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A syntactical study of the subminimal logic with Nelson negation

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Abstract

Several properties, including disjunction property and Craig interpolation property, of the subminimal logic with Nelson negation which has been introduced by D. Vakarelov are shown syntactically via the cut-elimination theorem. Some extensions of the logic are mentioned, too.

0 Introduction

The subminimal logic with Nelson negation, $\mathbf{SUBMIN^N}$, has been investigated in Vakarelov [5]. It is a sub-logic of the intuitionistic logic with Nelson negation (or strong negation), and was introduced with his intention of weakening as far as possible the function of the intuitionistic negation. He gave several semantics for $\mathbf{SUBMIN^N}$ as well as some of its extensions, and showed completeness with respect to each of his semantics and decidability of these logics.

This paper is devoted to a syntactical investigation into $\mathbf{SUBMIN^N}$ as well as some of its extensions. In the first section, $\mathbf{SUBMIN^N}$ is reviewed. A sequent calculus for $\mathbf{SUBMIN^N}$ is introduced in Section 2, and the cut-elimination theorem (Theorem 3.1) for the calculus is shown in the next section. In Section 4, corollaries to the theorem including disjunction property (Theorem 4.1) and Craig interpolation property (Theorem 4.4) of $\mathbf{SUBMIN^N}$ are deduced. Among the extensions of $\mathbf{SUBMIN^N}$ which Vakarelov introduced, the "intuitionistic" ones are briefly mentioned in the last section. We don't refer to the "classical" ones, since it suffices for them to merely modify the notion of a sequent to those with plural succedent formulas from those with single succedent formula.

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1 Subminimal logic with Nelson negation

Formulas are constructed from propositional letters as well as the propositional constants $\top$ (truth) and $\bot$ (falsity) by the use of the propositional connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication), $\neg$ (negation) and $\sim$ (Nelson negation or strong negation), and are denoted by $A, B, C, A_1, \ldots$. $A \equiv B$ is an abbreviation for $(A \supset B) \wedge (B \supset A)$.

The subminimal logic with Nelson negation, $\mathbf{SUBMIN^N}$, which was introduced in Vakarelov [5] is the following Hilbert style axiomatic system.

(I) Axioms for the positive logic with the constants $\top$ and $\bot$:

P1 $A \supset (B \supset A)$.

P2 $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$.

P3 $(A \wedge B) \supset A$.

P4 $(A \wedge B) \supset B$.

P5 $(C \supset A) \supset ((C \supset B) \supset (C \supset (A \wedge B)))$.

P6 $A \supset (A \vee B)$.

P7 $B \supset (A \vee B)$.

P8 $(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$.

P9 $A \supset \top$.

P10 $\bot \supset A$.

(II) Axioms for the Nelson negation $\sim$:

($\sharp$) $\sim A \supset (\neg B \supset \neg A)$.

($\sim\supset$) $\sim(A \supset B) \equiv (A \wedge \sim B)$.

($\sim\wedge$) $\sim(A \wedge B) \equiv (\sim A \vee \sim B)$.

($\sim\vee$) $\sim(A \vee B) \equiv (\sim A \wedge \sim B)$.

($\sim\neg$) $\sim\neg A \equiv A$.

($\sim\sim$) $\sim\sim A \equiv A$.

($\sim\top$) $\sim\top \supset A$.

($\sim\bot$) $A \supset \sim\bot$.

(III) Rules of inference:

$$\text{(MP)}\ \frac{A \quad A \supset B}{B}\,. \qquad \text{(Ext}-\neg)\ \frac{A \equiv B}{\neg A \equiv \neg B}\,.$$

In describing proofs in $\mathbf{SUBMIN^N}$, inferences in the positive logic will not be referred to, for simplicity.

The following facts are shown in Vakarelov [5].

Fact 1.1. (Vakarelov [5, Proposition 2.1]) *The following formulas are provable in* $\mathbf{SUBMIN^N}$:

P11 $(A \supset B) \supset (\neg B \supset \neg A)$.

P12 $\neg A \supset \neg\neg\top$. □

Fact 1.2. (Vakarelov [5, Corollary 5.3]) *The axiomatic system*

$$\mathbf{P1} + \mathbf{P2} + \cdots + \mathbf{P10} + \mathbf{P11} + \mathbf{P12} + \text{(MP)},$$

namely, the one which has **P1**–**P12** *as the axioms, and* (MP) *as the sole rule of inference, constitutes the* $\{\sim\}$*-less fragment of* $\mathbf{SUBMIN^N}$. □

An axiomatic system for the $\{\top, \bot, \sim\}$-less fragment of $\mathbf{SUBMIN^N}$ will be given afterward (Theorem 4.3).

The following lemma will be used in the proof of Lemma 2.3.

Lemma 1.3. *The following formulas are provable in* $\mathbf{SUBMIN^N}$:

Q1 $\neg A \supset (A \supset \neg C)$.

Q2 $\neg D \supset ((A \supset \neg A) \supset \neg A)$.

Q3 $\neg D \supset (\sim A \supset (A \supset \neg C))$.

Proof. **Q1**: $A \supset (\neg A \supset \neg C)$ follows from $A \supset (C \supset A)$ by **P11**; and so $\neg A \supset (A \supset \neg C)$.

