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On the geodesics in the orthonormal frame bundle of a Riemannian manifold

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Abstract

For a given Riemannian manifold (M, g), it is known that the linear frame bundle $P(M)$ of M can be endowed with the Sasaki-Mok metric. In this paper, we shall study the orthonormal frame bundle $F(M)$ of M with the induced metric as a submanifold of $P(M)$. We shall first study the properties of the geodesics in $F(M)$ and then focus on characterization of the geometrical natures of the geodesics in the orthogonal group $O(n+1)$ $(n \geq 2)$ with the standard Riemannian structure induced by the negative Killing form. Finally, we shall present a fairly complete description of the geodesics in $O(4)$, especially about the closed geodesics.

1. Introduction and the statement of the main results

Let (M, g) be an n dimensional Riemannian manifold. The orthonormal frame bundle $F(M)$ of M are defined as follows:

$$F(M) := \{(p; (v_1, v_2, \dots v_n)) : p \in M, v_i \in M_p, \langle v_i, v_j \rangle_M = \delta_{ij}\}.$$

There have been various attempts to introduce Riemannian metrics on $F(M)$. (see [2]-[6], [8]-[11].) In [8], a Sasaki-type metric of the general linear frame bundle $P(M)$ (c.f. the definition (3)) was studied and investigated again later in [3]. It was called Sasaki-Mok in [3] since it has the advantage that the metric resembles closely the Sasaki metric of the tangent bundle $T(M)$ of M (see [12] and [13]). Playing as the counterpart of the unit tangent bundle in the Sasaki metric over the tangent bundle, the orthonormal frame bundle $F(M)$, considered as a submanifold of $P(M)$, is thus endowed with the induced metric ds^2_{SM} that shall also be called Sasaki-Mok metric.

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In this article, we propose to investigate the properties of geodesics in $F(M)$. We shall start with deriving a simple version of the geodesic equation. In order to do so, we shall carry out the calculation under a globally defined vector frame field of $F(M)$ that turns out to be a proper tool for our purpose (c.f. [14]). Our computation should be compared with those in [8], which was based on a so-called adapted frame. The geodesic equation in $F(M)$ tells how the projection curve r of a geodesic $\gamma = (r, (v_1, \cdots, v_n))$, the frame field $(v_1, \cdots v_n)$ along r and the curvatures of underlying manifold M are interweaved together (c.f. Proposition 3.2). We therefore can draw geometrical conclusions about the geodesics in $F(M)$. Let ∇ denote the Levi-Civita connection of (M, g).

Theorem 1.1. *Let $\gamma = (r, (v_1, \cdots, v_n))$ be a curve in $F(M)$.*

1. *If γ is a geodesic in $F(M)$, then $|\dot{r}|, \langle \nabla_{\dot{r}} v_i, v_j \rangle$ $(1 \leq i, j \leq n)$ and $|\nabla_{\dot{r}} v_i|$ are constant.*

2. *The base manifold M has a constant positive sectional curvature $\frac{1}{2}$ if and only if the angles between v_i and $\dot{r}$ are constant.*

Definition 1.2. *The projection curve r of a curve $\gamma \in F(M)$ is called the* base curve *of γ throughout this paper. The notations γ and r are always associated in this way except otherwise mentioned.*

The second part of the above theorem can be considered as a characterization of the standard round spheres $S^n(R)$ in Euclidean space $\mathbb{R}^{n+1}$ since one can properly scale the Sasaki-Mok metric in the vertical component of $F(S^n(R))$ such that the geodesics of $F(S^n(R))$ have the same property as described in the theorem. The case $R = \sqrt{2}$ is special since $(F(S^n(\sqrt{2})), ds^2_{SM})$ is conformal to $O(n+1)$ endowed with the standard metric induced by the negative Killing form (c.f. Proposition 3.1).

Treating $O(n+1)$ as $F(S^n(\sqrt{2}))$ offers us a useful viewpoint to study the geometrical properties of $O(n+1)$. The idea is motivated by Elie Cartan's philosophy that, via the use of moving frames, the theory of Lie groups constitutes a powerful and elegant method for studying uniqueness and existences questions for submanifolds in homogeneous spaces. (see [5]). We believe the opposite is also true, i.e. certain homogeneous spaces could be understood better when they are treated as the frame bundle of a well-understood underlying space. In this article, we shall focus on the application of our general results in the study of geodesics in $O(n+1)$ $(n \geq 2)$. We have

Theorem 1.3. *Let r be a curve in $S^n(\sqrt{2})$. Set $r^{(1)} = \dot{r}$ and inductively define $r^{(m+1)} = \nabla_{\dot{r}} r^{(m)}$ $(m \geq 1)$. Then the followings are equivalent to each other.*

1. *r can be the base curve of a geodesic in $F(S^n(\sqrt{2}))$;*

2. $|r^{(i)}|$ $(1 \leq i)$ *are constant;*

3. $|r^{(i)}|$ $(1 \leq i \leq n)$ *are constant;*

4. There exist constant c_i $(1 \leq i \leq n)$ *and* r *is a solution of the following system:*

$$r^{(n+1)} = \sum_{1 \leq k \leq n} c_k r^{(k)}, \tag{1}$$

together with the initial conditions at $t = 0$*:*

$$\begin{aligned} \sum_{1 \leq k \leq n} c_k \langle r^{(k)}, r^{(n)} \rangle &= 0, \\ \sum_{1 \leq k \leq n} c_k \langle r^{(j)}, r^{(k)} \rangle + \langle r^{(j+1)}, r^{(n)} \rangle &= 0, \quad 1 \leq j \leq n-1, \\ \langle r^{(i+1)}, r^{(j)} \rangle + \langle r^{(i)}, r^{(j+1)} \rangle &= 0, \quad 1 \leq i, j \leq n-1. \end{aligned} \tag{2}$$

Furthermore, if r *meets one of above conditions, then* $\gamma = (r, (v_1, \cdots, v_n))$ *is a geodesic in* $F(S^n(\sqrt{2}))$ *if and only if*

$$\langle \nabla_{\dot{r}} v_i, v_j \rangle = const \qquad 1 \leq i, j \leq n.$$

When M is the 2-sphere $S^2(\sqrt{2})$, the equation (1) is reduced to $\nabla_{\dot{r}} \nabla_{\dot{r}} \dot{r} + \alpha \dot{r} = 0$, which stands for a round circle in the sphere $S^2(\sqrt{2})$ (c.f. the Proposition 5.3). So Theorem 1.3 together with Theorem 1.1 can be considered as the generalization of a result of Klingenberg and Sasaki [7], which can be rewritten as follows:

THEOREM (Klingenberg-Sasaki) *Let* $\gamma = (r(t), (v_1, v_2))$ *be a geodesic in* $O(3)$*. Then* $r(t)$ *is a piece of round circle in* $S^2(1)$ *and the vector fields* v_1 *and* v_2 *along* $r(t)$ *make constant angles with* $r(t)$*.*

As a corollary of Klingenberg and Sasaki theorem, every complete geodesic of $O(3)$ is periodic. Notice that the simplicity of geodesics in $O(3)$ is not surprising since $SO(3)$, the connected component of $O(3)$ containing identity, is covered by S^3. The nature question is how to characterize the periodicity of geodesics in a general orthonormal group $O(n+1)$. It turns out that the problem becomes much more complicated in high dimensional orthonormal groups. We shall give a fairly complete solution to this problem for $O(4)$. More specifically, we shall give a criterion of a closed geodesic in $F(S^3(\sqrt{2}))$ based on its initial conditions (see Theorem 5.1). The advantage of discussion in $F(S^3(\sqrt{2}))$ is due to the existence of globally defined vector frame field over the sphere $S^3(\sqrt{2})$ that can be used to simplify the geodesic equation into a linear system.

