Some application of the cobordism decomposition

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1. Introduction

Let M^{4k} be an $4k$-dimensional compact orientable differentiable manifold, P_{2k} the $2k$-dimensional complex projective space, Ω the (oriented) cobordism ring, and $\mathbf{Q}$ the field of rational numbers.

It is known that M^{4k} is decomposed as follows:

$$M^{4k} \sim \sum_{i_1+\cdots+i_t=k} A_{i_1,\cdots,i_t} P_{2i_1}\times\cdots\times P_{2i_t}$$

where '$\sim$' means 'is equivalent to' in $\Omega\otimes\mathbf{Q}$ and $A_{i_1,\ldots,i_t}$'s are rational numbers. In [2], [3] and [4], similarly to [1], we determined the values of some $A_{i_1,\ldots,i_t}$'s and considered some applications to the embedding problem. In this paper, we will consider, similarly to [1], another application of the cobordism decomposition.

2. Cobordism decomposition

Let M^{4k} be an $4k$-dimensional compact orientable differentiable manifold, and $\sum_{i\geq 0} p_i t^i = \prod_{i\geq 1}(1+\gamma_i t)$ a formal factorization where $p_i=p_i(M^{4k})$ is the i-dimensional Pontrjagin class of M^{4k}. We defined in [2] a multiplicative series——

$$\sum_{i\geq 0} A_i(u, v; p_1, \ldots, p_i)z^i = \prod_{i\geq 1} \frac{\sqrt{\gamma_i z}}{\mathrm{tgh}\sqrt{\gamma_i z}}(1+u\,\mathrm{tgh}^2\sqrt{\gamma_i z})(1+v\,\mathrm{tgh}^2\sqrt{\gamma_i z}) \tag{2.1}$$

where u, v and z are indeterminates. Since

$$\frac{\sqrt{\gamma_i z}}{\mathrm{tgh}\sqrt{\gamma_i z}} = 1+\sum_{j\geq 1}(-1)^{j-1}\frac{2^{2j}}{(2j)!}B_j(\gamma_i z)^j,$$

$$\sqrt{\gamma_i z}\,\mathrm{tgh}\sqrt{\gamma_i z} = \sum_{j\geq 1}(-1)^{j-1}\frac{2^{2j}(2^{2j}-1)}{(2j)!}B_j(\gamma_i z)^j$$

where B_j is the j-th Bernoulli number, we have:

$$\Delta_1(u, v; p_1)=p_1(u+v)+\frac{1}{3}p_1,$$

$$\Delta_2(u, v; p_1, p_2)=p_2(u^2+v^2)+p_1^2uv+\frac{1}{3}(4p_2-p_1^2)(u+v)+\frac{1}{45}(7p_2-p_1^2),$$

$$\Delta_3(u, v; p_1, p_2, p_3)=p_3(u^3+v^3)+p_2p_1(u^2v+uv^2)+\frac{1}{3}(6p_3-p_2p_1)(u^2+v^2)$$
$$+\frac{1}{3}(8p_2p_1-3p_1^3)uv+\frac{1}{15}(17p_3-8p_2p_1+2p_1^3)(u+v)$$
$$+\frac{1}{3^3\cdot 5\cdot 7}(62p_3-13p_2p_1+2p_1^3),$$

$$\Delta_4(u, v; p_1, \dots, p_4)=p_4(u^4+v^4)+p_3p_1(u^3v+uv^3)+p_2^2u^2v^2$$
$$+\frac{1}{3}(8p_4-p_3p_1)(u^3+v^3)+\frac{1}{3}(6p_3p_1+4p_2^2-3p_2p_1^2)(u^2v+uv^2)$$
$$+\frac{1}{45}(108p_4-27p_3p_1-7p_2^2+6p_2p_1^2)(u^2+v^2)$$
$$+\frac{1}{45}(102p_3p_1+80p_2^2-135p_2p_1^2+33p_1^4)uv$$
$$+\frac{1}{3^3\cdot 5\cdot 7}(744p_4-325p_3p_1-176p_2^2+248p_2p_1^2-51p_1^4)(u+v) \tag{2. 2}$$
$$+\frac{1}{3^4\cdot 5^2\cdot 7}(381p_4-71p_3p_1-19p_2^2+22p_2p_1^2-3p_1^4),$$

