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ON GENERALIZED MANNHEIM CURVES IN EUCLIDEAN 4-SPACE

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ABSTRACT. We give a definition of generalized Mannheim curve in Euclidean 4-space E^4. We show some characterizations and examples of generalized Mannheim curves.

1. Introduction

A regular smooth curve C in Euclidean 3-space E^3 is called a Mannheim curve if there exist another regular smooth curve $\hat{C}$, distinct from C, and a bijection $\phi : C \to \hat{C}$ such that the principal normal line at each point of C is the binormal line of $\hat{C}$ at the corresponding point under ϕ ([2], [3], [4], [6]). Then $\hat{C}$ is called a Mannheim mate curve of C. It is well-known that a regular smooth curve C in E^3 is a Mannheim curve if and only if its curvature function k_1 and its torsion function k_2 satisfy the equality $k_1 = \alpha\{(k_1)^2 + (k_2)^2\}$ on each point of C, where α is a positive constant number ([2], [3], [4], [6]).

In the present paper, we try to construct a notion of Mannheim curve in E^4. That is, we give a definition of generalized Mannheim curve in E^4. We prove some characterizations of the generalized Mannheim curve and we show examples of generalized Mannheim curves in Euclidean 4-space E^4. We take "smooth" to mean "of class C^∞"

2. Mannheim curves in E^3

In many books, it is described that a regular smooth curve in E^3 is a Mannheim curve if and only if its curvature function k_1 and its torsion function k_2 satisfy the equality $k_1 = \alpha\{(k_1)^2 + (k_2)^2\}$ on each point of the curve, where α is a positive constant number. But, a formula of parametric equation of Mannheim curve in E^3 is not described. We found that Eisenhart's book ([2]) had given us a formula of parametric equation of Mannheim curve in E^3 as follows:

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Theorem 2.1 ([2, p.51]) *Let C be a curve defined by*

$$\mathbf{x}(u) = \begin{bmatrix} \alpha\displaystyle\int h(u)\sin u\, du \\ \alpha\displaystyle\int h(u)\cos u\, du \\ \alpha\displaystyle\int h(u)g(u)\, du \end{bmatrix}, \quad u \in U \subset \mathbf{R}.$$

Here $\mathbf{R}$ denotes the set of all real numbers, α is a positive constant number, $g : U \to \mathbf{R}$ is any smooth function and $h : U \to \mathbf{R}$ is given by

$$h(u) = \frac{\{1 + (g(u))^2 + (\dot{g}(u))^2\}^3 + \{1 + (g(u))^2\}^3\{\ddot{g}(u) + g(u)\}^2}{\{1 + (g(u))^2\}^{3/2}\{1 + (g(u))^2 + (\dot{g}(u))^2\}^{5/2}},$$

here the dot ($\dot{}$) denotes the derivative with respect to u. Then the curvature function κ and the torsion function τ of C satisfy

$$\kappa(u) = \alpha\{(\kappa(u))^2 + (\tau(u))^2\}$$

on each point $\mathbf{x}(u)$ of C.

Remark 2.1 (1) If $g(u) = c$ (constant), then $h(u) = 1$. Thus the curve C is a circular helix.

(2) If $g(u) = \tan u$ $(-\frac{\pi}{2} < u < \frac{\pi}{2})$, then a Mannheim curve C is given by

$$\mathbf{x}(u) = \begin{bmatrix} \alpha\displaystyle\int \frac{(5 + 3\cos^2 u)\cos u \sin u}{(1 + \cos^2 u)^{5/2}} du \\ \alpha\displaystyle\int \frac{(5 + 3\cos^2 u)\cos^2 u}{(1 + \cos^2 u)^{5/2}} du \\ \alpha\displaystyle\int \frac{(5 + 3\cos^2 u)\sin u}{(1 + \cos^2 u)^{5/2}} du \end{bmatrix},$$

where α is a positive constant number.

(3) If $g(u) = \sinh u$ $\quad(u \in \mathbf{R})$, then a Mannheim curve C is given by

$$\mathbf{x}(u) = \begin{bmatrix} \dfrac{\alpha}{\sqrt{2}}\displaystyle\int \frac{(1 + \cosh^2 u)\sin u}{\cosh^2 u} du \\ \dfrac{\alpha}{\sqrt{2}}\displaystyle\int \frac{(1 + \cosh^2 u)\cos u}{\cosh^2 u} du \\ \dfrac{\alpha}{\sqrt{2}}\displaystyle\int \frac{(1 + \cosh^2 u)\sinh u}{\cosh^2 u} du \end{bmatrix},$$

where α is a positive constant number.

