A remark on transformation group with four orbit types

By

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Introduction

The object of this note is to prove the following

THEOREM. *Let G be a compact connected Lie group, locally isomorphic to $T^r \times G_1 \times G_2 \times \ldots \times G_s$. where T^r is r-dimensional torus and each G_i is a simple compact connected Lie group of rank $\geqq 5$. Then the fixed point set of any effective differentiable action of G on a euclidean space R^m with four orbit types is non-empty.*

The fixed point set of differentiable action of compact connected Lie group on euclidean spaces with two or three orbit types have been proved to be non-empty by BOREL ([1]) and HSIANG, W. C. ([2]). Our result is a direct consequence of the works of HSIANG, W. C. and HSIANG, W. Y. ([3], [4]).

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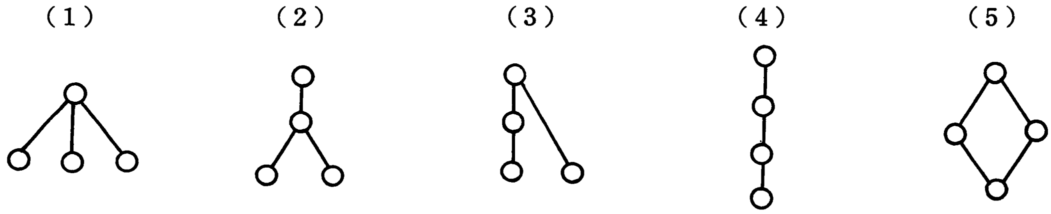
1. Statement of results

Let G be a compact connected Lie group and f an effective differentiable action of G on R^m, i.e. $f: G \times R^m \longrightarrow R^m$ is a differentiable mapping satisfying (1) $f(e, x)=x$ for every $x \varepsilon R^m$ (2) $f(g_1, f(g_2, x))=f(g_1 g_2, x)$ for $g_i \varepsilon G$, $x \varepsilon R^m$ and (3) if $f(g, x)=x$ for every $x \varepsilon R^m$, then $g=e$. We write $f(g, x)=gx$.

In the first place, we consider the case where G is locally isomorphic to a product $G_1 \times G_2$ of two simple compact connected Lie groups G_i of rank $\geqq 5$. Assume the number of orbit types of f is four. Then $G_1 \times G_2$ acts almost effectively on R^m with four orbit types. The set of all orbit types of a differenentiable action is an ordered set (i.e. $(G_x) \leqq (G_y)$ if every element of (G_x) is contained in some element of (G_y)). Hence we can define a graph for a differentiable action with finite orbit types as follows; points of the graph are orbit types and points a and b are jointed by a segment from a to b when $a<b$ and there is no point c such that $a<c<b$.

Then possible graphs of action with four orbit types are;



Consider the restricted action f_i of f to G_i. It is clear that the number of orbit types of f_i is at most four and hence principal isotropy subgroups of f_i are positive dimensional. Therefore a result in [2] shows that the fixed point set of f_i is non-empty. Choose $x_i \in F(G_i, R^m)$ (=the fixed point set of f_i) and fix them. By the following lemmas, it follows that the isotropy subgroups G_{x_1} and G_{x_2} are split, i.e. $G_{x_1}=G_1\times G_{2,x_1}$ and $G_{x_2}=G_{1,x_2}\times G_2$.

LEMMA 1. *Let $G=G_1\times G_2$ and $\overline{G}$ be a subgroup of G which containes G_1. Then $\overline{G}=G_1\times K_2$, where K_2 is a subgroup of G_2.*

LEMMA 2. *Let f be a differentiable action of $G_1\times G_2$. If $G_x=K_1\times K_2$, where K_i is subgroup of G_i, then $G_x=G_{1,x}\times G_{2,x}$.*

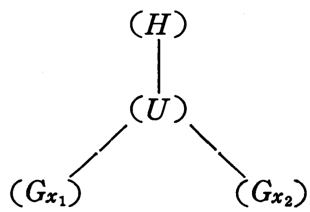
We shall show that the fixed point set of f is non-empty. It is suffifient to consider the case where $(G_{x_i})\neq(G_{x_{3-i}})(i=1, 2)$ and none of (G_{x_i}) is principal. Therefore the four orbit types are; (H)(=principal), (U), (G_{x_1}) and (G_{x_2}). Since the fixed point set of actions of graph (4) and (5) is non-empty, it suffices to consider the cases (1), (2) and (3).