We believe that orthonormal frame bundle $F(M)$ are interesting in its own place and is worth further studying. This bundle $F(M)$ and the Riemannian structure investigated in this paper are generically attached to each Riemannian manifold, and therefore should encode some information about the underlying Riemannian manifold M. The existence of the globally defined vector frame field and the geodesic flows determined by them (see [14]) might suggest that $F(M)$ offer us an interesting type of Riemannian manifolds. For some other applications of the frame bundle, see [14]),[15] and [16].

The rest of the paper is organized as follows. The section 2 is used to collect some known results about the frame bundle. Despite that the Levi-Civita connection has been carried out in [8], we shall calculate it again under the well-known globally defined vector frame fields, which turns out to be the right tool for our purpose. In section 3, we shall derive a simple version of geodesic equation in a general frame bundle of a Riemannian manifold endowed with Sasaki-Mok metric (see Proposition 3.2). The further discussion is based on the equations and we expect that the simplicity of the equation would have other applications. The sections 4 and 5 are used to discuss the properties of geodesics of $F(S^n(\sqrt{2}))$ and the closeness of the geodesics in $F(S^3(\sqrt{2}))$ respectively.

2. Preliminaries

In this section, we shall describe the notations and conventions and collect the related formulae we need for further computations. The readers are referred to [1] and [8] for detail. Let (M,g) be an n dimensional Riemannian manifold with the metric tensor $g_{ij}du^i \otimes du^j$ and ∇ be the Levi-Civita connection on M. Let $(u^1, u^2, \cdots, u^n)$ be a local coordinate system. Denote the local vector field $\frac{\partial}{\partial u^i}$ by ∂_i. We shall agree that repeated indices are summed over their range and the indices denoted by low case letter has the range from 1 to n: $1 \leq a, b, c, \cdots, \leq n$. Locally we can express the connection

$$\nabla \partial_i = \omega_i^j \partial_j = \Gamma_{ki}^j du^k \partial_j,$$

where (ω_j^i) be the connection matrix. The curvature operator R is defined as

$$R(\partial_i, \partial_j) = [\nabla_{\partial_i}, \nabla_{\partial_j}] - \nabla_{[\partial_i, \partial_j]}.$$

Let

$$R(\partial_i, \partial_j)\partial_k = R_{ijk}^l \partial_l.$$

The Riemannian curvature tensor is defined by

$$R(X, Y, Z, W) = \langle R(X, Y)W, Z \rangle,$$

and

$$R_{ijkl} = R(\partial_i, \partial_j, \partial_k, \partial_l) = R_{ijl}^h g_{hk}.$$

Let $P(M)$ be the general linear frame bundle over M:

$$P(M) := \{(p; (v_1, v_2, \dots v_n)) : p \in M, v_i \in M_p, v_1 \wedge \cdots \wedge v_n \neq 0\}. \tag{3}$$

If $(u^1, \cdots, u^n)$ are local coordinates, we can locally express an element v in $P(M)$ by

$$v := (v_1, \cdots, v_n) = (\partial_1, \cdots, \partial_n)X$$

with $X = (x^i_j) \in GL(n)$. The inverse matrix X^{-1} shall be denoted as $Y = (y^i_j)$. We define $n(n+1)$ forms on $P(M)$ as follows:

$$\theta^i = y^i_k du^k, \qquad \theta^i_j = y^i_k \delta x^k_j,$$

where $\delta x^k_j = dx^k_j + \omega^k_l x^l_j$ is the covariant differential. It is well known that the above $n(n+1)$ forms are globally defined and form a frame field of the cotangent bundle of $P(M)$ (c.f. [1]). The dual vector fields (e_i, e_{kl}) of (θ^i, θ^k_l), which form a globally defined frame field of the tangent bundle of $P(M)$, can be explicitly expressed as follows:

$$e_i = x^j_i D_j, \quad e_{jk} = x^m_j D_{mk}, \quad 1 \leq i, j, k \leq n.$$

where $D_j = \partial_j - x^q_p \Gamma^h_{jq} \partial / \partial x^h_p$ and $D_{mk} = \partial / \partial x^m_k$.

Remark 1. *Note that D_j are horizontal vectors and D_{mk} are vertical vectors of the standard linear connection on $P(M)$. The frame D_j, D_{mk} was called the adapted frame on $P(M)$ in [8] where the calculation was carried out under this frame.*

For a curve $\gamma(t) = (r(t), (v_1(t), \cdots, v_n(t)))$, we define $\dot\omega^i, \dot\omega^i_j, \dot\theta^i, \dot\theta^i_j$ as follows:

$$\begin{aligned} \dot\omega^i &= \omega^i(\dot r), \quad \dot\omega^i_j = \omega^i_j(\dot r), \\ \dot\theta^i &= \theta^i(\dot\gamma) = y^i_j \dot u^j, \quad \dot\theta^i_j = \theta^i_j(\dot\gamma) = y^i_k \dot x^k_j + y^i_k \dot\omega^k_l x^l_j. \end{aligned}$$

By the definition, we have

$$\dot r = \dot u^i \partial_i = \dot\theta^i v_i, \tag{4}$$

$$\dot\gamma = \dot u^i \partial_i + \dot x^i_j \partial_{ij} = \dot\theta^i e_i + \dot\theta^i_j e_{ij}, \tag{5}$$

$$\dot\theta^i_j = \langle \nabla_{\dot r} v_j, v_i \rangle \quad \text{or} \quad \nabla_{\dot r} v_j = \dot\theta^i_j v_i. \tag{6}$$

In [8], the Sasaki-Mok metric of $P(M)$ is defined as

$$ds^2_{SM} := g_{ij} du^i du^j + g_{ij} \delta x^i_k \delta x^j_k = g_{ij} x^i_k x^j_l (\theta^k \theta^l + \theta^k_h \theta^l_h).$$

One can directly check that the right action by the orthogonal group $O(n)$ is isometric.

Definition 2.1. *Let $\gamma_1 = (r, (v_1, \cdots, v_n))$ and $\gamma_2 = (r, (w_1, \cdots, w_n))$ be two curves with the same base curve r. They are said to be conjugate with each other if there exists $Q \in O(n)$ such that $(w_1, \cdots, w_n) = (v_1, \cdots, v_n)Q$. So conjugate curves are congruent to each other.*

By definition, we have

$$\begin{aligned}\langle e_i, e_j\rangle_{P(M)} &= \langle v_i, v_j\rangle_M, \quad \langle e_i, e_{jk}\rangle_{P(M)} = 0,\\ \langle e_{ij}, e_{kl}\rangle_{P(M)} &= v\langle v_i, v_k\rangle_M \delta_{jl}.\end{aligned}$$

The levi-Civita connection $\tilde{\nabla}$ of Sasaki-Mok metric was computed out under the adapted base (D_i, D_{ij}) in [8] (c.f. the equations (4.12)). For our purpose, we rewrite them under the globally defined frame e_i, e_{ij} as follows:

$$\begin{aligned}\tilde{\nabla}_{e_i} e_j &= -\frac{1}{2} s_{ijkl} W^{kh} e_{hl}, \quad \tilde{\nabla}_{e_{ij}} e_{kl} = \delta_{jk} e_{il},\\ \tilde{\nabla}_{e_i} e_{kl} &= \frac{1}{2} s_{ijkl} W^{jh} e_h, \quad \tilde{\nabla}_{e_{kl}} e_i = \delta_{il} e_k + \frac{1}{2} s_{ijkl} W^{jh} e_h,\end{aligned} \tag{7}$$

where $W^{st} = y_p^s y_q^t g^{pq}$ and $s_{ijkl} = R(v_i, v_j, v_k, v_l)$.