3. Special Frenet curves in E^4

Let C be a regular smooth curve in Euclidean 4-space E^4 defined by $\mathbf{x} : L \ni s \mapsto \mathbf{x}(s) \in E^4$, where L denotes a subset of the set $\mathbf{R}$ of all real numbers, and s is the arc-length parameter of C. The curve C is called a *special Frenet curve* if there exist three smooth functions k_1, k_2, k_3 on C and smooth frame field $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ along the curve C such that these satisfy the following properties:

- The formulas of Frenet-Serret holds:

$$\begin{cases} \mathbf{e}_1(s) = & \mathbf{x}'(s) & & & \\ \mathbf{e}_1'(s) = & & k_1(s)\mathbf{e}_2(s) & & \\ \mathbf{e}_2'(s) = & -k_1(s)\mathbf{e}_1(s) & & k_2(s)\mathbf{e}_3(s) & \\ \mathbf{e}_3'(s) = & & -k_2(s)\mathbf{e}_2(s) & & k_3(s)\mathbf{e}_4(s) \\ \mathbf{e}_4'(s) = & & & -k_3(s)\mathbf{e}_3(s) & \end{cases}$$

  for $s \in L$, where the prime ($'$) denotes differentiation with respect to s.

- The frame field $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ is of orthonormal positive orientation.
- The functions k_1 and k_2 are of positive, and the function k_3 doesn't vanish.
- The functions k_1 , k_2 , and k_3 are called the first, the second, and the third curvature function of C, respctively. The frame field $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ is called the Frenet frame field on C. We refer this notion to [8].

Remark 3.1 For $s \in L$, the frame field $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ and curvature functions k_1, k_2 and k_3 are determined by the following steps:
(1st step)
$\mathbf{e}_1(s) := \mathbf{x}'(s)$.
(2nd step)
$k_1(s) := \|\mathbf{e}_1'(s)\| > 0$,
$\mathbf{e}_2(s) := \dfrac{1}{k_1(s)} \cdot \mathbf{e}_1'(s)$.
(3rd step)
$k_2(s) := \|\mathbf{e}_2'(s) + k_1(s) \cdot \mathbf{e}_1(s)\| > 0$,
$\mathbf{e}_3(s) := \dfrac{1}{k_2(s)} \cdot (\mathbf{e}_2'(s) + k_1(s) \cdot \mathbf{e}_1(s))$.
(4th step)
$\mathbf{e}_4(s) := \varepsilon \cdot \dfrac{1}{\|\mathbf{e}_3'(s) + k_2(s) \cdot \mathbf{e}_2(s)\|} \cdot (\mathbf{e}_3'(s) + k_2(s) \cdot \mathbf{e}_2(s))$ $(\varepsilon = \pm 1)$, and sign of ε is determined by the fact that the orthonormal frame field $\{\mathbf{e}_1(s), \mathbf{e}_2(s), \mathbf{e}_3(s), \mathbf{e}_4(s)\}$ is of positive orientation, and
$k_3(s) := \langle \mathbf{e}_3'(s), \mathbf{e}_4(s) \rangle$, and the function k_3 doesn't vanish.
In order to make sure that the curve C is a special Frenet curve, we must check the above steps (from (1st step) to (4th step)) for $s \in L$.

At each point of C, a line ℓ_1 in the direction of $\mathbf{e}_2$ is called *the first normal line*, a line ℓ_2 in the direction of $\mathbf{e}_3$ is called *the second normal line*, and a line ℓ_3 in the direction of $\mathbf{e}_4$ is called *the third normal line*. We remark that, at each point of a curve in Euclidean 3-space E^3, the first normal line and the second normal line are called the principal normal line and the binormal line, respectively.

4. Generalized Mannheim curves in E^4 and results

We already had a generalized Bertrand curve in E^4 ([5]). By the same idea, we have a generalization in E^4 for the notion of Mannheim curve in E^3.

Definition 4.1 A special Frenet curve C in E^4 is a *generalized Mannheim curve* if there exists a special Frenet curve $\hat{C}$ in E^4 such that the first normal line at each point of C is included in the plane generated by the second normal line and the third normal line of $\hat{C}$ at corresponding point under ϕ. Here ϕ is a bijection from C to $\hat{C}$. The curve $\hat{C}$ is called the *generalized Mannheim mate curve* of C.

Hereafter, a special Frenet curve C in E^4 is parametrized by the arc-length parameter s, that is, C is given by $\mathbf{x} : L \ni s \mapsto \mathbf{x}(s) \in E^4$. Let C be a generalized Mannheim curve in E^4. Then, by the definition, a generalized Mannheim mate curve $\hat{C}$ is given by the map $\hat{\mathbf{x}} : L \to E^4$ such that

$$\hat{\mathbf{x}}(s) = \mathbf{x}(s) + \alpha(s) \cdot \mathbf{e}_2(s), \quad s \in L, \tag{4.1}$$

where α is a smooth function on L. We remark that the parameter s generally is not an arc-length parameter of $\hat{C}$. Let $\hat{s}$ be the arc-length of $\hat{C}$ defined by

$$\hat{s} = \int_0^s \left\| \frac{d\hat{\mathbf{x}}(s)}{ds} \right\| ds.$$

We can consider a smooth function $f : L \mapsto \hat{L}$ given by $f(s) = \hat{s}$. Then we have