The case (1). Consider the slice representation f_{i,x_i}. By a result of Borel ([1]) and the following proposition, it follows that a principal isotropy subgroup H is split.

PROPOSITION 1. *Let $G=G_1\times G_2$, where each G_i is a simple compact connected Lie group with rank $\geqq 5$ and H is a closed subgroup of G. Assume that G/H is a positive dimensional sphere. Then $H=G_1\times H_2$ or $H_1\times G_2$ and G_1/H_1 or G_2/H_2 is equal to G/H, respectively. Moreover principal isotropy subgrops are conjugate to $H_1\times H_2$, where G_1/H_1 and G_2/H_2 are spheres.*

Consider the induced action of G_2 on $X=F(G_1, R^m)$. Since $F(G, R^m)$ is empty, G_2 acts on X with only one orbit type (G_{2,x_1}) such that $G_2/G_{2,x_1}$ is sphere. The Z_2-Gysin sequence of fibering $G_2/G_{2,x_1}\longrightarrow X\longrightarrow X'$ induces a contradiction. Thus the graph (1) is impossible.

The case (2). In this case, we may assume the graph is; consider the induced action of G_2 on $X=F(G_1, R^m)$. It is clear that this action has only one orbit type and hence the following proposition, which is proved by similar arguments in the case (1), implies that $F(G_2, X)$ is non-empty. This is a contradiction.



PROPOSITION 2. *Let X be a Z_2-acyclic manifold and f be a differentiable action of a connected Lie group G on X with only one orbit type. Then G acts trivally on X.*

The case (3). The same arguments as in the case (2) show that this case is also impossible.

Thus we have proved that any almost effective differentiable action of $G_1 \times G_2$, where G_i's are simple compact connected Lie groups with rank $\geqq 5$ has fixed points.

We shall investigate the acyclicity of the fixed points set. Let $X = F(G, R^m)$ and $X_i = F(G_i, R^m)$. Consider the restricted action $\bar{f}_i$ of G_i on $X_{3-i}(i=1, 2)$. When one of $\bar{f}_i$'s has only one orbit type, say $\bar{f}_1$, proposition 2 implies that $X = X_2$. Hence X is Z_2-acyclic. Assume that both $\bar{f}_1$ and $\bar{f}_2$ have at least two orbit types. Then it is clear that both $\bar{f}_1$ and $\bar{f}_2$ must have two orbit types and hence both f_1 and f_2 have two orbit types.

PROPOSITION 3. *Let G be a simple compact connected Lie group of rank $\geqq 5$ and f be differentiable action of G on R^m with two orbit types. Then G is a classical Lie group and all isotropy subgroups are conjugate to standardly embedded subgroup.*

From this proposition, we can prove the following

PROPOSITION 4. *Let $G_1 \times G_2$ act almost effectively on R^m with four orbit types. Then the fixed poin set is Z_2-acyclic.*

Summing up above arguments, we have proved the following

THEOREM 1. *Let G be a compact connected Lie group, locally isomorphic to $G_1 \times G_2$, where each G_i is a simple compact connected Lie group of rank $\geqq 5$, and f be an almost effective differentiable action of G on R^m with four orbit types. Then the fixed point set of f is Z_2-acyclic.*

Next we shall consider the case G is locally isomorphic to $G_1 \times G_2$, where G_1 is a semi-simple compact connected Lie group and G_2 is a simple compact connected Lie group of rank $\geqq 5$. Let f be an effective differentiable action of G on R^m with four orbit types. Assume that the fixed point set of the restricted action of f to G_1 is Z_2-acyclic. By the same arguments used in the proof of Theorem 1, we can prove the fixerd point set of f is Z_2-acyclic. By the induction on the number of simple factors of G, we can prove the following.