$$\Delta_5(u, v; p_1, \dots, p_5)=p_5(u^5+v^5)+p_4p_1(u^4v+uv^4)+p_3p_2(u^3v^2+u^2v^3)$$
$$+\frac{1}{3}(10p_5-p_4p_1)(u^4+v^4)+\frac{1}{3}(8p_4p_1+4p_3p_2-3p_3p_1^2)(u^3v+uv^3)$$
$$+(4p_3p_2-p_2^2p_1)u^2v^2+\frac{1}{45}(185p_5-37p_4p_1-7p_3p_2+6p_3p_1^2)(u^3+v^3)$$
$$+\frac{1}{15}(36p_4p_1+57p_3p_2-29p_3p_1^2-26p_2^2p_1+11p_2p_1^3)(u^2v+uv^2)$$
$$+\frac{1}{3^3\cdot 5\cdot 7}(2120p_5-620p_4p_1-336p_3p_2+221p_3p_1^2+111p_2^2p_1$$
$$-51p_2p_1^3)(u^2+v^2)+\frac{1}{3^3\cdot 5\cdot 7}(1488p_4p_1+2856p_3p_2-2140p_3p_1^2$$
$$-2648p_2^2p_1+2329p_2p_1^3-440p_1^5)uv+\frac{1}{3^4\cdot 5^2\cdot 7}(6910p_5-2809p_4p_1$$
$$-3171p_3p_2+2142p_3p_1^2+2272p_2^2p_1-1808p_2p_1^3+310p_1^5)(u+v)$$
$$+\frac{1}{3^5\cdot 5^2\cdot 7\cdot 11}(5110p_5-919p_4p_1-336p_3p_2+237p_3p_1^2+127p_2^2p_1$$

$$-83p_2p_1^3+10p_1^5), \quad \ldots \quad ([2]\,(1.9)).$$

Let $x[M^{4k}]$ denote the value of an $4k$-dimensional cohomology class x on the fundamental class of M^{4k}. Since

$$p_i(P_{2k})=\binom{2k+1}{i}g_{2k}^{2i} \quad (i=0, 1, \ldots, k)$$

$$g_{2k}^{2k}[P_{2k}]=1$$

where g_{2k} is the generator of $H^2(P_{2k}; \mathbf{Z})$, we have:

$$(2.\ 3) \qquad \Delta_k(u, v; p_1, \ldots, p_k)[P_{2k}]=\sum_{i+j=k}\binom{2k+1}{i}\binom{2k+1}{j}u^iv^j.$$

Since M^{4k} is decomposed in $\Omega\otimes\mathbf{Q}$ as follows:

$$(2.\ 4) \qquad M^{4k}\sim\sum_{i_1+\cdots+i_t=k} A_{i_1, \ldots, i_t}P_{2i_1}\times\cdots\times P_{2i_t}$$

where $A_{i_1, \ldots, i_t}$'s are rational numbers, and $\Delta_k(u, v; p_1, \ldots, p_k)[M^{4k}]$ is a cobordism invariant, we have ([2] (3. 2)):

$$\Delta_k(u, v; p_1, \ldots, p_k)[M^{4k}]$$

$$(2.\ 5) \qquad =\sum_{i_1+\cdots+i_t=k} A_{i_1, \ldots, i_t}\Delta_{i_1}(u, v; p_1, \ldots, p_{i_1})[P_{2i_1}]\cdots\Delta_{i_t}(u, v; p_1, \ldots, p_{i_t})[P_{2i_t}].$$

