi\cup\theta|,\ i_C(\phi))$ and $(i_C)_*:\pi_1(C,\ \phi\cup\theta,\ \phi)\to\pi_1(S|C|,\ S|\phi\cup\theta|,\ i_C(\phi))$ is one-to-one onto (Theorem 1-1°) where $\phi\cup\theta$ means the c. s. s. complex generated by ϕ and θ. Therefore we have a simplex $\sigma_0\in\Gamma_1(C,\ \phi\cup\theta,\ \phi)$ such that $\sigma_0\varepsilon^0=\theta$ and $i_C(\sigma_0)\sim h_0\varepsilon^2$ lsd. $S|\phi\cup\theta|$. Donote this homotopy by $k_0\in(S|C|)_2$. It is clear that $k_0\varepsilon^0=i(\theta)\eta^0$. We have also a simplex $\sigma_1\in\Gamma_1(D,\ \phi\cup\theta,\ \phi)$ such that $\sigma_1\varepsilon^0=\theta$ and $i_D(\sigma_1)\sim h_1\varepsilon^2$ lsd. $S|\phi\cup\theta|$. Denote this homotopy by $k_1\in(S|D|)_2$. We see that $k_1\varepsilon^0=i_D(\theta)\eta^0$. Let $f_0\in(S|C|)_2$ and $f_1\in(S|D|)_2$ be solutions of the following equations

$$[\overset{(0)}{g\eta^0},\ \overset{(1)}{\square},\ \overset{(2)}{h_0},\ \overset{(3)}{k_0}] \text{ and } [\overset{(0)}{g\eta^0},\ \overset{(1)}{\square},\ \overset{(2)}{h_1},\ \overset{(3)}{k_1}] \text{ respectively.}$$

Then we have $f\sim f_3$ lsd. $S|C|,\ S|D|$ where $f_3\in(S|U|)_2$ is a solution of the following equation

$$[\overset{(0)}{f_0},\ \overset{(1)}{f_1},\ \overset{(2)}{f},\ \overset{(3)}{\square}].$$

On the other hand, let $\Upsilon\in U_2$ be a solvent of the equation

$$[\overset{(0)}{\sigma_0},\ \overset{(1)}{\sigma_1},\ \overset{(2)}{\square}]$$

and let $\upsilon=\Upsilon\varepsilon^2$. Then we have $\upsilon\varepsilon^0=\upsilon\varepsilon^1=\phi$, and since $\pi_1(U,\ \phi)=0$ there exists a simplex $\Omega\in U_2$ such that $\Omega\varepsilon^0=\Omega\varepsilon^1=\phi\eta^0$ and $\Omega\varepsilon^2=\upsilon$. A solution σ of the equation in U:

$$[\overset{(0)}{\sigma_0\eta^0},\ \overset{(1)}{\square},\ \overset{(2)}{\Upsilon},\ \overset{(3)}{\Omega}]$$

is a simplex contained in $\Gamma_2(U;\ C,\ D,\ \phi)$, i.e. $\sigma\varepsilon^0=\sigma_0$, $\sigma\varepsilon^1=\sigma_1$ and $\sigma\varepsilon^2=\phi\eta^0$. Let $F_3\in(S|U|)_3$ be a solvent of the equation in $S|U|$:

$$[\overset{(0)}{i_U(\sigma_0)\eta^0},\ \overset{(1)}{f_3},\ \overset{(2)}{i_U(\sigma)},\ \overset{(3)}{\square}]$$

and let $f_4=F_3\varepsilon^3$. Then we have $f_4\varepsilon^0=f_4\varepsilon^1=f_4\varepsilon^2=i_U(\phi)\eta^0$, and since $\pi_2(S|U|,\ i_U(\phi))=\pi_2(U,\ \phi)=0$ there exists $F_4\in(S|U|)_3$ such that $F_4\varepsilon^0=F_4\varepsilon^1=F_4\varepsilon^2=i_U(\phi)\eta^0\eta^1$ and $F_4\varepsilon^3=f_4$. Let $F\in(S|U|)_3$ be a solution of the equation in $S|U|$:

$$\overset{(0)}{[i_U(\sigma_0)\eta^0\eta^1}, \overset{(1)}{\square}, \overset{(2)}{i_U(\sigma)\eta^1}, \overset{(3)}{F_3}, \overset{(4)}{F_4}]$$

and let $G \in (S|U|)_3$ be a solution of the equation in $S|U|$:

$$\overset{(0)}{[i_U(\sigma_0)\eta^0\eta^2}, \overset{(1)}{i_U(\sigma)\eta^2}, \overset{(2)}{\square}, \overset{(3)}{i_U(\sigma)\eta^1}, \overset{(4)}{F}].$$

Then we have $G\varepsilon^0 = i_U(\sigma_0)\eta^1 \in S|C|$, $G\varepsilon^1 = i_D(\sigma_1)\eta^1 \in S|D|$, $G\varepsilon^2 = i_U(\sigma)$ and $G\varepsilon^3 = f_3$, i.e. $i_U(\sigma) \sim f_3$ lsd. $S|C|$, $S|D|$. Thus we have $i_U(\sigma) \sim f$ lsd. $S|C|$, $S|D|$, i.e. $(i_U)_*$ is onto.