3. The geodesics in the orthonormal frame bundle $F(M)$

We shall consider the orthonormal frame bundle $F(M)$ as a Riemannian submanifold of $P(M)$ with the induced metric. Direct computations show that $\theta^i, \frac{1}{\sqrt{2}}(\theta_l^k - \theta_k^l)$ $(1 \le i \le n, 1 \le k < l \le n)$ form a frame field for the cotangent bundle over $F(M)$ with the dual orthonormal vector frame field $E_i := e_i, E_{kl} := \frac{1}{\sqrt{2}}(e_{kl} - e_{lk})$. Set $E_{ii} = 0$ and $E_{kl} = -E_{lk}$ if $k > l$ for convenience. Let $\bar{\nabla}$ denote the Levi-Civita connection for $F(M)$. Using connection expression (7) under the base (e_i, e_{jk}), one can express the connection $\bar{\nabla}$ under the base E_i, E_{kl} $(1 \le i \le n, 1 \le k < l \le n)$ as follows:

$$\begin{aligned}\bar{\nabla}_{E_i} E_j &= -\frac{\sqrt{2}}{4} s_{ijkl} E_{kl}, \quad \bar{\nabla}_{E_{ij}} E_{kl} = \frac{\sqrt{2}}{4}\delta_{ijlm} E_{mk} + \frac{\sqrt{2}}{4}\delta_{ijmk} E_{ml},\\ \bar{\nabla}_{E_i} E_{kl} &= \frac{\sqrt{2}}{2} s_{ijkl} E_j, \quad \bar{\nabla}_{E_{kl}} E_i = -\frac{\sqrt{2}}{2}\delta_{ijkl} E_j + \frac{\sqrt{2}}{2} s_{ijkl} E_j,\end{aligned} \tag{8}$$

where $\delta_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$. If M is the sphere $S^2(\sqrt{2})$, then $s_{ijkl} = \frac{1}{2}\delta_{ijkl}$. One can check directly that E_i, E_{kl} forms a base of left invariant vector fields and $\sqrt{\frac{2}{n-1}} E_i, \sqrt{\frac{2}{n-1}} E_{kl}$ forms an orthonormal frame field under the negative Killing form. So we have

Proposition 3.1. *$F(S^n(\sqrt{2}))$ is conformal to the orthogonal group $O(n+1)$ endowed with the standard metric induced by negative Killing form.*

We are now ready to derive the geodesic equation of $F(M)$.

Proposition 3.2. *Let (M, g) be a general Riemannian manifold and $F(M)$ be its orthonormal frame bundle endowed the Sasaki-Mok metric. Then a curve $\gamma = (r; (v_1, \cdots, v_n))$ in $F(M)$ is a geodesic if and only if*

$$\begin{aligned} \nabla_{\dot r}\dot r \quad &+ \quad R(\dot r, v_j, v_s, v_t)\dot\theta_t^s v_j = 0, & (9)\\ \dot\theta_j^k \quad &= \quad constant, \quad 1 \le j, k \le n. & (10) \end{aligned}$$

Furthermore, for any point

$$P_0 = (p_0, (v_1, \cdots, v_n)) \in F(M)$$

and initial values

$$T_0 \in TM_{p_0}, \quad \eta_l^k = -\eta_k^l, \qquad 1 \le i \le n, 1 \le k, l \le n,$$

there exists uniquely a geodesic γ of $F(M)$ such that

$$\gamma(0) = P_0, \quad \dot r(0) = T_0, \quad \dot\theta_l^k = \eta_l^k.$$

Proof. By (5), $\dot\gamma = \dot\theta^k E_k + \frac{\sqrt2}{2}\dot\theta_n^t E_{tn}$. We have by (8)

$$\begin{aligned} \bar\nabla_{\dot\gamma}\gamma &= \bar\nabla_{\dot\gamma}(\dot\theta^k E_k + \frac{\sqrt2}{2}\dot\theta_n^t E_{tn}) \\ &= \frac{d\dot\theta^k}{dt}E_k + \frac{\sqrt2}{2}\frac{d\dot\theta_n^t}{dt}E_{tn} + \dot\theta^k\bar\nabla_{\dot\gamma}E_k + \dot\theta_n^t\bar\nabla_{\dot\gamma}E_{tn} \\ &= y_l^k(\ddot u^l + \dot u^j\dot\omega_j^l + x_j^l R(\dot r, v_j, v_s, v_t)\dot\theta_t^s)E_k + \frac{\sqrt2}{2}\frac{d\dot\theta_n^t}{dt}E_{tn}. \end{aligned}$$

Hence $\bar\nabla_{\dot\gamma}\gamma = 0$ if and only if

$$(\ddot u^l + \dot u^j\dot\omega_j^l + x_j^l R(\dot r, v_j, v_s, v_t)\dot\theta_t^s)\partial_l = 0, \quad \frac{d\dot\theta_n^t}{dt} = 0,$$

or equivalently (9) and (10) hold. The rest of the proposition follows from the existence and uniqueness of the geodesic in $F(M)$ with given initial condition. □

Definition 3.3. *We call $(\dot\theta_j^i)$ the* fundamental matrix *of the curve γ. So a geodesic has a constant fundamental matrix.*

The proof of Theorem 1.1.

By Proposition 3.2 and (6), we see that $\langle\nabla_{\dot r}v_i, v_j\rangle$ are constant. A simple calculation shows that (9) is equivalent to

$$\nabla_{\dot r}\dot r = \langle v_i, \nabla_{\dot r}v_j\rangle R(v_i, v_j)\dot r.$$

which implies $\dot{r}\langle\dot{r},\dot{r}\rangle = 0$. We may assume that $|\dot{r}(t)| \equiv c \neq 0$. So by use of (9), (4) and (6) we have

$$\begin{aligned}\dot{r}\langle\dot{r}, v_i\rangle_M &= -R(\dot{r}, v_i, v_s, v_t)\dot{\theta}^s_t + \langle\dot{r}, \nabla_{\dot{r}} v_i\rangle \\ &= -R(v_j, v_i, v_s, v_t)\dot{\theta}^j\dot{\theta}^s_t + \dot{\theta}^j\dot{\theta}^j_i \\ &= \dot{\theta}^j\dot{\theta}^s_t(\frac{1}{2}\delta_{jist} - R(v_j, v_i, v_s, v_t)).\end{aligned}$$

Taking any point as the initial point $r(0)$, since the initial values

$$\dot{\theta}^s_t, \dot{\theta}^j(0), \qquad 1 \leq s < t, j \leq n$$

can be arbitrarily chosen, the theorem follows. □

4. The geodesics in $O(n+1)$

The aim of this section is to prove Theorem 1.3.

The proof of Theorem 1.3.