For some small k's, by comparisons of coefficients of u^iv^j's in both sides of (2. 5), we have ([2](3. 4), (3. 6), (3. 8), (3. 10) and (3. 12)):

$$A_1=\frac{1}{3}p_1[M^4],$$

$$A_2=\frac{1}{5}(-2p_2+p_1^2)[M^8]$$

$$A_{11}=\frac{1}{9}(5p_2-2p_1^2)[M^8],$$

$$A_3=\frac{1}{7}(3p_3-3p_2p_1+p_1^3)[M^{12}]$$

$$A_{21}=\frac{1}{15}(-21p_3+19p_2p_1-6p_1^3)[M^{12}]$$

$$A_{111}=\frac{1}{27}(28p_3-23p_2p_1+7p_1^3)[M^{12}],$$

$$A_4=\frac{1}{9}(-4p_4+4p_3p_1+2p_2^2-4p_2p_1^2+p_1^4)[M^{16}]$$

$$(2.\ 6) \qquad A_{31}=\frac{1}{21}(36p_4-33p_3p_1-18p_2^2+33p_2p_1^2-8p_1^4)[M^{16}]$$

$$A_{22}=\frac{1}{25}(18p_4-18p_3p_1-7p_2^2+16p_2p_1^2-4p_1^4)[M^{16}]$$

$$A_{211}=\frac{1}{45}(-180p_4+159p_3p_1+80p_2^2-150p_2p_1^2+36p_1^4)[M^{16}]$$

$$A_{1111}=\frac{1}{81}(165p_4-137p_3p_1-70p_2^2+127p_2p_1^2-30p_1^4)[M^{16}],$$

$$A_5=\frac{1}{11}(5p_5-5p_4p_1-5p_3p_2+5p_3p_1^2+5p_2^2p_1-5p_2p_1^3+p_1^5)[M^{20}]$$

$$A_{41}=\frac{1}{27}(-55p_5+51p_4p_1+55p_3p_2-51p_3p_1^2-53p_2^2p_1+51p_2p_1^3-10p_1^5)[M^{20}]$$

$$A_{32}=\frac{1}{35}(-55p_5+55p_4p_1+49p_3p_2-52p_3p_1^2-49p_2^2p_1+50p_2p_1^3-10p_1^5)[M^{20}]$$

$$A_{311}=\frac{1}{63}(330p_5-294p_4p_1-315p_3p_2+288p_3p_1^2+297p_2^2p_1-283p_2p_1^3$$
$$+55p_1^5)[M^{20}]$$

$$A_{221}=\frac{1}{75}(330p_5-312p_4p_1-288p_3p_2+291p_3p_1^2+281p_2^2p_1-279p_2p_1^3$$
$$+55p_1^5)[M^{20}]$$

$$A_{2111}=\frac{1}{135}(-1430p_5+1250p_4p_1+1269p_3p_2-1180p_3p_1^2-1189p_2^2p_1$$
$$+1136p_2p_1^3-220p_1^5)[M^{20}]$$

$$A_{11111}=\frac{1}{243}(1001p_5-836p_4p_1-861p_3p_2+780p_3p_1^2+791p_2^2p_1-745p_2p_1^3$$
$$+143p_1^5)[M^{20}].$$

Let HP_k be the k-dimensional quaternion projective space. Since

$$p(HP_k)=(1+\alpha_k)^{2(k+1)}(1+4\alpha_k)^{-1}$$

where α_k is the generator of $H^4(HP_k;\mathbf{Z})$, from (2. 6) we have ([2] (3. 14), (3. 15), (3. 16), (3. 17) and (3. 18)):

$$HP_1\sim 0$$

$$HP_2\sim -2P_4+3P_2^2$$

(2. 7) $$HP_3\sim\lambda_3(-8P_6+24P_4P_2-16P_2^3)$$

$$HP_4\sim\lambda_4\left(-\frac{82}{3}P_8+90P_6P_2+45P_4^2-200P_4P_2^2+\frac{280}{3}P_2^4\right)$$

$$HP_5\sim\lambda_5(-92P_{10}+340P_8P_2+324P_6P_4-816P_6P_2^2-792P_4^2P_2+1616P_4P_2^3$$
$$-580P_2^5)$$

where $\lambda_i=\alpha_i^i\,[HP_i]$ $(i=3, 4, 5)$.