To show that $(i_U)_*$ is one-to-one, consider two simplices σ and $\tau \in \Gamma_2(U; C, D, \phi)$ such that there exists a homotopy $F \in (S|U|)_3 : i_U(\sigma) \sim i_U(\tau)$ lsd. $S|C|$, $S|D|$, i.e. $F\varepsilon^0 \in S|C|$, $F\varepsilon^1 \in S|D|$, $F\varepsilon^2 = i_U(\sigma)$ and $F\varepsilon^3 = i_U(\tau)$. For the sake of simplicity, put $\phi_0 = \sigma\varepsilon^0\varepsilon^0$ and $\phi_1 = \tau\varepsilon^0\varepsilon^0$. Since $(i_{C\cap D})_* : \pi_1(C|D|, \phi_0 \cup \phi_1, \phi_1) \to \pi_1(S|C| \cap S|D|, S|\phi_0 \cup \phi_1|, i(\phi_1))$ is one-to-one onto (THEOREM 1-1°) where $\phi_0 \cup \phi_1$ means the c. s. s. complex generated by ϕ_0 and ϕ_1 and since $F\varepsilon^0\varepsilon^0 \in \Gamma_1(S|C| \cap S|D|, S|\phi_0 \cup \phi_1|, i(\phi_1))$, there exists a simplex $\gamma \in (C \cap D)_1$ such that $\gamma\varepsilon^0 = \phi_0$, $\gamma\varepsilon^1 = \phi_1$ and $i(\gamma) \sim F\varepsilon^0\varepsilon^0$ lsd. $S|\phi_0 \cup \phi_1|$. Denote this homotopy by $g \in (S|C| \cap S|D|)_2$, i.e. $g\varepsilon^0 = i(\phi_0)\eta^0$, $g\varepsilon^1 = i(\gamma)$ and $g\varepsilon^2 = F\varepsilon^0\varepsilon^0$. Let $h_0 \in (S|C|)_3$ and $h_1 \in (S|D|)_3$ be solvents of the following equations

$$\overset{(0)}{[g}, \overset{(1)}{F\varepsilon^0\varepsilon^1\eta^1}, \overset{(2)}{\square}, \overset{(3)}{F\varepsilon^0}] \text{ and } \overset{(0)}{[g}, \overset{(1)}{F\varepsilon^1\varepsilon^1\eta^1}, \overset{(2)}{\square}, \overset{(3)}{F\varepsilon^1}] \text{ respectively.}$$

Consider a solution $F' \in (S|U|)_3$ of the equation in $S|U|$:

$$\overset{(0)}{[h_0}, \overset{(1)}{h_1}, \overset{(2)}{F\varepsilon^2\eta^2}, \overset{(3)}{\square}, \overset{(4)}{F}].$$

Then we have $F'\varepsilon^0 = h_0\varepsilon^2 \in S|C|$, $F'\varepsilon^1 = h_1\varepsilon^2 \in S|D|$, $F'\varepsilon^2 = i_U(\sigma)$ and $F'\varepsilon^3 = i(\tau)$, i.e. $F' : i_U(\sigma) \sim i_U(\tau)$ lsd. $S|C|$, $S|D|$. Moreover we have $F'\varepsilon^0\varepsilon^0 = h_0\varepsilon^2\varepsilon^0 = g\varepsilon^1 = i(\gamma)$.