"$1 \Rightarrow 2$". Assuming that r is the base curve of a geodesic γ. By Theorem 1.1(2), we can express $\dot{r} = \sum_{1\leq i\leq n} c_i v_i$ as a linear combination of v_i with constant coefficient c_i, and also we can express

$$r^{(2)} = \sum_i c_i \nabla_{\dot{r}} v_i = \sum_i \dot{\theta}^j_i v_j$$

as the combination of v_i. This implies that all $|r^{(2)}|$ are constant because $\dot{\theta}^i_j$ are constant. One can inductively show that all $|r^{(i)}|$ $(i \geq 1)$ are constant.

"$2 \Rightarrow 3$" is trivial.

"$3 \Rightarrow 4$". Assuming that all $|r^{(i)}|$ $(1 \leq i \leq n)$ are constant. Without lose of generality, we may assume that $|\dot{r}| = 1$. Since

$$\langle r^{(i)}, r^{(j)}\rangle = \dot{r}\langle r^{(i-1)}, r^{(j)}\rangle - \langle r^{(i-1)}, r^{(j+1)}\rangle,$$

by induction on $|i-j|$, it is easy to see that all $\langle r^{(i)}, r^{(j)}\rangle$ $(1 \leq i, j \leq n)$ are constant. Therefore for any linear combination $W = \sum_i c_i r^{(i)}$ with constant coefficient c_i, we have

$$|W| \equiv 0 \quad \textit{or} \quad |W| \neq 0. \tag{11}$$

Note that $\langle r^{(n+1)}, r^{(i)}\rangle = -\langle r^{(n)}, r^{(i+1)}\rangle = \textit{const}$ $(i \leq n-1)$ and $\langle r^{(n+1)}, r^{(n)}\rangle = 0$. By (11), it is easy to see that $r^{(n+1)}$ is a linear combination of $r^{(1)}, \cdots, r^{(n)}$ whether

$r^{(1)}, \cdots, r^{(n)}$ are linearly independent or not. The initial conditions in (2) correspond to the following equations at $t = 0$:

$$\begin{aligned}
\dot{r}\langle r^{(n)}, r^{(n)}\rangle &= 0, \qquad \dot{r}\langle r^{(j)}, r^{(n)}\rangle = 0, \quad 1 \leq j \leq n-1, \\
\dot{r}\langle r^{(i)}, r^{(j)}\rangle &= 0, \quad 1 \leq i, j \leq n-1.
\end{aligned}$$

"4 $\Rightarrow$ 1". Assuming that r is the solution of the equation (1) with the initial condition (2). Set $f_{ij} = \langle r^{(i)}, r^{(j)}\rangle$ $(i \leq j \leq n)$. We have

$$\begin{aligned}
\dot{f}_{ij} &= f_{i+1,j} + f_{i,j+1}, \qquad 1 \leq i, j \leq n-1, \\
\dot{f}_{i,n} &= f_{i+1,n} + \sum_j c_j f_{i,j}, \quad i < n, \\
\dot{f}_{n,n} &= 2\sum_j c_j f_{j,n}.
\end{aligned}$$

The second order linear system about $\ddot{f}_{ij}$ has the initial value $\dot{f}_{ij}(0) = 0$ by (2), which implies that f_{ij} are constant functions and therefore $|r^{(i)}|$ $(1 \leq i \leq n)$ are constant. It is not hard to see that there exist $r^{(1)}, \cdots, r^{(s)}$ that they are linearly independent but $r^{(1)}, \cdots, r^{(s)}, r^{(s+1)}$ are linearly dependent with $1 \leq s \leq n$. Let $v_1, \cdots, v_s$ be orthonormal vector fields obtained by the Schmidt's orthogonalization of $r^{(1)}, r^{(2)} \cdots$. If $s < n$, let $v_{s+1}, \cdots, v_n$ be parallel vector fields along r such that they are orthonormal and perpendicular to $r^{(1)}, \cdots, r^{(s)}$ at $t = 0$. For any fixed $1 + s \leq k \leq n$, we put $f_j = \langle v_k, r^{(j)}\rangle$ $(1 \leq j \leq s)$. It is clear that f_j are the solution of the following linear system:

$$\dot{f}_s = \sum_{1 \leq i \leq s} \alpha_i f_i, \qquad \dot{f}_i = f_{i+1}, \quad 1 \leq i < s,$$

with the constant coefficients α_i and the initial condition $f_j(0) = 0$. Therefore by the uniqueness of the solution, we have $f_j \equiv 0$, which implies that $v_{s+1}, \cdots, v_n$ are all perpendicular to $r^{(j)}$ and therefore perpendicular to $v_1, \cdots, v_s$. Hence $\gamma = (r, (v_1, \cdots, v_n))$ is a curve in $F(M)$. Note that $v_1 = \dot{r}$, so we have

$$\nabla_{\dot{r}}\dot{r} + R(\dot{r}, v_j, v_s, v_t)\dot{\theta}^s_t v_j = r^{(2)} + \dot{\theta}^1_j v_j = r^{(2)} - \langle r^{(2)}, v_j\rangle v_j = 0$$

and so γ satisfies the equation (9). It is easy to see (10) holds too, which implies that γ is a geodesic in $F(M)$. The rest of the assertions follows directly from Proposition 3.2. $\square$

5. The geodesics in $O(4)$

Let us first characterize the geodesics of $F(S^3(\sqrt{2}))$. Recall that a curve r is called *normal* if $|\dot{r}| \equiv 1$.

Theorem 5.1. *Let $r(t)$ be a curve in $S^3(\sqrt{2})$.*

1. *A curve $r(t)$ can be a normal base curve of a geodesic γ in $F(S^3(\sqrt{2}))$ if and only if it is a solution of the following system.*

$$\begin{aligned} 0 &= \nabla_{\dot r}\nabla_{\dot r}\dot r + \alpha\dot r + \beta\dot r \times \nabla_{\dot r}\dot r, && (12)\\ \langle \dot r, \dot r\rangle|_{t=0} &= 1, \quad \langle \nabla_{\dot r}\dot r, \nabla_{\dot r}\dot r\rangle_{t=0} = \alpha, \quad \langle \nabla_{\dot r}\dot r, \dot r\rangle_{t=0} = 0, && (13)\end{aligned}$$

where $\alpha \geq 0, \beta$ are constant and β is set to be 0 when $\alpha = 0$. Furthermore, for such a normal base curve r of a geodesic in $F(M)$, choosing $e_1 = \dot r$, $e_2 = \frac{1}{\sqrt{\alpha}}\nabla_{\dot r}\dot r$ if $\alpha \neq 0$, otherwise e_2 being a unit parallel vector field along r and perpendicular to $\dot r$, and $e_3 = e_1 \times e_2$, we see $\Gamma = (r, (e_1, e_2, e_3))$ is a geodesic of $F(S^3(\sqrt{2}))$. We shall call this a generic geodesic *of the base curve r.*

2. *Any two normal base curves of geodesics in $F(S^3(\sqrt{2}))$ that meets the equation (12) and (13) can be transformed by an orthonormal transformation of $S^3(\sqrt{2})$ to each other.*

3. *Let $r \in S^3(\sqrt{2})$ meet (12) and (13). A curve $\gamma = (r, (v_1, v_2, v_3))$ is a geodesic if and only if there exists $Q \in O(3)$ such that*

$$(v_1(t), v_2(t), v_3(t))Q = (e_1, e_2, e_3)\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_0 t & -\sin\theta_0 t \\ 0 & \sin\theta_0 t & \cos\theta_0 t \end{pmatrix}, \qquad (14)$$

where e_1, e_2, e_3 are defined as above and θ_0 is a constant.