THEOREM 2. *Let G be a semi-simle compact connected Lie group, locally iosmorphic to $G_1 \times \ldots \times G_s$, where each G_i is simple of rank $\geqq 5$, and f be an effective differerentiable action of G on R^m with four orbit types. Thne the fixed point set of f is Z_2-acyclic.*

Every compact connected Lie group G is locally isomorphic to $T^r \times G_1 \times \ldots \times G_s$, where T^r is r-dimensional torus and each G_i is a simple compact connected Lie group. From theorem 2 and Smith's theorem, it follows immeadiately that the fixed point set of any effective differentiable action of G on R^m with four orbit types is Z_2-acyclic. This completes the proof of the theorem mentioned in Introduction.

2. Proof of lemmas and propositions

Proof of Lemma 1. Note that G_1 is a normal subgroup of $\bar{G}$. Define a map $p : \bar{G}/G_1 \longrightarrow G_2$ by $p(gG)$, where $p_2 : G \longrightarrow G_2$ is the projection. Then p is a well defined homomorphism and the following diagram is commutative;

$$\begin{tikzcd} 1 \arrow[r] & G_1 \arrow[r] \arrow[d, equal] & \overline{G} \arrow[r] \arrow[d] & \overline{G}/G_1 \arrow[r] \arrow[d, "p"] & 1 \\ 1 \arrow[r] & G_1 \arrow[r] & G \arrow[r, "p_2"'] & G_2 \arrow[r] & 1. \end{tikzcd}$$

Hence p is injective. Put $K_2=$ the image of p. Then K_2 is a subgroup of G_2. Define a map $h : G_1\times K_2 \longrightarrow \overline{G}$ by $h(g_1, p(gG))=(g_1,p_2(g))$. Then it is clear that h is a well defined isomorphism.

We omit the proof of lemma 2 since it is elementary.

Proof of Proposition 1. It is known that a compact connected Lie group which acts transitively and effectively on a sphere is one of the followings; classical groups, exceptional group of rank 2, $K\times L/N$, where K is classical, $L=(e)$, $SO(2)$ or $Sp(1)$ and N is a finite group ([5]). Hence $G=G_1\times G_2$, where each G_i is simple, cannot act on the sphere effectively and the ineffective kernel W is not a finite group. Therefore W containes G_1 or G_2 and hence H must contain G_1 or G_2. Then $H=G_1\times H_2$ or $H_1\times G_2$, where H_i is a subgroup of G_i.

Proof of proposition 3. Choose $x\varepsilon F(G\ R^m)$ and consider the local representation f_x at x. By a result in [1]., the non-trivial orbits are spheres. Since G is simple of rank $\geqq 5$, G is $SU(n)$, $Sp(n)$ or $SO(n)$ and non-trivial isotropy subgroups are conjugate to standardly embedded subgroups $SU(k)$, $Sp(k)$ or $SO(k)$ respectivey (*cf.* [3], [4]).

Proof of Proposition 4. It suffices to prove that if a classical Lie group G acts on Z_2-acyclic manifold X with two orbit types and standardly embedded subgroups as non-trival isotropy subgroups, then the fixed point set is also Z_2-acyclic. Since the proofs for the four cases of $Su(n)$, $Sp(n)$ and $SO(n)$ are almost parallel, we shall only prove the $SO(n)$ case. First consider the case of $SO(2k)$. By assumption, all isotropy subgroups are conjugate to $SO(2k)$ or $SO(2k-1)$. Let T be a maximal torus and $F=F(T, X)$. It is known that F is Z_2-acyclic manifold. It is easy to see that $F(SO(2k), X)=F$. Next consider the case of $SO(2k+1)$. Let $(Z_2)^{2k}$ be a Z_2-maximal torus of $SO(2k+1)$. It is not difficult to see that $F(SO(2k+1), X)=F((Z_2)^{2k}, X)$, which is Z_2-acyclic by Smıth s theorem.

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