Q2: By **Q1**, $\neg A \supset (A \supset \neg\top)$, and hence $(A \supset \neg A) \supset (A \supset \neg\top)$; so by **P11**, $(A \supset \neg A) \supset (\neg\neg\top \supset \neg A)$, and hence $\neg\neg\top \supset ((A \supset \neg A) \supset \neg A)$; and so by **P12**, $\neg D \supset ((A \supset \neg A) \supset \neg A)$.

Q3: By **Q1**, $\neg A \supset (A \supset \neg D)$. On the other hand, by ($\sharp$), $\neg C \supset (\sim A \supset \neg A)$. Hence $\neg C \supset (\sim A \supset (A \supset \neg D))$. □

2 Sequent calculus for SUBMIN$^{\mathrm{N}}$

Sequents are expressions of the form $A_1, A_2, \ldots, A_n \to C$, where $n \geq 0$. Possibly empty finite sequences of formulas separated by commas are denoted by $\Gamma, \Pi, \Gamma_1, \ldots$. For sequent calculi, see Takeuti [4].

Our sequent calculus **G-SUBMIN$^{\mathbf{N}}$** consists of the following initial sequents and rules of inference.

(i) Initial sequents:

$$A \to A; \quad \to \top; \quad \bot \to C; \quad \sim\top \to C; \quad \to \sim\bot.$$

(ii) Structural rules of inference:

$$\textit{Weakening}\ \frac{\Gamma \to C}{A, \Gamma \to C}. \qquad \textit{Exchange}\ \frac{\Gamma, A, B, \Pi \to C}{\Gamma, B, A, \Pi \to C}.$$

$$\textit{Contraction}\ \frac{A, A, \Gamma \to C}{A, \Gamma \to C}. \qquad \textit{Cut}\ \frac{\Gamma \to A \quad A, \Pi \to C}{\Gamma, \Pi \to C}.$$

(iii) Logical rules of inference for the positive logic:

$$(\wedge \to)\ \frac{A, \Gamma \to C}{A \wedge B, \Gamma \to C}, \quad \frac{B, \Gamma \to C}{A \wedge B, \Gamma \to C}. \qquad (\to \wedge)\ \frac{\Gamma \to A \quad \Gamma \to B}{\Gamma \to A \wedge B}.$$

$$(\vee \to)\ \frac{A, \Gamma \to C \quad B, \Gamma \to C}{A \vee B, \Gamma \to C}. \qquad (\to \vee)\ \frac{\Gamma \to A}{\Gamma \to A \vee B}, \quad \frac{\Gamma \to B}{\Gamma \to A \vee B}.$$

$$(\supset \to)\ \frac{\Gamma \to A \quad B, \Pi \to C}{A \supset B, \Gamma, \Pi \to C}. \qquad (\to \supset)\ \frac{A, \Gamma \to B}{\Gamma \to A \supset B}.$$

(iv) Logical rules of inference for $\neg$:

$$(\neg\rightarrow)\ \frac{\Gamma\rightarrow A}{\neg A,\Gamma\rightarrow\neg C}. \qquad (\rightarrow\neg)\ \frac{A,\Gamma\rightarrow\neg A}{\neg D,\Gamma\rightarrow\neg A}.$$

(v) Logical rule of inference $(\sim\rightarrow)$:

$$(\sim\rightarrow)\ \frac{\Gamma\rightarrow A}{\sim A,\neg D,\Gamma\rightarrow\neg C}.$$

(vi) Other logical rules of inference for $\sim$:

$$(\sim\wedge\rightarrow)\ \frac{\sim A,\Gamma\rightarrow C \quad \sim B,\Gamma\rightarrow C}{\sim(A\wedge B),\Gamma\rightarrow C}. \quad (\rightarrow\sim\wedge)\ \frac{\Gamma\rightarrow\sim A}{\Gamma\rightarrow\sim(A\wedge B)},\ \frac{\Gamma\rightarrow\sim B}{\Gamma\rightarrow\sim(A\wedge B)}.$$

$$(\sim\vee\rightarrow)\ \frac{\sim A,\Gamma\rightarrow C}{\sim(A\vee B),\Gamma\rightarrow C},\ \frac{\sim B,\Gamma\rightarrow C}{\sim(A\vee B),\Gamma\rightarrow C}. \quad (\rightarrow\sim\vee)\ \frac{\Gamma\rightarrow\sim A \quad \Gamma\rightarrow\sim B}{\Gamma\rightarrow\sim(A\vee B)}.$$

$$(\sim\supset\rightarrow)\ \frac{A,\Gamma\rightarrow C}{\sim(A\supset B),\Gamma\rightarrow C},\ \frac{\sim B,\Gamma\rightarrow C}{\sim(A\supset B),\Gamma\rightarrow C}. \quad (\rightarrow\sim\supset)\ \frac{\Gamma\rightarrow A \quad \Gamma\rightarrow\sim B}{\Gamma\rightarrow\sim(A\supset B)}.$$

$$(\sim\neg\rightarrow)\ \frac{A,\Gamma\rightarrow C}{\sim\neg A,\Gamma\rightarrow C}. \qquad (\rightarrow\sim\neg)\ \frac{\Gamma\rightarrow A}{\Gamma\rightarrow\sim\neg A}.$$

$$(\sim\sim\rightarrow)\ \frac{A,\Gamma\rightarrow C}{\sim\sim A,\Gamma\rightarrow C}. \qquad (\rightarrow\sim\sim)\ \frac{\Gamma\rightarrow A}{\Gamma\rightarrow\sim\sim A}.$$

In describing proofs in **G-SUBMIN**$^{\mathbf{N}}$, applications of the structural rules except cut will not be referred to, for simplicity.