Proof.

1. First, assume that r is a normal base curve of a geodesic $\gamma = (r, (v_1, v_2, v_3))$ in $F(M)$. By Theorem 1.3, $|\nabla_{\dot r}\dot r| = \sqrt{\alpha}$ is a constant number. When $\alpha = 0$, the curve r itself is a geodesic and the equation (12) and (13) hold for arbitrary β. We set β to be 0 in the case $\alpha = 0$. We now assume $\alpha > 0$. According to Theorem 1.1, we see $\langle \dot r, v_1\rangle$ is constant and therefore, replacing γ by a proper conjugate geodesic of γ if necessary, we may assume $v_1 = \dot r$. Similarly, since $\nabla_{\dot r}\dot r$ makes constant angles with v_1 and v_2, we may assume $v_2 = \frac{1}{\sqrt{\alpha}}\nabla_{\dot r}\dot r$ and

$$v_3 = \frac{1}{\sqrt{\alpha}}\dot r \times \nabla_{\dot r}\dot r.$$

We have

$$\begin{aligned} \nabla_{\dot r}\nabla_{\dot r}\dot r &= \sqrt{\alpha}\nabla_{\dot r}v_2 = \sqrt{\alpha}\langle \nabla_{\dot r}v_2, v_1\rangle v_1 + \sqrt{\alpha}\langle \nabla_{\dot r}v_2, v_3\rangle v_3 \\ &= -\sqrt{\alpha}\langle v_2, \nabla_{\dot r}v_1\rangle v_1 + \sqrt{\alpha}\langle \nabla_{\dot r}v_2, v_3\rangle v_3 \\ &= -\alpha\dot r - \beta\dot r \times \nabla_{\dot r}\dot r, \end{aligned}$$

where $\beta = \dot{\theta}_3^2$ is constant, which leads us to (12). The initial condition (13) is clearly satisfied.

To prove the other direction, we assume that r satisfies (12) and (13). Set

$$f(t) = \langle \dot{r}, \dot{r} \rangle, \qquad g(t) = \langle \nabla_{\dot{r}} \dot{r}, \nabla_{\dot{r}} \dot{r} \rangle.$$

By (12) we have

$$\frac{1}{2} f'' - g + \alpha f = 0, \qquad\qquad g' + \alpha f' = 0. \tag{15}$$

Solving the above equations, we obtain

$$f = a \cos \sqrt{\alpha} t + b \sin \sqrt{\alpha} t + c, \quad g = -\alpha f + d,$$

where a, b, c, d are constant. By the initial values $f(0) = 1$, $g(0) = \alpha$ and (15), we have $a = 0$. Together with $f'(0) = 0$, we find f and g must be constant. By (13), we have $\langle \nabla_{\dot{r}} \dot{r}, \nabla_{\dot{r}} \dot{r} \rangle \equiv \alpha$. For the case $\alpha = 0$ it is easy to verify that Γ is a geodesic. We shall assume $\alpha > 0$. Then $\Gamma = (r, (e_1, e_2, e_3))$ defined in Theorem 5.1 is a curve in $F(M)$. By definition, we have

$$\dot{\theta}_1^2 = \sqrt{\alpha}, \quad \dot{\theta}_1^3 = 0, \quad \dot{\theta}_2^3 = -\beta.$$

The equation (9) is now reduced to $\nabla_{\dot{r}} \dot{r} - \sqrt{\alpha} v_2 = 0$, which holds now according to the choice of v_2 of Γ, therefore Γ is a geodesic.

2. Let r_1 and r_2 meet the same equations (12) and (13). The conclusion is clear for the case $\alpha = 0$. When $\alpha \neq 0$, there is unique transformation T of $\mathbb{R}^4$ such that it maps the orthonormal frame

$$(r_1, (\dot{r}_1, \frac{1}{\sqrt{\alpha}} \nabla_{\dot{r}_1} \dot{r}_1, \dot{r}_1 \times \frac{1}{\sqrt{\alpha}} \nabla_{\dot{r}_1} \dot{r}_1))|_{t=0}$$

to the orthonormal frame

$$(r_2, (\dot{r}_2, \frac{1}{\sqrt{\alpha}} \nabla_{\dot{r}_2} \dot{r}_2, \dot{r}_2 \times \frac{1}{\sqrt{\alpha}} \nabla_{\dot{r}_2} \dot{r}_2))|_{t=0}.$$

The conclusion follows since both $T(r_1)$ and r_2 meet the equation (12) and have the same initial conditions on the initial position, first and second derivatives.

3. Let $\gamma = (r, (v_1, v_2, v_3))$ be a geodesic in $F(S^3(\sqrt{2}))$. With a right action induced by a proper matrix in $O(3)$, we can assume that $v_1 = \dot{r}$, $v_2(0) = e_2(0)$ and $v_3(0) = e_3(0)$. So there is a function $\theta = \theta(t)$ such that $\theta(0) = 0$ and

$$v_2 = \cos\theta e_2 + \sin\theta e_3, \quad v_3 = -\sin\theta e_2 + \cos\theta e_3.$$

By the definition, we have

$$\begin{aligned} \dot{\theta}_2^3 &= \langle \nabla_{\dot{r}} (\cos\theta e_2 + \sin\theta e_3), -\sin\theta e_2 + \cos\theta e_3 \rangle \\ &= \dot{\theta} - \beta \end{aligned} \tag{16}$$

and $\theta = (\dot{\theta}_2^3 + \beta)t$, which leads to (14).

We now complete the proof of Theorem 5.1. $\square$

The following lemma simplifies the equation (12) and initial condition (13) into a linear system.

Lemma 5.2. *Let $r(t) = (u_1, u_2, u_3, u_4)$ be a curve in $\mathbb{R}^4$. Then it can be a normal base curve of a geodesic in $F(S^3(\sqrt{2}))$ if and only if it is a solution of the system:*

$$\begin{aligned} (\dot{u}_1, \dot{u}_2, \dot{u}_3, \dot{u}_4)^T &= \frac{1}{\sqrt{2}} Y (u_1, u_2, u_3, u_4)^T \qquad (17) \\ \sum_{1 \le i \le 4} u_i^2|_{t=0} &= 2, \qquad (18) \end{aligned}$$

where

$$Y = \begin{pmatrix} 0 & -y_1 & -y_2 & -y_3 \\ y_1 & 0 & y_3 & -y_2 \\ y_2 & -y_3 & 0 & y_1 \\ y_3 & y_2 & -y_1 & 0 \end{pmatrix}$$

and y_1, y_2, y_3 is a solution of the system:

$$\begin{aligned} \ddot{y}_1 + (\beta + \frac{1}{\sqrt{2}})(y_2\dot{y}_3 - y_3\dot{y}_2) + \alpha y_1 &= 0, \\ \ddot{y}_2 + (\beta + \frac{1}{\sqrt{2}})(y_3\dot{y}_1 - y_1\dot{y}_3) + \alpha y_2 &= 0, \qquad (19) \\ \ddot{y}_3 + (\beta + \frac{1}{\sqrt{2}})(y_1\dot{y}_2 - y_2\dot{y}_1) + \alpha y_3 &= 0, \end{aligned}$$

with the initial conditions:

$$\sum_{1 \le i \le 3} y_i^2(0) = 1, \quad \sum_{1 \le i \le 3} \dot{y}_i^2(0) = \alpha, \quad \sum_{1 \le i \le 3} y_i \dot{y}_i|_{t=0} = 0. \qquad (20)$$