3. Cobordism decomposition of a submanifold

Let N^{4k+2r} be an $(4k+2r)$-dimensional compact orientable differentiable manifold and M^{4k} the submanifold of N^{4k+2r} determined by a sequence of cohomology classes $x_1, \dots, x_r \in H^2(N^{4k+2r}; \mathbf{Z})$. Then we have ([3] (1. 1)):

$$\Delta_k(u, v; p_1(M), \dots, p_k(M))[M^{4k}]$$

(3. 1)
$$= \kappa^{4k+2r}\left\{\prod_{i=1}^{r}\frac{\mathrm{tgh}x_i}{(1+u\,\mathrm{tgh}^2x_i)(1+v\,\mathrm{tgh}^2x_i)}\sum_{j\geqq 1}\Delta_j(u, v; p_1(N), \dots, p_j(N))\right\}[N^{4k+2r}]$$

where $\kappa^{4k+2r}(X)$ denotes the $(4k+2r)$-dimensional component of the element $X\in\sum_i H^i(N^{4k+2r}; \mathbf{Z})$. Since M^{4k} is decomposed in $\Omega\otimes\mathbf{Q}$ as follows:

(3. 2)
$$M^{4k}\sim\sum_{i_1+\dots+i_t=k} A^{2r}_{i_1, \dots, i_t} P_{2i_1}\times\dots\times P_{2i_t}$$

where $A^{2r}_{i_1, \dots, i_t}$'s are rational numbers, we have:

(3. 3)
$$\Delta_k(u, v; p_1(M), \dots, p_k(M))[M^{4k}]$$
$$=\sum_{i_1+\dots+i_t=k} A^{2r}_{i_1, \dots, i_t}\Delta_{i_1}(u, v; p_1, \dots, p_{i_1})[P_{2i_1}]\cdots\Delta_{i_t}(u, v; p_1, \dots, p_{i_t})[P_{2i_t}].$$

For some small (k, r)'s, by substitutions of (2.3) and (3.1) into (3.3), and comparisons of coefficients of u^iv^j's in both sides of (3.3), we can obtain the representations in terms of $p_i(N)$'s and $x_1, \dots, x_r$ of $A^{2r}_{i_1, \dots, i_t}$'s. For example, when $r=1$, we have (where we will use x instead of x_1. [3] (2. 2), (2. 3), (2. 4), (2. 5) and (2. 6)):

$$A^2_1=\frac{1}{3}(-x^3+xp_1(N))[N^6],$$

$$A^2_2=\frac{1}{5}(-x^5-x(2p_2-p_1^2))[N^{10}]$$

$$A^2_{11}=\frac{1}{9}(3x^5-x^3p_1+x(5p_2-2p_1^2))[N^{10}],$$

$$A^2_3=\frac{1}{7}(-x^7+x(3p_3-3p_2p_1+p_1^3))[N^{14}]$$

$$A^2_{21}=\frac{1}{15}(8x^7-x^5p_1+x^3(2p_2-p_1^2)-x(21p_3-19p_2p_1+6p_1^3))[N^{14}]$$

$$A^2_{111}=\frac{1}{27}(-12x^7+3x^5p_1-x^3(5p_2-2p_1^2)+x(28p_3-23p_2p_1+7p_1^3))[N^{14}],$$

$$A^2_4=\frac{1}{9}(-x^9-x(4p_4-4p_3p_1-2p_2^2+4p_2p_1^2-p_1^4))[N^{18}]$$

$$A^2_{31}=\frac{1}{21}(10x^9-x^7p_1-x^3(3p_3-3p_2p_1+p_1^3)+x(36p_4-33p_3p_1-18p_2^2$$
$$+33p_2p_1^2-8p_1^4))[N^{18}]$$

$$A^2_{22}=\frac{1}{25}(5x^9+x^5(2p_2-p_1^2)+x(18p_4-18p_3p_1-7p_2^2+16p_2p_1^2-4p_1^4))[N^{18}]$$

$$A^2_{211}=\frac{1}{45}(-55x^9+8x^7p_1-x^5(11p_2-5p_1^2)+x^3(21p_3-19p_2p_1+6p_1^3)$$
$$-x(180p_4-159p_3p_1-80p_2^2+150p_2p_1^2-36p_1^4))[N^{18}]$$

$$A^2_{1111}=\frac{1}{81}(55x^9-12x^7p_1+3x^5(5p_2-p_1^2)-x^3(28p_3-23p_2p_1+7p_1^3)$$
$$+x(165p_4-137p_3p_1-70p_2^2+127p_2p_1^2-30p_1^4))[N^{18}],$$