By the following theorem, **G-SUBMIN**$^{\mathbf{N}}$forms a sequent calculus for **SUBMIN**$^{\mathbf{N}}$.

Theorem 2.1. *A formula C is provable in* **SUBMIN**$^{\mathbf{N}}$*, iff the sequent $\rightarrow C$ is provable in* **G-SUBMIN**$^{\mathbf{N}}$*.*

This follows from the following two lemmas.

Lemma 2.2. *If a formula C is provable in* **SUBMIN**$^{\mathbf{N}}$*, then the sequent $\rightarrow C$ is provable in* **G-SUBMIN**$^{\mathbf{N}}$*.*

Proof. By easy induction on the length of the proof in **SUBMIN**$^{\mathbf{N}}$. We will only mention the axiom ($\sharp$) and the rule (Ext-$\neg$).

For the axiom ($\sharp$), the following proof in **G-SUBMIN**$^{\mathbf{N}}$suffices.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\sim A, \neg B, A \to \neg A}\,(\sim\to)}{\sim A, \neg B \to \neg A}\,(\to \neg)}{\sim A \to \neg B \supset \neg A}}{\to \sim A \supset (\neg B \supset \neg A)}$$

On the other hand, the proof which follows serves for the rule (Ext-$\neg$).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\to A \equiv B \qquad \dfrac{\dfrac{B \to B \quad A \to A}{B \supset A, B \to A}}{A \equiv B, B \to A}}{B \to A}}{\neg A, B \to \neg B}\,(\neg\to)}{\neg A \to \neg B}\,(\to\neg)}{\to \neg A \supset \neg B} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\to A \equiv B \qquad \dfrac{\dfrac{A \to A \quad B \to B}{A \supset B, A \to B}}{A \equiv B, A \to B}}{A \to B}}{\neg B, A \to \neg A}\,(\neg\to)}{\neg B \to \neg A}\,(\to\neg)}{\to \neg B \supset \neg A}}{\to \neg A \equiv \neg B}$$

□

Lemma 2.3. *If a sequent* $\Gamma \to C$ *is provable in* **G-SUBMIN**$^{\mathbf{N}}$, *then the formula* C *or* $(\bigwedge \Gamma) \supset C$ *is provable in* **SUBMIN**$^{\mathbf{N}}$, *according as* Γ *is empty or not.*

Proof. By induction on the length of the proof in **G-SUBMIN**$^{\mathbf{N}}$. The rules ($\neg \to$), ($\to \neg$) and ($\sim \to$) are justified by Lemma 1.3, and other rules and initial sequents are easy to check. □

3 Cut-elimination for G-SUBMIN$^{\mathbf{N}}$

This section is devoted to the proof of the following cut-elimination theorem for **G-SUBMIN**$^{\mathbf{N}}$.

Theorem 3.1. *Every proof in* **G-SUBMIN**$^{\mathbf{N}}$*can be transformed into another proof in* **G-SUBMIN**$^{\mathbf{N}}$*such that the endsequent of the latter is the same as that of the former, cut is not applied in the latter, and every logical rule applied in the latter is applied in the former as well.*

As usual, this theorem is proved by showing the following lemma for the calculus **G*-SUBMIN**$^{\mathbf{N}}$that is obtained from **G-SUBMIN**$^{\mathbf{N}}$by replacing cut with the following mix rule, where A, the *mix formula* of this inference, occurs in Π at least once, and Π_A denotes the result of deleting all the occurrences of A in Π.

$$Mix\ \frac{\Gamma \to A \quad \Pi \to C}{\Gamma, \Pi_A \to C}$$

Lemma 3.2. *Every proof in* **G*-SUBMIN**$^{\mathbf{N}}$ *in which mix is applied as the last inference only can be transformed into another proof in* **G*-SUBMIN**$^{\mathbf{N}}$ *such that the endsequent of the latter is the same as that of the former, mix is not applied in the latter, and every logical rule applied in the latter is applied in the former as well.*

This lemma is shown by double induction on the grade and rank of the given proof, as usual. The course of proof is to reduce first the right rank of the given proof, secondly the left rank, and lastly the grade. Only a few typical cases of **G*-SUBMIN**$^{\mathbf{N}}$ will be mentioned. Let M denote the sole application of mix in the given proof, while $\boldsymbol{S}_{\mathrm{L}}$ and $\boldsymbol{S}_{\mathrm{R}}$ the left and right upper sequents of M, respectively.