Proof. Let $e = (e_1, e_2, e_3, e_4)$ be the standard frame in $\mathbb{R}^4$ and set

$$U = \begin{pmatrix} -u_2 & -u_3 & -u_4 & u_1 \\ u_1 & -u_4 & u_3 & u_2 \\ u_4 & u_1 & -u_2 & u_3 \\ -u_3 & u_2 & u_1 & u_4 \end{pmatrix}$$

and $W = (W_1, W_2, W_3, W_4) := \frac{1}{\sqrt{2}} eU$ that forms a frame field defined on $\mathbb{R}^4 - \{(0,0,0,0)\}$. It is easy to see that $UU^T = \sum_{i \le 4} u_i^2 E$ where E is the identity matrix. Let $\tilde{\nabla}$ be the standard Levi-Civita connection of $\mathbb{R}^4$. Direct computation shows

$$\tilde{\nabla}_{W_i} W_j = \frac{1}{\sqrt{2}} \tau(i,j,k) W_k, \quad \tilde{\nabla}_{W_i} W_i = -\frac{1}{\sqrt{2}} W_4, \qquad 1 \le i, j, k \le 3,$$

where $\tau(i,j,k)$ is 1 or -1 corresponding whether (i,j,k) is an even or odd permutation of $(1,2,3)$. If we assume that $r=(u_1,u_2,u_3,u_4)$ is a curve in $S^3(\sqrt{2})$, there exist y_1,y_2,y_3 such that

$$\dot{r}=\sum_{1\le i\le 3} y_iW_i=\frac{1}{\sqrt{2}}eU\begin{pmatrix} y_1\\ y_2\\ y_3\\ 0\end{pmatrix}. \tag{21}$$

So

$$\begin{aligned}\nabla_{\dot{r}}\dot{r} &= \sum_{1\le i\le 3}\dot{y}_iW_i+\sum_{1\le i\le 3}y_i\nabla_{\dot{r}}W_i\\ &= \sum_{1\le i\le 3}\dot{y}_iW_i+\frac{1}{\sqrt{2}}\sum_{1\le i,j,k\le 3}\tau(i,j,k)y_iy_jW_k=\sum_{1\le i\le 3}\dot{y}_iW_i.\end{aligned}$$

Hence

$$\begin{aligned}\nabla_{\dot{r}}\nabla_{\dot{r}}\dot{r} &= (\ddot{y}_1+\frac{1}{\sqrt{2}}(y_2\dot{y}_3-y_3\dot{y}_2))W_1+(\ddot{y}_2+\frac{1}{\sqrt{2}}(y_3\dot{y}_1-y_1\dot{y}_3))W_2\\ &+ (\ddot{y}_3+\frac{1}{\sqrt{2}}(y_1\dot{y}_2-y_2\dot{y}_1))W_3. \end{aligned}\tag{22}$$

To prove the lemma, first assuming that r is a normal base curve of a geodesic in $F(S^3(\sqrt{2}))$, we find (22) and (12) imply (19). It is easy to see that (21) implies (17) and the initial conditions (20), and it is also easy to check the initial condition (18).

To prove the "if" part of the theorem, let (y_1,y_2,y_3) be a solution of (19) and (20) and assume that $r(t)=(u_1,u_2,u_3,u_4)$ is a solution of (17) and (18). Set

$$f=\sum_{1\le i\le 3}y_i^2,\qquad g=\sum_{1\le i\le 3}\dot{y}_i^2.$$

and the initial condition (20) turns to $g(0)=\alpha f(0)$ and $\dot{f}(0)=0$. By multiplying y_i on the i-th equation in (19) and then summing them up, we have $\frac{1}{2}\ddot{f}-g+\alpha f=0$. Repeating the same trick by multiplying $\dot{y}_i$, we obtain $\dot{g}+\alpha\dot{f}=0$. By the same argument as in the proof of Theorem 5.1, f and g must be constant. We rewrite (17) as

$$\begin{pmatrix} y_1\\ y_2\\ y_3\\ 0\end{pmatrix}=\frac{\sqrt{2}}{\sum_{i\le 4}u_i^2}U^T\begin{pmatrix}\dot{u}_1\\ \dot{u}_2\\ \dot{u}_3\\ \dot{u}_4\end{pmatrix}, \tag{23}$$

which implies $\sum_{i\le 4}u_i\dot{u}_i=0$ and so $r(t)\in S^3(\sqrt{2})$. We have

$$\dot{r}=e(\dot{u}_1,\dot{u}_2,\dot{u}_3,\dot{u}_4)^T=\frac{1}{\sqrt{2}}(W_1,W_2,W_3,W_4)U^T(\dot{u}_1,\dot{u}_2,\dot{u}_3,\dot{u}_4)^T. \tag{24}$$

From (24) and (23), we have

$$\dot{r} = y_1 W_1 + y_2 W_2 + y_3 W_3.$$

Following the same computation as in the first part of the proof, one can check that r satisfies the equation (12) and the initial condition (13). $\square$

For our purpose, we need characterize the round circle in a sphere. The following result can be proved by the uniqueness of the solution of a differential equation with a given initial condition (see [9]).

Proposition 5.3. *Take three vectors $v, w, z \in \mathbb{R}^n$ $(n \geq 3)$ such that*

$$v \cdot w = w \cdot z = z \cdot v = 0, \quad |w| = 1, \quad |v| = R > 0, \tag{25}$$

and set $\alpha = |z|^2 \geq 0$. Let ∇ be the Levi-Civita connection on the sphere $S^{n-1}(R)$ with radius R in $\mathbb{R}^n$. Then the unique solution of the following system

$$\begin{aligned} 0 &= \nabla_{\dot{r}} \nabla_{\dot{r}} \dot{r} + \alpha \dot{r}, & (26) \\ r(0) &= v, \quad \dot{r}(0) = w, \quad \nabla_{\dot{r}} \dot{r}|_{t=0} = z & (27) \end{aligned}$$

is the normal circle in $S^{n-1}(R)$ intersected by the 2-dimensional plane generated by w and $z - \frac{1}{R^2} v$ with the initial position v and the initial tangent vector w. Furthermore, the radius of the circle is equal to $\frac{1}{\sqrt{\alpha + (1/R^2)}}$.

For the further discussion of the closeness of geodesics in $F(S^3(\sqrt{2}))$, we need prove the following result about the closeness of solutions of a linear system, which might be interesting in its own place.

Theorem 5.4. *Let $\alpha \geq 0$ and β be two constant. Let (y_1, y_2, y_3) be a solution of the equation (19) and initial condition (20) and (u_1, u_2, u_3, u_4) be a solution of the system (17) and (18). Then*

1. *The solution (y_1, y_2, y_3) stands for a round circle in $S^2(1) \subset \mathbb{R}^3$ with the period $2\pi/\sqrt{b}$.*

2. *The solution (u_1, u_2, u_3, u_4) is periodic if and only if the number*

$$I_{\alpha,\beta} = \frac{\sqrt{\alpha + (\beta - \frac{1}{\sqrt{2}})^2}}{\sqrt{\alpha + (\beta + \frac{1}{\sqrt{2}})^2}} \tag{28}$$

is rational. Furthermore if we denote the number $I_{\alpha,\beta}$ by $2\frac{m}{n} + 1$ with primitive integers m, n, then the minimal period is $2\pi n/\sqrt{b}$.