(3. 4) $$A^2_5=\frac{1}{11}(-x^{11}+x(5p_5-5p_4p_1-5p_3p_2+5p_3p_1^2+5p_2^2p_1-5p_2p_1^3+p_1^5))[N^{22}]$$

$$A^2_{41}=\frac{1}{27}(12x^{11}-x^9p_1+x^3(4p_4-4p_3p_1-2p_2^2+4p_2p_1^2-p_1^4)$$
$$-x(55p_5-51p_4p_1-55p_3p_2+51p_3p_1^2+53p_2^2p_1-51p_2p_1^3+10p_1^5))[N^{22}]$$

$$A^2_{32}=\frac{1}{35}(12x^{11}+x^7(2p_2-p_1^2)-x^5(3p_3-3p_2p_1+p_1^3)-x(55p_5-55p_4p_1$$
$$-49p_3p_2+52p_3p_1^2+49p_2^2p_1-50p_2p_1^3+10p_1^5))[N^{22}]$$

$$A^2_{311}=\frac{1}{63}(-78x^{11}+10x^9p_1-x^7(5p_2-2p_1^2)+3x^5(3p_3-3p_2p_1+p_1^3)$$
$$-x^3(36p_4-33p_3p_1-18p_2^2+33p_2p_1^2-8p_1^4)+x(330p_5-294p_4p_1$$
$$-315p_3p_2+288p_3p_1^2+297p_2^2p_1-283p_2p_1^3+55p_1^5))[N^{22}]$$

$$A^2_{221}=\frac{1}{75}(-78x^{11}+5x^9p_1-8x^7(2p_2-p_1^2)+x^5(21p_3-19p_2p_1+6p_1^3)$$
$$-x^3(18p_4-18p_3p_1-7p_2^2+16p_2p_1^2-4p_1^4)+x(330p_5-312p_4p_1$$
$$-288p_3p_2+291p_3p_1^2+281p_2^2p_1-279p_2p_1^3+55p_1^5))[N^{22}]$$

$$A^2_{2111}=\frac{1}{135}(364x^{11}-55x^9p_1+4x^7(16p_2-7p_1^2)-x^5(91p_3-80p_2p_1+25p_1^3)$$
$$+x^3(180p_4-159p_3p_1-80p_2^2+150p_2p_1^2-36p_1^4)-x(1430p_5-1250p_4p_1$$

$$-1269p_3p_2+1180p_3p_1^2+1189p_2^2p_1-1136p_2p_1^3+220p_1^5))[N^{22}]$$

$$A^2_{11111}=\frac{1}{243}(-273x^{11}+55x^9p_1-12x^7(5p_2-2p_1^2)+3x^5(28p_3-23p_2p_1+7p_1^3)$$

$$-x^3(165p_4-137p_3p_1-70p_2^2+127p_2p_1^2-30p_1^4)$$

$$+x(1001p_5-836p_4p_1-861p_3p_2+780p_3p_1^2+791p_2^2p_1$$

$$-745p_2p_1^3+143p_1^5))[N^{22}].$$

Now, we consider the case of $N^{4k+2r}=P_{2k+r}$ and $x_i=c_ig_{2k+r}(i=1,\ldots,r)$ where c_i is a non-zero integer, that is, M^{4k} is the submanifold of P_{2k+r} determined by $c_1g_{2k+r},\ldots,c_rg_{2k+r}$. When $r=1$, from (3.4) we have ([**3**] §2. 1), . . . , 5)):