(A) First, suppose that the right rank of the proof is greater than 1. Only the cases where $\boldsymbol{S}_{\mathrm{R}}$ is the lower sequent of $(\neg \to)$, $(\to \neg)$ or $(\sim \to)$ are problematic; among them, the case where $\boldsymbol{S}_{\mathrm{R}}$ is the lower sequent $\sim A, \neg D, \Pi \to \neg C$ of $(\sim \to)$ and the mix formula of M is $\sim A$ is exemplified. The proof has the following form, where $\sim A$ occurs in Π.

$$\cfrac{\Gamma \to \sim A \qquad \cfrac{\Pi \to A}{\sim A, \neg D, \Pi \to \neg C}\,(\sim \to)}{\Gamma, \neg D, \Pi_{\sim A} \to \neg C}\,M$$

Only the case where $\sim A$ does not occur in Γ is problematic. In this case, transform the proof into the following one.

$$\cfrac{\Gamma \to \sim A \qquad \cfrac{\cfrac{\Gamma \to \sim A \quad \Pi \to A}{\Gamma, \Pi_{\sim A} \to A}\,M_1}{\sim A, \neg D, \Gamma, \Pi_{\sim A} \to \neg C}\,(\sim \to)}{\Gamma, \neg D, \Pi_{\sim A} \to \neg C}\,M_2$$

The mixes M_1 and M_2 are eliminable by the hypothesis of induction on rank.

(B) Secondly, suppose that the right rank of the proof is equal to 1, while the left one is greater than 1. The only problematic cases are the ones where $\boldsymbol{S}_{\mathrm{L}}$ is the lower sequent of $(\to \neg)$, while $\boldsymbol{S}_{\mathrm{R}}$ is the lower sequent of $(\neg \to)$, $(\to \neg)$ or $(\sim \to)$. As an example, suppose that $\boldsymbol{S}_{\mathrm{R}}$ is the lower sequent of $(\neg \to)$. The proof has the following form, where $\neg A$ does not occur in Π.

$$\cfrac{\cfrac{A, \Gamma \to \neg A}{\neg D, \Gamma \to \neg A}\,(\to \neg) \qquad \cfrac{\Pi \to A}{\neg A, \Pi \to \neg C}\,(\neg \to)}{\neg D, \Gamma, \Pi \to \neg C}\,M$$

Transform this into the following one.

$$\dfrac{\Pi \to A \qquad \dfrac{A, \Gamma \to \neg A \qquad \dfrac{A \to A}{\neg A, A \to \neg C}\,(\neg \to)}{A, \Gamma \to \neg C}\,M_1}{\neg D, \Gamma, \Pi \to \neg C}\,M_2$$

The mixes M_1 and M_2 are eliminable by the hypothesis of induction on rank and grade, respectively.

(C) Lastly, suppose that the rank of the proof is equal to 2. The characteristic cases of **G*-SUBMIN**$^{\mathbf{N}}$are the ones where the outermost logical symbol of the mix formula is $\neg$ or $\sim$.

Case 1: The outermost logical symbol of the mix formula is $\neg$. Suppose, for example, that $\boldsymbol{S}_{\mathrm{L}}$ and $\boldsymbol{S}_{\mathrm{R}}$ are the lower sequents of $(\neg \to)$ and $(\to \neg)$, respectively. The proof has the following form, where $\neg C$ does not occur in Π.

$$\dfrac{\dfrac{\Gamma \to A}{\neg A, \Gamma \to \neg C}\,(\neg \to) \qquad \dfrac{\Pi \to C}{\neg C, \Pi \to \neg D}\,(\to \neg)}{\neg A, \Gamma, \Pi \to \neg D}\,M$$

Transform this into the following mix-free one.

$$\dfrac{\Gamma \to A}{\neg A, \Gamma, \Pi \to \neg D}\,(\neg \to)$$

Case 2: The outermost logical symbol of the mix formula is $\sim$. The problematic cases are the ones where $\boldsymbol{S}_{\mathrm{R}}$ is the lower sequent of $(\sim \to)$, while $\boldsymbol{S}_{\mathrm{L}}$ is the initial sequent $\to \sim\perp$ or the lower sequent of $(\to \sim\wedge)$, $(\to \sim\vee)$, $(\to \sim\supset)$, $(\to \sim\neg)$ or $(\to \sim\sim)$. Among them, the case where $\boldsymbol{S}_{\mathrm{L}}$ is the lower sequent of $(\to \sim\supset)$ is exemplified. In this case, the proof has the following form, where $\sim(A \supset B)$ does not occur in Π.

$$\dfrac{\dfrac{\Gamma \to A \qquad \Gamma \to \sim B}{\Gamma \to \sim(A \supset B)}\,(\to \sim\supset) \qquad \dfrac{\Pi \to A \supset B}{\sim(A \supset B), \neg D, \Pi \to \neg C}\,(\sim \to)}{\Gamma, \neg D, \Pi \to \neg C}\,M$$

Transform this into the following one.

$$\dfrac{\Pi \to A \supset B \qquad \dfrac{\Gamma \to A \qquad \dfrac{\Gamma \to \sim B \qquad \dfrac{B \to B}{\sim B, \neg D, B \to \neg C}\,(\sim \to)}{\Gamma, \neg D, B \to \neg C}\,M_1}{A \supset B, \Gamma, \neg D \to \neg C}}{\Gamma, \neg D, \Pi \to \neg C}\,M_2$$

The mixes M_1 and M_2 are eliminable by the hypothesis of induction on grade.

This concludes the proof of Lemma 3.2, and so ends that of our cut-elimination theorem.

4 Some corollaries

In this section, corollaries to the cut-elimination theorem (Theorem 3.1) are shown.

Since cut-free proofs in **G-SUBMIN**$^{\mathbf{N}}$enjoy a kind of subformula property, decidability of **SUBMIN**$^{\mathbf{N}}$immediately follows; while, by inspection of the rules of inference of **SUBMIN**$^{\mathbf{N}}$except for cut, the following theorems are evident.