Proof. In order to prove the first part, let $\mu(t) = (y_1, y_2, y_3)$ be a solution of (19) and (20). The equation (19) can be written as

$$\frac{d^2\mu}{dt^2} + (\beta + \frac{1}{\sqrt{2}})\mu \times \dot{\mu} + \alpha\mu = 0. \tag{29}$$

By the same argument in the proof of Lemma 5.2, $\sum_i y_i^2$ and $\sum_i \dot{y}_i^2$ are constant. Therefore μ is a curve in $S^2(1)$. Let $\tilde{\nabla}$ be the Riemannian connection of $S^2(1)$. We have by (29)

$$\tilde{\nabla}_{\dot{\mu}}\dot{\mu} + (\beta + \frac{1}{\sqrt{2}})\mu \times \dot{\mu} = 0,$$

hence

$$\tilde{\nabla}_{\dot{\mu}}\tilde{\nabla}_{\dot{\mu}}\dot{\mu} + (\beta + \frac{1}{\sqrt{2}})(\mu \times \frac{d^2\mu}{dt^2})^T = \tilde{\nabla}_{\dot{\mu}}\tilde{\nabla}_{\dot{\mu}}\dot{\mu} + (\beta + \frac{1}{\sqrt{2}})^2\dot{\mu} = 0.$$

By Proposition 5.3, μ is a circle. Set $\mu_1(t) = \mu(t/\sqrt{\alpha})$. It satisfies the equation

$$\tilde{\nabla}_{\dot{\mu}_1}\tilde{\nabla}_{\dot{\mu}_1}\dot{\mu}_1 + \frac{(\beta + \frac{1}{\sqrt{2}})^2}{\alpha}\dot{\mu}_1 = 0,$$

and the initial conditions (25) and (27) in the Proposition 5.3. So the radius of the round circle is $\sqrt{\frac{\alpha}{b}}$ and the period of μ is $2\pi\sqrt{\frac{\alpha}{b}}/|\dot{\mu}| = 2\pi/\sqrt{b}$.

To prove the second part, we note that, by Theorem 5.1(2) and Lemma 5.2, the solutions (u_1, u_2, u_3, u_4) of the system (19)-(18) can be transformed by orthonormal transformations to each other as long as the pair (α, β) in (19) are same. Fortunately the system (17) and (18) can be solved for each $(\alpha \geq 0, \beta)$ if y_1, y_2, y_3 are chosen as follows:

$$y_1 = c\cos\sqrt{b}t, \quad y_2 = c\sin\sqrt{b}t, \quad y_3 = \pm\sqrt{1-c^2}, \tag{30}$$

where $c^2 = \frac{\alpha}{b}$, b is defined as

$$b = (\beta + \frac{1}{\sqrt{2}})^2 + \alpha, \tag{31}$$

and the signature in the expression about y_3 corresponds $+$ or $-$ depending whether $\beta + \frac{1}{\sqrt{2}}$ is negative or nonnegative. We need the following lemma.

Lemma 5.5. *With y_1, y_2, y_3 taken as in (30), a solution (u_1, u_2, u_3, u_4) of (17) satisfies the following linear equation with constant coefficients.*

$$F(D)u_i = 0, \qquad 1 \leq i \leq 4 \tag{32}$$

where $D = d/dt$ and

$$\begin{aligned} F(\lambda) \ &:= \ \left((\lambda^2 + \frac{1}{2})^2 - b(\lambda^2 + \frac{1}{2}) + \frac{\alpha}{2}\right)^2 \\ &+ \ 4b(\lambda^2 + \frac{1}{2})^2\lambda^2. \end{aligned}$$

Proof. In fact, differentiating the equation (17) and noticing that $YY = -E$, we have

$$(\ddot{u}_1, \ddot{u}_2, \ddot{u}_3, \ddot{u}_4)^T = (\frac{1}{\sqrt{2}}\dot{Y} - \frac{1}{2}E)(u_1, u_2, u_3, u_4)^T. \tag{33}$$

Set

$$A := \begin{pmatrix} \sin\sqrt{b}t & \cos\sqrt{b}t \\ -\cos\sqrt{b}t & \sin\sqrt{b}t \end{pmatrix}, \qquad B := \begin{pmatrix} \cos\sqrt{b}t & \sin\sqrt{b}t \\ -\sin\sqrt{b}t & \cos\sqrt{b}t \end{pmatrix}.$$

Then the equation (33) can be rewritten as

$$\begin{pmatrix} \ddot{u}_1 + \frac{1}{2}u_1 \\ \ddot{u}_4 + \frac{1}{2}u_4 \end{pmatrix} = \sqrt{\frac{\alpha}{2}}A^T \begin{pmatrix} u_2 \\ u_3 \end{pmatrix}, \tag{34}$$

$$\begin{pmatrix} \ddot{u}_2 + \frac{1}{2}u_2 \\ \ddot{u}_3 + \frac{1}{2}u_3 \end{pmatrix} = -\sqrt{\frac{\alpha}{2}}A \begin{pmatrix} u_1 \\ u_4 \end{pmatrix}. \tag{35}$$

By (34), we have

$$\begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \sqrt{\frac{2}{\alpha}}A \begin{pmatrix} \ddot{u}_1 + \frac{1}{2}u_1 \\ \ddot{u}_4 + \frac{1}{2}u_4 \end{pmatrix}. \tag{36}$$

Differentiating (36) twice, we have

$$\begin{aligned} \begin{pmatrix} \ddot{u}_2 \\ \ddot{u}_3 \end{pmatrix} &= \frac{\sqrt{2}\sqrt{b}}{c}A \begin{pmatrix} \ddot{u}_1 + \frac{1}{2}u_1 \\ \ddot{u}_4 + \frac{1}{2}u_4 \end{pmatrix} - \frac{2\sqrt{2}}{c}B \begin{pmatrix} u_1^{(3)} + \frac{1}{2}\dot{u}_1 \\ u_4^{(3)} + \frac{1}{2}\dot{u}_4 \end{pmatrix} \\ &+ \sqrt{\frac{2}{\alpha}}A \begin{pmatrix} u_1^{(4)} + \frac{1}{2}\ddot{u}_1 \\ u_4^{(4)} + \frac{1}{2}\ddot{u}_4 \end{pmatrix}. \end{aligned} \tag{37}$$

Substituting (36) and (37) into (35), we have

$$f(D)u_1 + g(D)u_4 = 0, \qquad f(D)u_4 - g(D)u_1 = 0$$

where

$$\begin{aligned} f(\lambda) &= (\lambda^2 + \frac{1}{2})^2 - b(\lambda^2 + \frac{1}{2}) + \frac{1}{2}\alpha, \\ g(\lambda) &= 2\sqrt{b}\lambda(\lambda^2 + \frac{1}{2}). \end{aligned}$$