$$3A^2_1=-c^3+4c,$$

$$5A^2_2=-c^5+6c$$

$$3A^2_{11}=c^5-2c^3+c,$$

$$7A^2_3=-c^7+8c$$

$$15A^2_{21}=8(c^7-c^5-c^3+c)$$

$$9A^2_{111}=4(-c^7+2c^5-c^3),$$

$$9A^2_4=-c^9+10c$$

$$21A^2_{31}=10(c^9-c^7-c^3+c)$$

$$5A^2_{22}=c^9-2c^5+c \tag{3. 5}$$

$$9A^2_{211}=-11c^9+16c^7+c^5-6c^3$$

$$81A^2_{1111}=5(11c^9-24c^7+15c^5-2c^3),$$

$$11A^2_5=-c^{11}+12c$$

$$9A^2_{41}=4(c^{11}-c^9-c^3+c)$$

$$35A^2_{32}=12(c^{11}-c^7-c^5+c)$$

$$21A^2_{311}=2(-13c^{11}+20c^9-7c^7+6c^5-6c^3)$$

$$25A^2_{221}=2(-13c^{11}+10c^9+16c^7-10c^5-3c^3)$$

$$135A^2_{2111}=4(91c^{11}-165c^9+48c^7+35c^5-9c^3)$$

$$81A^2_{11111}=-91c^{11}+220c^9-168c^7+40c^5-c^3.$$

When $r=2$, similarly we have (where we will use c, d instead of c_1, c_2. [3] §2. 6), . . . , 10)):

$$3A^4_1=-(c^3d+cd^3)+5cd,$$

$$5A^4_2=-(c^5d+cd^5)+7cd$$

$$9A^4_{11}=3c^5d+c^3d^3+3cd^5-7(c^3d+cd^3)+7cd,$$

$$7A^4_3=-(c^7d+cd^7)+9cd$$

$$15A^4_{21}=8(c^7d+cd^7)+c^5d^3+c^3d^5-9(c^5d+cd^5)-9(c^3d+cd^3)+18cd$$

$$9A^4_{111}=-4(c^7d+cd^7)-(c^5d^3+c^3d^5)+9(c^5d+cd^5)+3c^3d^3-6(c^3d+cd^3)+cd,$$

$$9A^4_4=-(c^9d+cd^9)+11cd$$

$$21A^4_{31}=10(c^9d+cd^9)+c^7d^3+c^3d^7-11(c^7d+cd^7)-11(c^3d+cd^3)+22cd$$

$$25A^4_{22}=5(c^9d+cd^9)+c^5d^5-11(c^5d+cd^5)+11cd$$

$$45A^4_{211}=-55(c^9d+cd^9)-8(c^7d^3+c^3d^7)-6c^5d^5+88(c^7d+cd^7)+11(c^5d^3+c^3d^5)+11c^3d^3-44(c^3d+cd^3)+11cd$$

(3. 6)
$$81A^4_{1111}=55(c^9d+cd^9)+12(c^7d^3+c^3d^7)+9c^5d^5-132(c^7d+cd^7)-33(c^5d^3+c^3d^5)+99(c^5d+cd^5)+33c^3d^3-22(c^3d+cd^3),$$

$$11A^4_5=-(c^{11}d+cd^{11})+13cd$$

$$27A^4_{41}=12(c^{11}d+cd^{11})+c^9d^3+c^3d^9-13(c^9d+cd^9)-13(c^3d+cd^3)+26cd$$

$$35A^4_{32}=12(c^{11}d+cd^{11})+c^7d^5+c^5d^7-13(c^7d+cd^7)-13(c^5d+cd^5)+26cd$$

$$63A^4_{311}=-78(c^{11}d+cd^{11})-10(c^9d^3+c^3d^9)-3(c^7d^5+c^5d^7)+130(c^9d+cd^9)+13(c^7d^3+c^3d^7)-52(c^7d+cd^7)+39(c^5d+cd^5)+13c^3d^3-52(c^3d+cd^3)+13cd$$

$$75A^4_{221}=-78(c^{11}d+cd^{11})-5(c^9d^3+c^3d^9)-8(c^7d^5+c^5d^7)+65(c^9d+cd^9)+13c^5d^5+104(c^7d+cd^7)+13(c^5d^3+c^3d^5)-78(c^5d+cd^5)-26(c^3d+cd^3)+13cd$$