Theorem 4.1. *If* $A \vee B$ *is provable in* **SUBMIN**$^{\mathbf{N}}$, *then either* A *or* B *is provable in* **SUBMIN**$^{\mathbf{N}}$. □

Theorem 4.2. *There is no formula* A *such that* $\neg A$ *is provable in* **SUBMIN**$^{\mathbf{N}}$. □

As another corollary to the cut-elimination theorem, an axiomatic system for the $\{\top, \bot, \sim\}$-less fragment of **SUBMIN**$^{\mathbf{N}}$can be obtained. Recall that **Q1** and **Q2** are introduced in Lemma 1.3.

Theorem 4.3. *The axiomatic system*

$$\mathbf{P1} + \mathbf{P2} + \cdots + \mathbf{P8} + \mathbf{Q1} + \mathbf{Q2} + (\mathrm{MP})$$

constitutes the $\{\top, \bot, \sim\}$*-less fragment of* **SUBMIN**$^{\mathbf{N}}$.

Proof. Suppose that a $\{\top, \bot, \sim\}$-less formula C is provable in **SUBMIN**$^{\mathbf{N}}$. Then, the sequent $\to C$ has a cut-free proof in **G-SUBMIN**$^{\mathbf{N}}$, and only the initial sequents of the form $A \to A$ and the rules of the classes (ii), (iii) and (iv) are applied in it. So, by reviewing the proof of Lemma 2.3, C can be obtained by using the axioms **P1**–**P8**, **Q1**–**Q2** and the rule (MP). □

Note that Vakarelov's proof of Fact 1.2 is semantical, but our proof of the above theorem is completely syntactical.

The next by-product of the cut-elimination theorem is a **SUBMIN**$^{\mathbf{N}}$-version of Craig's interpolation theorem. It is what is called the Maehara method to deduce the latter theorem from the former (cf. Takeuti [4, Lemma 6.5]).

Theorem 4.4. (1) *If* $A \supset B$ *is provable in* **SUBMIN**$^{\mathbf{N}}$, *then there exists a formula* I *such that* $A \supset I$ *and* $I \supset B$ *are provable in* **SUBMIN**$^{\mathbf{N}}$*and* I *contains only those propositional letters which occur in both* A *and* B.

(2) *If A and B are $\{\sim\}$-less, in addition, then I can be taken from among the $\{\sim\}$-less formulas.*

(3) *If A and B are $\{\top, \bot, \sim\}$-less, and some propositional letters are contained in both A and B, in addition, then I can be taken from among the $\{\top, \bot, \sim\}$-less formulas.*

This theorem follows immediately from the following lemma; where with regard to (3), let p be a propositional letter which is contained in both A and B.

Lemma 4.5. (1) *Let $\Gamma \to C$ be provable in* **G-SUBMIN$^{\mathbf{N}}$**, *and let $(\Gamma_1; \Gamma_2)$ be an arbitrary partition of Γ. Then there exists a formula I such that $\Gamma_1 \to I$ and $I, \Gamma_2 \to C$ are provable in* **G-SUBMIN$^{\mathbf{N}}$**, *and satisfies the following property $(*_1)$:*

$(*_1)$ *I contains only those propositional letters which occur in both Γ_1 and Γ_2, C.*

(2) *If Γ is a sequence of $\{\sim\}$-less formulas, and C is also $\{\sim\}$-less, an addition, then I can be taken to satisfy $(*_2)$ below instead of $(*_1)$:*

$(*_2)$ *I is $\{\sim\}$-less, and contains only those propositional letters which occur in both Γ_1 and Γ_2, C.*

(3) *If Γ is a sequence of $\{\top, \bot, \sim\}$-less formulas, C is also $\{\top, \bot, \sim\}$-less, and if a propositional letter p is given, in addition, then I can be taken to satisfy $(*_3)$ below instead of $(*_1)$ and $(*_2)$:*

$(*_3)$ *I is $\{\top, \bot, \sim\}$-less, and contains only those propositional letters which occur in both Γ_1 and Γ_2, C or are p.*

Proof. The proof of (1) is sketched below; the $\{\sim\}$-less part of the proof forms that of (2), while just take $p \supset p$ instead of $\top$ in Case 1 below for the proof of (3).

The proof is by induction on the number of inferences applied in a cut-free proof of $\Gamma \to C$. The demanded formula I will be called an *interpolant of* $(\Gamma_1 \to / \Gamma_2 \to C)$. Only the characteristic cases of **G-SUBMIN$^{\mathbf{N}}$** are dealt with.

Case 1: $\Gamma \to C$ is an initial sequent $A \to A$. In this case, A and $\top$ form interpolants of $(A \to / \to A)$ and $(\to / A \to A)$, respectively.

Case 2: The last inference is $(\neg \to)$ with $\Gamma \to A$ and $\neg A, \Gamma \to \neg C$ as the upper and lower sequents, respectively. By the hypothesis of induction, let I and J be interpolants of $(\Gamma_2 \to / \Gamma_1 \to A)$ and $(\Gamma_1 \to / \Gamma_2 \to A)$, respectively. Then $\neg I$ and J form interpolants of $(\neg A, \Gamma_1 \to / \Gamma_2 \to \neg C)$ and $(\Gamma_1 \to / \neg A, \Gamma_2 \to \neg C)$, respectively.