Let $\tau_0 \in C_3$ and $\tau_1 \in D_3$ be solvents of the following equations

$$\overset{(0)}{[\gamma}, \overset{(1)}{\square}, \overset{(2)}{\tau\varepsilon^0}] \text{ and } \overset{(0)}{[\gamma}, \overset{(1)}{\square}, \overset{(2)}{\tau\varepsilon^1}] \text{ respectively.}$$

Then $\tau \sim \tau'$ lsd. C, D where $\tau' \in \Gamma_2(U; C, D, \phi)$ is a solution of the following equation in U:

$$\overset{(0)}{[\tau_0}, \overset{(1)}{\tau_1}, \overset{(2)}{\square}, \overset{(3)}{\tau}].$$

Therefore, to complete this proof it sufficies to show that $\tau' \sim \sigma$ lsd. C, D. Let $k \in (S|C|)_2$ be a solution of the equation in $S|C|$:

$$\overset{(0)}{[i_C(\gamma)\eta^1}, \overset{(1)}{\square}, \overset{(2)}{F'\varepsilon^0}, \overset{(3)}{i_C(\tau_0)}].$$

Then we have $k\varepsilon^0 = i_C(\phi_0)\eta^0$, $k\varepsilon^1 = i_C(\sigma\varepsilon^0)$, $k\varepsilon^2 = i_C(\tau'\varepsilon^0)$. Therefore $i_C(\sigma\varepsilon^0) \sim i_C(\tau'\varepsilon^0)$

lsd. $i_C(\phi_0)$. Hence we have $\sigma\varepsilon^0 \sim \tau'\varepsilon^0$ lsd. $\phi \cup \phi_0$, for $(i_C)_*: \pi_1(C, \phi\cup\phi_0, \phi) \to \pi_1(S|C|, S|\phi\cup\phi_0|, i_C(\phi))$ is one-to-one onto. Namely there exists a simplex $\rho_0 \in C_2$ such that $\rho_0\varepsilon^0 = \phi_0\eta^0$, $\rho_0\varepsilon^1 = \sigma\varepsilon^0$ and $\rho_0\varepsilon^2 = \tau'\varepsilon^0$. Similarly we have a simplex $\rho_1 \in D_2$ such that $\rho_1\varepsilon^0 = \phi_0\eta^0$, $\rho_1\varepsilon^1 = \sigma\varepsilon^1$ and $\rho_1\varepsilon^2 = \tau'\varepsilon^1$. Then $\tau' \sim \tau''$ lsd. C, D where $\tau'' \in \Gamma_2(U; C, D, \phi)$ is a solution of the equation in U:

$$[\overset{(0)}{\rho_0}, \overset{(1)}{\rho_1}, \overset{(2)}{\square}, \overset{(3)}{\tau'}].$$

On the other hand, for a solvent $\Xi \in U_3$ of the following equation in U:

$$[\overset{(0)}{\sigma}, \overset{(1)}{\tau''}, \overset{(2)}{\sigma\varepsilon^1\eta^0}, \overset{(3)}{\square}],$$

each face of $\xi = \Xi\varepsilon^3$ degenerates at ϕ. Therefore there exists a simplex $\Omega \in U_3$ such that $\Omega\varepsilon^0 = \Omega\varepsilon^2 = \Omega\varepsilon^3 = \phi\eta^0\eta^1$ and $\Omega\varepsilon^1 = \xi$, for $\pi_2(U, \phi) = 0$. Let $\rho \in U_3$ be a solution of the equation in U:

$$[\overset{(0)}{\square}, \overset{(1)}{\tau''\eta^2}, \overset{(2)}{\zeta}, \overset{(3)}{\tau''\eta^0}, \overset{(4)}{\rho'}]$$

where $\rho' \in U_3$ is a solution of the equation in U:

$$[\overset{(0)}{\square}, \overset{(1)}{\Xi}, \overset{(2)}{\tau''\eta^0}, \overset{(3)}{\sigma\varepsilon^1\eta^0\eta^1}, \overset{(4)}{\Omega}]$$

and $\zeta \in D_3$ is a solvent of the equation in D:

$$[\overset{(0)}{\square}, \overset{(1)}{\tau''\eta^2\varepsilon^1}, \overset{(2)}{\tau''\eta^0\varepsilon^2}, \overset{(3)}{\sigma\varepsilon^1\eta^0}].$$

Then we have $\rho\varepsilon^0 = \tau''\varepsilon^0\eta^1 \in C$, $\rho\varepsilon^1 = \zeta\varepsilon^0 \in D$, $\rho\varepsilon^2 = \tau''$ and $\rho\varepsilon^3 = \sigma$. Thus we have $\sigma \sim \tau'' \sim \tau' \sim \tau$ lsd. C, D.

References

[1] K. Aoki, E. Honma and T. Kaneko: *Homotopy theory of c. s. s. pairs and triads.* J. Faculty Sci. Niigata Univ., Ser I, Vol. 2, No. 2(1959), 67–95.

[2] D. M. Kan: *A combinatorial definition of homotopy groups*, Ann. Math., 67(1958) 282–312.

[3] J. Milnor: *The geometric realization of a semi-simplicial complex*, Ann. Math., 65(1957). 357–362.

Department of Mathematics
Niigata University