$$135A^4_{2111}=364(c^{11}d+cd^{11})+55(c^9d^3+c^3d^9)+36(c^7d^5+c^5d^7)-715(c^9d+cd^9)$$

$$-104(c^7d^3+c^3d^7)-78c^5d^5+260(c^7d+cd^7)+13(c^5d^3+c^3d^5)+169(c^5d+cd^5)+78c^3d^3-78(c^3d+cd^3)$$

$$243A^5_{11111}=-273(c^{11}d+cd^{11})-55(c^9d^3+c^3d^9)-36(c^7d^5+c^5d^7)+715(c^9d+cd^9)+156(c^7d^3+c^3d^7)+117c^5d^5-624(c^7d+cd^7)-156(c^5d^3+c^3d^5)+195(c^5d+cd^5)+65c^3d^3-13(c^3d+cd^3).$$

4. Application

In this section, we will consider whether HP_k is cobordant to M^{4k} which is a submanifold of P_{2k+r}, or not, for $r=1$, 2 and $k=1,\dots,5$.

CASE 1. $r=k=1$. From (2.7) and (3.5),

$$3A^2_1=-c(c^2-4)=0.$$

Hence, HP_1 is not cobordant to any 4-dimensional submanifold of P_3 other than those which are determined by $\pm 2g_3$.

CASE 2. $r=1$, $k=2$. From (2.7) and (3.5),

$$5A^2_2+3A^2_{11}=-2c^3+7c=-1.$$

Since $2c^3-7c-1\neq 0$ for any integer c, HP_2 is not cobordant to any 8-dimensional submanifold of P_5. ([**1**] §7)

CASE 3. $r=1$, $k=3$. From (2.7) and (3.5),

$$2\cdot 15A^2_{21}+5\cdot 9A^2_{111}=-4c(c^2-4)(c^2-1)^2=0.$$

When $c=\pm 1$, there are no λ_3's such that $A^2_3=-8\lambda_3$, $A^2_{21}=24\lambda_3$, $A^2_{111}=-16\lambda_3$ because $A^2_3\neq 0$, $A^2_{21}=A^2_{111}=0$ in (3.5). Hence, HP_3 is not cobordant to any 12-dimensional submanifold of P_7 other than those which are determined by $\pm 2g_7$. ([**1**] §7)

CASE 4. $r=1$, $k=4$. From (2.7) and (3.5),

$$5\cdot 21A^2_{31}-50\cdot 5A^2_{22}-9A^2_{211}=11c^3(c^2-4)(c^2-1)^2=0.$$

Similarly to CASE 3, we have that HP_4 is not cobordant to any 16-dimensional submanifold of P_9 other than those which are determined by $\pm 2g_9$.

CASE 5. $r=1$, $k=5$. From (2.7) and (3.5),

$$165\cdot 9A^2_{41}-55\cdot 35A^2_{32}-6\cdot 25A^2_{221}=156c^3(c^2+1)(c^2-4)(c^2-1)^2=0.$$

Similarly to CASE 3, we have that HP_5 is not cobordant to any 20-dimensional submanifold of P_{11} other than those which are determined by $\pm 2g_{11}$.

CASE 6. $r=2$, $k=1$. From (2. 7) and (3. 6),

$$3A_1^4=-cd(c^2+d^2-5)=0.$$

Hence, HP_1 is not cobordant to any 4-dimensional submanifold of P_4 other than those which are determined by $(\pm 2g_4, \pm g_4)$.

CASE 7. $r=k=2$. Frm (2.7) and (3.6),

$$5A_2^4=-cd(c^4+d^4-7)=-10.$$

When c and d are prime to 5, $c^4+d^4-7\equiv 0 \pmod 5$. Hence, $c^4+d^4-7=\pm 5$ and $cd=\pm 2$, or $c^4+d^4-7=\pm 10$ and $cd=\pm 1$. But, there are no (c, d)'s which satisfy above equations. On the other hand, when $c\equiv d\equiv 0 \pmod 5$, clearly $5A_2^4\neq -10$. Thus, HP_2 is not cobordant to any 8-dimensional submanifold of P_6. ([1] §8)