Case 3: The last inference is $(\to \neg)$ with $A, \Gamma \to \neg A$ and $\neg D, \Gamma \to \neg A$ as the upper and lower sequents, respectively. By the hypothesis of induction, let I be an interpolant of $(\Gamma_1 \to /A, \Gamma_2 \to \neg A)$. Then $\neg\neg I$ and I form interpolants of $(\neg D, \Gamma_1 \to /\Gamma_2 \to \neg A)$ and $(\Gamma_1 \to /\neg D, \Gamma_2 \to \neg A)$, respectively.

Case 4: The last inference is $(\sim\to)$ with $\Gamma \to A$ and $\sim A, \neg D, \Gamma \to \neg C$ as the upper and lower sequents, respectively. By the hypothesis of induction, let I and J be interpolants of $(\Gamma_2 \to /\Gamma_1 \to A)$ and $(\Gamma_1 \to /\Gamma_2 \to A)$, respectively. Then $\neg I$, $\neg\bot \supset \neg I$, $\neg\bot \wedge J$ and J form interpolants of $(\sim A, \neg D, \Gamma_1 \to /\Gamma_2 \to \neg C)$, $(\sim A, \Gamma_1 \to /\neg D, \Gamma_2 \to \neg C)$, $(\neg D, \Gamma_1 \to / \sim A, \Gamma_2 \to \neg C)$ and $(\Gamma_1 \to /\sim A, \neg D, \Gamma_2 \to \neg C)$, respectively. □

A logic $\boldsymbol{L}$ is called *normalizable*, iff there is a finite set $\{A_1, A_2, \ldots, A_n\}$ of formulas such that

$$\boldsymbol{L} = A_1 + A_2 + \cdots + A_n + (\mathrm{MP}),$$

and the propositional connectives which occur in each of $A_1, A_2, \ldots,$ and A_n are the implication $\supset$ and at most one of the others; where in the above equation, $A_1, A_2, \ldots,$ and A_n work as the axiom schemes rather than axioms (cf. Hosoi-Ono [1]). It is well-known that the classical and intuitionistic logics are normalizable.

Theorem 4.6. *The logic* $\mathbf{SUBMIN^N}$ *is not normalizable.*

Proof. Suppose that $\mathbf{SUBMIN^N}$ were normalizable, and $\{A_1, A_2, \ldots, A_n\}$ the set of formulas in question. Then, since $\sim p \supset (\neg q \supset \neg p)$ is an instance of the axiom (#) and so is provable in $\mathbf{SUBMIN^N}$, it must be obtained from some substitution instances $B_1, B_2, \ldots, B_m$ of $A_1, A_2, \ldots, A_n$ by applications of (MP). So, the sequent $\sim p, \neg q, B_1, B_2, \ldots, B_m \to \neg p$ has a proof in $\mathbf{G\text{-}SUBMIN^N}$ in which $(\sim\to)$ is not applied. On the other hand, since A_i is provable in $\mathbf{SUBMIN^N}$, the sequent $\to A_i$ has a cut-free proof $(i = 1, 2, \ldots, n)$. Since it is not the case that both $\neg$ and $\sim$ occur in A_i, $(\sim\to)$ is not applied in the cut-free proof, too. It follows that $\to B_j$ also has a proof in which $(\sim\to)$ is not applied $(j = 1, 2, \ldots, m)$. So, $\sim p, \neg q \to \neg p$ has a proof in which $(\sim\to)$ is not applied. By the cut-elimination theorem, it has a cut-free proof in which $(\sim\to)$ is not applied. But this is impossible. □

Consider the following property for a logic $\boldsymbol{L}$:

(#) *If $A \supset B$ is provable in $\boldsymbol{L}$, and if A and B contain no propositional letter in common, then either $\neg A$ or B is provable in $\boldsymbol{L}$.*

The classical and intuitionisitic logics enjoy this property.

Theorem 4.7. *The logic* $\mathbf{SUBMIN^N}$ *lacks the property* (#).

Proof. $(p \wedge \neg p) \supset \neg q$ forms a counterexample. □

5 Extensions of $\mathbf{SUBMIN^N}$

Let **(Nor)** and **(Nor*)** be the following candidates for additional axioms:

(Nor) $\neg\neg\top$.

(Nor*) $\neg\top \supset A$.

The extensions $\mathbf{MIN^N}$, $\mathbf{CO\text{-}MIN^N}$ and $\mathbf{INT^N}$ of $\mathbf{SUBMIN^N}$ are defined as follows:

$$\begin{aligned} \mathbf{MIN^N} &= \mathbf{SUBMIN^N} + \mathbf{(Nor)}, \\ \mathbf{CO\text{-}MIN^N} &= \mathbf{SUBMIN^N} + \mathbf{(Nor^*)}, \\ \mathbf{INT^N} &= \mathbf{SUBMIN^N} + \mathbf{(Nor)} + \mathbf{(Nor^*)}, \end{aligned}$$

namely, $\mathbf{MIN^N}$ is $\mathbf{SUBMIN^N}$ added by **(Nor)** as an additional axiom, and so on.