Similarly, we have

$$f(D)u_2 - g(D)u_3 = 0, \qquad f(D)u_3 + g(D)u_2 = 0.$$

So each u_i $(1 \leq i \leq 4)$ satisfies the equation (32). $\square$

To continue the proof of the theorem, we need solve the equation $F(\lambda) = 0$. Fortunately, the roots of the equation $F(\lambda) = 0$ have the simple format $\pm\sqrt{-\lambda_i}$ $(1 \leq i \leq 4)$ where

$$\begin{aligned} \lambda_1 &= \frac{1}{2}(h + b - \sqrt{b^2 + 2bh}), \quad \lambda_2 = \frac{1}{2}(h + b + \sqrt{b^2 + 2bh}) \\ \lambda_3 &= \frac{1}{2}(s + b - \sqrt{b^2 + 2bs}), \quad \lambda_4 = \frac{1}{2}(s + b + \sqrt{b^2 + 2bs}) \end{aligned} \tag{38}$$

where $s = 1 + |\sqrt{2}\beta + 1|$ and $h = 1 - |\sqrt{2}\beta + 1|$ [1]. Note that all λ_i are nonnegative. Thus one can find a solution of the equation (17) and initial condition (18) for each pair $\alpha \geq 0, \beta$ as follows:

1. If $\beta \leq -\frac{1}{\sqrt{2}}$, then

$$r = (-p \sin\sqrt{\lambda_3}t, q\cos\sqrt{\lambda_4}t, q\sin\sqrt{\lambda_4}t, p\cos\sqrt{\lambda_3}t). \tag{39}$$

   where

$$q = \frac{cp}{\sqrt{2\lambda_4} + \sqrt{1 - c^2}}, \quad p^2 + q^2 = 2.$$

2. If $-\frac{1}{\sqrt{2}} \leq \beta \leq 0$, then

$$r = (-p \sin\sqrt{\lambda_1}t, q\cos\sqrt{\lambda_2}t, q\sin\sqrt{\lambda_2}t, p\cos\sqrt{\lambda_1}t), \tag{40}$$

   where

$$q = \frac{cp}{\sqrt{2\lambda_2} - \sqrt{1 - c^2}}, \quad p^2 + q^2 = 2.$$

3. If $\beta \geq 0$, then

$$r = (p \sin\sqrt{\lambda_1}t, q\cos\sqrt{\lambda_2}t, q\sin\sqrt{\lambda_2}t, p\cos\sqrt{\lambda_1}t), \tag{41}$$

   where

$$q = \frac{cp}{\sqrt{2\lambda_2} - \sqrt{1 - c^2}}, \quad p^2 + q^2 = 2.$$

When $\alpha = 0$, these constant p and q in (39) and (40) should be considered to be 0 and $\sqrt{2}$ respectively. For the verification, the following two sets of algebraic identities are needed.

$$\sqrt{\lambda_4} - \sqrt{\lambda_3} = \sqrt{b}, \tag{42}$$

$$\sqrt{\lambda_2} - \sqrt{\lambda_1} = \sqrt{b}, \quad \text{if } h \geq 0 \tag{43}$$

$$\sqrt{\lambda_2} + \sqrt{\lambda_1} = \sqrt{b}, \quad \text{if } h \leq 0, \tag{44}$$

[1]Thanks to my student Hu Jie for helping me to solve the equation

and

$$\begin{aligned}
(\sqrt{2\lambda_3}-\sqrt{1-c^2})(\sqrt{2\lambda_4}+\sqrt{1-c^2}) &= c^2, \quad \text{if } \beta \le -\tfrac{1}{\sqrt{2}} \\
(\sqrt{2\lambda_1}+\sqrt{1-c^2})(\sqrt{2\lambda_2}-\sqrt{1-c^2}) &= c^2 \quad \text{if } -\tfrac{1}{\sqrt{2}} \le \beta \le 0 \\
(\sqrt{2\lambda_1}-\sqrt{1-c^2})(\sqrt{2\lambda_2}-\sqrt{1-c^2}) &= -c^2 \quad \text{if } \beta \ge 0
\end{aligned}$$

When $\beta \le -\frac{1}{\sqrt{2}}$, it is clear that the solution (39) is periodic if and only if $\sqrt{\frac{\lambda_3}{\lambda_4}}$ is a rational number, which is equivalent to $\sqrt{\frac{\lambda_4}{b}}$ (and therefore $\sqrt{\frac{\lambda_3}{b}}$) is a rational number according to (42). Note that the identity $m^2 = \frac{\lambda_4}{b}$ for some $m > 1$ is equivalent to $\frac{s}{2b} = m(m \pm 1)$, that is same as

$$I^2_{\alpha,\beta} = \frac{\alpha + (\beta - \frac{1}{\sqrt{2}})^2}{\alpha + (\beta + \frac{1}{\sqrt{2}})^2} = (2m \pm 1)^2.$$

So the solution of (39) is periodic if and only if $I_{\alpha,\beta}$ is a rational number. Now if $I_{\alpha,\beta} = 2\frac{m}{n} + 1$ and m, n are relatively prime integers, by the equation (42), one can check that $(\sqrt{\lambda_4}, \sqrt{\lambda_3})$ is equal to $(\frac{m}{n}\sqrt{b}, \frac{m-n}{n}\sqrt{b})$ or $(\frac{m+n}{n}\sqrt{b}, \frac{m}{n}\sqrt{b})$. It is easy to see that the minimal period of the solution (39) is $\frac{2\pi n}{\sqrt{b}}$. Similarly, one can prove the other two cases. □

Note that every geodesic in $F(S^3(\sqrt{2}))$ is conjugate to a geodesic with a given initial frame $(v_1(0), v_2(0), v_3(0))$ (c.f. definition 3.3). We are now in a position to prove our main theorem which establishes a criterion of the periodicity of a complete geodesic in $F(S^3(\sqrt{2}))$ based on the initial condition of the geodesic.

Theorem 5.6. *Let $\gamma = (r, (v_1, v_2, v_3)) \in F(S^3(\sqrt{2}))$ be a complete geodesic whose base curve $r(t)$ satisfies the equation (12).*

1. *The base curve $r(t)$ is periodic if and only if $I_{\alpha,\beta}$ is a rational number. Furthermore if we denote the number $I_{\alpha,\beta}$ by $2\frac{m}{n} + 1$ with primitive integers m, n, then the minimal period is $2\pi n/\sqrt{b}$, where b is as defined in (31).*
2. *Let $(\dot{\theta}^i_j)$ be the fundamental matrix of γ, which is part of the initial values of the geodesic equations (9), (10) (c.f. the definition 3.3 and Proposition (3.2)) and $\gamma(t)$ have the same initial point as a generic geodesic $\Gamma(t)$ determined by $r(t)$. Then $\gamma(t)$ is periodic if and only if $I_{\alpha,\beta}$ and $\frac{\dot{\theta}^3_2+\beta}{\sqrt{b}}$ are both rational numbers.*

Proof. The first part of the theorem comes from Lemma 5.2 and Theorem 5.4. To prove the second part, note that the generic geodesic is periodic if and only if the base curve is periodic. If the base curve r of a geodesic γ is periodic, then the period is $2\pi n/\sqrt{b}$ for some integer n by Theorem 5.4. Since the θ_0 in (14) is $\dot{\theta}^3_2 + \beta$ by (16), so γ is periodic if and only if $\frac{2n\pi}{\sqrt{b}} \times \frac{\dot{\theta}_2^3+\beta}{2\pi}$ is rational, or equivalently $\frac{\dot{\theta}_2^3+\beta}{\sqrt{b}}$ is rational, and the theorem follows. $\square$

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