CASE 8. $r=2$, $k=3$. From (2.7) and (3.6),

$$2\cdot 15A_{21}^4+5\cdot 9A_{111}^4=cd((9(c^2+d^2)-39)c^2d^2-4(c^2+d^2)^3+27(c^2+d^2)^2$$
$$-48(c^2+d^2)+41)=0$$

$$45\cdot 7A_3^4+7\cdot 15A_{21}^4=cd((-26(c^2+d^2)+126)c^2d^2+11(c^2+d^2)^3-63(c^2+d^2)^2$$
$$-63(c^2+d^2)+531)=0.$$

Eliminating c^2d^2,

$$(c^2+d^2-5)^2(c^2+d^2-9)(c^2+d^2-23)=0.$$

Hence, $(c, d)=(\pm 2, \pm 1)$. Thus, HP_3 is not cobordant to any 12-dimensional submanifold of P_8 other than those which are determined by $(\pm 2g_8, \pm g_8)$.

CASE 9. $r=2$, $k=4$. Since

$$375\cdot 9A_4^4+82\cdot 25A_{22}^4=100\cdot 21A_{31}^4+21\cdot 45A_{211}^4=8\cdot 25A_{22}^4+45A_{211}^4=0$$

from (2. 7), by (3. 6) we have:

$$35(c^2+d^2)^4-902(c^2+d^2)^2+5027$$
$$=-152(c^2d^2)^2+(140(c^2+d^2)^2-1804)c^2d^2$$

$$155(c^2+d^2)^4-748(c^2+d^2)^3+2024(c^2+d^2)-2431$$
$$=-300(c^2d^2)^2+(552(c^2+d^2)^2-2013(c^2+d^2)+231)c^2d^2$$

$$15(c^2+d^2)^4-88(c^2+d^2)^3+88(c^2+d^2)^2+44(c^2+d^2)-99$$
$$=-12(c^2d^2)^2+(52(c^2+d^2)^2-253(c^2+d^2)+187)c^2d^2.$$

Eliminating c^2d^2,

$$(c^2+d^2-5)^2(c^2+d^2-11)((c^2+d^2)^3-3(c^2+d^2)^2-39(c^2+d^2)+55)=0.$$

Hence, $(c, d)=(\pm 2, \pm 1)$. Thus, HP_4 is not cobordant to any 16-dimensional submanifold of P_{10} other than those which are determined by $(\pm 2g_{10}, \pm g_{10})$.

CASE 10. $r=2$, $k=5$. Since

$$21\cdot 27A^4_{41}-17\cdot 35A^4_{32}=28\cdot 27A^4_{41}+5\cdot 63A^4_{311}=110\cdot 35A^4_{32}+21\cdot 75A^4_{221}=0$$

from (2. 7), by (3. 6) we have:

$$48B^5-273B^4+221B^3+221B^2-273B+104$$
$$=(160B-546)C^2+(-219B^3+1092B^2-663B-442)C$$
$$-54B^5+286B^4-260B^3+195B^2-624B+793$$
$$=(-219B+442)C^2+(248B^3-1079B^2+780B-325)C$$
$$-318B^5+1365B^4+754B^3-3068B^2-546B+3133$$
$$=(-1333B+3003)C^2+(1485B^3-5460B^2-1989B+6136)C$$

where $B=c^2+d^2$, $C=c^2d^2$. Eliminating C,

$$(B-2)^3(B-5)^2(B-13)(160B^4-306B^3-259B^2-6471B+1561)=0.$$

Hence, $(c, d)=(\pm 1, \pm 1)$ or $(\pm 2, \pm 1)$. But, when $(c, d)=(\pm 1, \pm 1)$, there are no λ_5's such that $A^4_5=-92\lambda_5$, $A^4_{41}=340\lambda_5, \ldots, A^4_{11111}=-580\lambda_5$ because $A^4_5\neq 0$, $A^4_{41}=0$ in (3. 6). Thus, HP_5 is not cobordant to any 20-dimensional submanifold of P_{12} other than those which are determined by $(\pm 2g_{12}, \pm g_{12})$.

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