For the corresponding sequent calculi, consider the following rules:

$$(\neg\rightarrow)_{\mathrm{R}}\ \frac{\Gamma \rightarrow A}{\neg A, \Gamma \rightarrow C}. \qquad (\rightarrow\neg)_{\mathrm{L}}\ \frac{A, \Gamma \rightarrow \neg A}{\Gamma \rightarrow \neg A}.$$

$$(\sim\rightarrow)_{\mathrm{L}}\ \frac{\Gamma \rightarrow A}{\sim A, \Gamma \rightarrow \neg C}. \quad (\sim\rightarrow)_{\mathrm{R}}\ \frac{\Gamma \rightarrow A}{\sim A, \neg D, \Gamma \rightarrow C}. \quad (\sim\rightarrow)_{\mathrm{LR}}\ \frac{\Gamma \rightarrow A}{\sim A, \Gamma \rightarrow C}.$$

Note that $\sim$ as well as $\neg$ is introduced in the lower sequents of $(\sim\rightarrow)_{\mathrm{L}}$ and $(\sim\rightarrow)_{\mathrm{R}}$, while only $\sim$ is introduced in that of $(\sim\rightarrow)_{\mathrm{LR}}$.

The sequent calculi $\mathbf{G\text{-}MIN^N}$, $\mathbf{G\text{-}CO\text{-}MIN^N}$ and $\mathbf{G\text{-}INT^N}$ are defined as follows:

$$\begin{aligned} \mathbf{G\text{-}MIN^N} &= \text{(i)} + \text{(ii)} + \text{(iii)} + (\neg\rightarrow) + (\rightarrow\neg)_{\mathrm{L}} + (\sim\rightarrow)_{\mathrm{L}} + \text{(vi)}, \\ \mathbf{G\text{-}CO\text{-}MIN^N} &= \text{(i)} + \text{(ii)} + \text{(iii)} + (\neg\rightarrow)_{\mathrm{R}} + (\rightarrow\neg) + (\sim\rightarrow)_{\mathrm{R}} + \text{(vi)}, \\ \mathbf{G\text{-}INT^N} &= \text{(i)} + \text{(ii)} + \text{(iii)} + (\neg\rightarrow)_{\mathrm{R}} + (\rightarrow\neg)_{\mathrm{L}} + (\sim\rightarrow)_{\mathrm{LR}} + \text{(vi)}, \end{aligned}$$

namely, $\mathbf{G\text{-}MIN^N}$ is obtained from $\mathbf{G\text{-}SUBMIN^N}$ by replacing $(\rightarrow\neg)$ and $(\sim\rightarrow)$ with $(\rightarrow\neg)_{\mathrm{L}}$ and $(\sim\rightarrow)_{\mathrm{L}}$, and so on. Remember that the $\{\top, \bot\}$-less fragment of $\mathbf{G\text{-}INT^N}$ is essentially the same as the calculus introduced in Ishimoto [2], [3] for the intuitionistic logic with strong negation.

Decidability as well as the subsequent theorems are proved similarly to Theorems 2.1, 3.1, 4.1, 4.2 ($\mathbf{CO\text{-}MIN^N}$ alone), 4.3, 4.4, 4.6 ($\mathbf{MIN^N}$ and $\mathbf{CO\text{-}MIN^N}$ alone), and 4.7 ($\mathbf{CO\text{-}MIN^N}$ alone), respectively.

Theorem 5.1. *Let $\boldsymbol{L} \in \{\mathbf{MIN^N}, \mathbf{CO\text{-}MIN^N}, \mathbf{INT^N}\}$. A formula C is provable in $\boldsymbol{L}$, iff the sequent $\to C$ is provable in* $\mathbf{G}$-$\boldsymbol{L}$. □

Theorem 5.2. *Let $\boldsymbol{L} \in \{\mathbf{MIN^N}, \mathbf{CO\text{-}MIN^N}, \mathbf{INT^N}\}$. Every proof in* $\mathbf{G}$-$\boldsymbol{L}$ *can be transformed into another proof in* $\mathbf{G}$-$\boldsymbol{L}$ *such that the endsequent of the latter is the same as that of the former, cut is not applied in the latter, and every logical rule applied in the latter is applied in the former as well.* □

Theorem 5.3. *Let $\boldsymbol{L} \in \{\mathbf{MIN^N}, \mathbf{CO\text{-}MIN^N}, \mathbf{INT^N}\}$. If $A \vee B$ is provable in $\boldsymbol{L}$, then either A or B is provable in $\boldsymbol{L}$.* □

Theorem 5.4. *There is no formula A such that $\neg A$ is provable in* $\mathbf{CO\text{-}MIN^N}$. □

Recall that $\neg\neg\top$ is an axiom and so is a theorem of $\mathbf{MIN^N}$and $\mathbf{INT^N}$.

Theorem 5.5. *The axiomatic systems*

$$\mathbf{P1} + \mathbf{P2} + \cdots + \mathbf{P8} + \mathbf{Q1} + \mathbf{Q2}_{\mathrm{L}} + (\mathrm{MP}),$$

$$\mathbf{P1} + \mathbf{P2} + \cdots + \mathbf{P8} + \mathbf{Q1}_{\mathrm{R}} + \mathbf{Q2} + (\mathrm{MP}), \text{ and}$$

$$\mathbf{P1} + \mathbf{P2} + \cdots + \mathbf{P8} + \mathbf{Q1}_{\mathrm{R}} + \mathbf{Q2}_{\mathrm{L}} + (\mathrm{MP})$$

constitute the $\{\top, \bot, \sim\}$-less fragments of $\mathbf{MIN^N}$, $\mathbf{CO\text{-}MIN^N}$, *and* $\mathbf{INT^N}$, *respectively, where $\mathbf{Q1}_{\mathrm{R}}$ and $\mathbf{Q2}_{\mathrm{L}}$ are as below:*

$\mathbf{Q1}_{\mathrm{R}}$ $\neg A \supset (A \supset C)$.

$\mathbf{Q2}_{\mathrm{L}}$ $(A \supset \neg A) \supset \neg A$. □

Theorem 5.6. *Let $\boldsymbol{L} \in \{\mathbf{MIN^N}, \mathbf{CO\text{-}MIN^N}, \mathbf{INT^N}\}$.*

(1) *If $A \supset B$ is provable in $\boldsymbol{L}$, then there exists a formula I such that I contains only those propositional letters which occur in both A and B, and such that $A \supset I$ and $I \supset B$ are provable in $\boldsymbol{L}$.*

(2) *If A and B are $\{\sim\}$-less, in addition, then I can be taken from among the $\{\sim\}$-less formulas.*

(3) *If A and B are $\{\top, \bot, \sim\}$-less, and some propositional letters are contained in both A and B, in addition, then I can be taken from among the $\{\top, \bot, \sim\}$-less formulas.* □

Theorem 5.7. *The logics* $\mathbf{MIN^N}$*and* $\mathbf{CO\text{-}MIN^N}$*are not normalizable.* □

In contrast to $\mathbf{MIN^N}$and $\mathbf{CO\text{-}MIN^N}$, the logic $\mathbf{INT^N}$is normalizable.

Theorem 5.8. *The logic* $\mathbf{INT^N}$ *is normalizable.*

Proof.

$$\mathbf{INT^N} = \mathbf{P1} + \mathbf{P2} + \cdots + \mathbf{P8} + \mathbf{Q1}_\mathrm{R} + \mathbf{Q2}_\mathrm{L} + \big({\sim}A \supset (A \supset C)\big) + (\mathrm{MP}).$$

□

Theorem 5.9. *The logic* $\mathbf{CO\text{-}MIN^N}$ *lacks the property* (#). □

In contrast to $\mathbf{CO\text{-}MIN^N}$, the logics $\mathbf{MIN^N}$ and $\mathbf{INT^N}$ enjoy the property (#) (Theorem 5.12).

Lemma 5.10. *Let* $\boldsymbol{L} \in \{\mathbf{MIN^N}, \mathbf{INT^N}\}$. *If the sequent* $A, \Gamma \to \neg\top$ *is provable in* $\mathbf{G}\text{-}\boldsymbol{L}$, *then so too is* $\Gamma \to \neg A$.

Proof. By the following proof.

$$\dfrac{\dfrac{A, \Gamma \to \neg\top \qquad \dfrac{\to \top}{\neg\top \to \neg A}\ (\neg \to)\ [or\ (\neg \to)_\mathrm{R}]}{A, \Gamma \to \neg A}}{\Gamma \to \neg A}\ (\to \neg)_\mathrm{L}$$

□

Lemma 5.11. *Let* $\boldsymbol{L} \in \{\mathbf{MIN^N}, \mathbf{INT^N}\}$. *If* C *contains no propositional letter, then either* C *or* $\neg C$ *is provable in* $\boldsymbol{L}$.

Proof. It suffices to show that either $\to C$ or $C \to \neg\top$ is provable in $\mathbf{G}\text{-}\boldsymbol{L}$; for, in the latter case, $\to \neg C$ is provable by Lemma 5.10. We prove the above claim by induction on the grade of C. Only some typical cases are mentioned.

Case 1: C is a constant or its Nelson negation. The sequents

$$\to \top, \quad \bot \to \neg\top, \quad {\sim}\top \to \neg\top, \quad \textit{and} \quad \to {\sim}\bot$$

are initial sequents, and so are provable.

Case 2: C is $\neg A$. Either $\to A$ or $A \to \neg\top$ is provable by the hypothesis of induction. Hence, either $\neg A \to \neg\top$ or $\to \neg A$ is provable by $(\neg \to)$ [or $(\neg \to)_\mathrm{R}$] or Lemma 5.10, respectively.

Case 3: C is ${\sim}(A \supset B)$. Either $\to A$ or $A \to \neg\top$ is provable, and either $\to {\sim}B$ or ${\sim}B \to \neg\top$ is provable by the hypothesis of induction. If $\to A$ and $\to {\sim}B$ are provable, $\to {\sim}(A \supset B)$ is provable by $(\to {\sim}\supset)$; otherwise, ${\sim}(A \supset B) \to \neg\top$ is provable by $({\sim}\supset\to)$. □

Theorem 5.12. *Let* $\boldsymbol{L} \in \{\mathrm{MIN^N}, \mathrm{INT^N}\}$. *The logic* $\boldsymbol{L}$ *enjoys the property* (#).

Proof. Suppose that $A \supset B$ is provable in $\boldsymbol{L}$, and A and B contain no propositional letter in common. By Theorem 5.6 (1), there exists a formula I such that I contains only those propositional letters which occur in both A and B, and such that $A \supset I$ and $I \supset B$ are provable. It follows by the assumption that I contains no propositional letter; so by Lemma 5.11, either I or $\neg I$ is provable; and so either B or $\neg A$ is provable. □

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