$$f'(s) = \sqrt{\{1 - \alpha(s) \cdot k_1(s)\}^2 + \{\alpha'(s)\}^2 + \{\alpha(s) \cdot k_2(s)\}^2}$$

for $s \in L$. The representation of $\hat{C}$ by arc-length parameter $\hat{s}$ is denoted by $\hat{\mathbf{x}} : \hat{L} \ni \hat{s} \mapsto \hat{\mathbf{x}}(\hat{s}) \in E^4$, here we use the same letter "$\hat{\mathbf{x}}$" for simplicity. Then we can simply write

$$\hat{\mathbf{x}}(f(s)) = \mathbf{x}(s) + \alpha(s) \cdot \mathbf{e}_2(s)$$

for curve $\hat{C}$, and a bijection $\phi : C \to \hat{C}$ is given by $\phi(\mathbf{x}(s)) = \hat{\mathbf{x}}(f(s))$. This notation is used in section **5**. Thus we have

$$\begin{aligned} \frac{d\hat{\mathbf{x}}(f(s))}{ds} &= \left.\frac{d\hat{\mathbf{x}}(\hat{s})}{d\hat{s}}\right|_{\hat{s}=f(s)} f'(s) \\ &= f'(s) \cdot \hat{\mathbf{e}}_1(f(s)) \end{aligned}$$

for $s \in L$.

In the present paper, our results are Theorems 4.1, 4.2, 4.3, 4.4, and 4.5.

Theorem 4.1 *If a special Frenet curve C in E^4 is a generalized Mannheim curve, then the first curvature function k_1 and second curvature functions k_2 of C satisfy the equality:*

$$k_1(s) = \alpha\{(k_1(s))^2 + (k_2(s))^2\}, \; s \in L, \tag{4.2}$$

where α is a positive constant number.

Theorem 4.2 *Let C be a special Frenet curve in E^4 whose curvature functions k_1 and k_2 are non-constant functions and satisfy the equality: $k_1(s) = \alpha\{(k_1(s))^2 + (k_2(s))^2\}$, $s \in L$. Here α is a positive constant number. If the curve $\hat{C}$ given by $\hat{\mathbf{x}}(s) = \mathbf{x}(s) + \alpha \cdot \mathbf{e}_2(s)$, $s \in L$ is a special Frenet curve, then C is a generalized Mannheim curve and $\hat{C}$ is the generalized Mannheim mate curve of C.*

Remark 4.1 For a special Frenet curve C with (4.2), it is not clear that the smooth curve $\hat{C}$ given by (4.1) is a special Frenet curve. It is unknown whether the reverse of Theorem 4.1 is true or false.

Theorem 4.3 *Let C be a special Frenet curve in E^4 such that its third curvature function k_3 doesn't vanish. The curvature functions k_1 and k_2 of C are constant functions if and only if there exists a special Frenet curve $\hat{C}$ in E^4 such that the first normal line at each point of C is the third normal line of $\hat{C}$ at corresponding each point under a bijection $\phi : C \to \hat{C}$.*

Remark 4.2 In n-dimensional Euclidean space, curves with constant curvatures are given in [1].

The following theorem gives a parametric representation of generalized Mannheim curves in E^4. We have this theorem under the referee's advice.

Theorem 4.4 *Let C be a curve defined by*

$$\mathbf{x}(u) = \begin{bmatrix} \alpha \displaystyle\int f(u) \sin u \, du \\ \alpha \displaystyle\int f(u) \cos u \, du \\ \alpha \displaystyle\int f(u) g(u) \, du \\ \alpha \displaystyle\int f(u) h(u) \, du \end{bmatrix}, \quad u \in U \subset \mathbf{R}.$$

Here α is a positive constant number, g and h are any smooth functions : $U \to \mathbf{R}$, *and* $f : U \to \mathbf{R}$ *is given by*

$$
\begin{aligned}
& f(u) \\
= \ & \left(1+(g(u))^2+(h(u))^2\right)^{-\frac{3}{2}} \\
& \times\Big\{1+(g(u))^2+(h(u))^2+(\dot{g}(u))^2+(\dot{h}(u))^2 \\
& \qquad +\Big(g(u)\dot{h}(u)-\dot{g}(u)h(u)\Big)^2\Big\}^{-\frac{5}{2}} \\
& \times\Big[\Big\{1+(g(u))^2+(h(u))^2+(\dot{g}(u))^2+(\dot{h}(u))^2 \\
& \quad +\Big(g(u)\dot{h}(u)-\dot{g}(u)h(u)\Big)^2\Big\}^3 \\
& \quad +\left(1+(g(u))^2+(h(u))^2\right)^3 \times \Big\{(g(u)+\ddot{g}(u))^2+\Big(h(u)+\ddot{h}(u)\Big)^2 \\
& \quad +\Big(\Big(g(u)\dot{h}(u)-\dot{g}(u)h(u)\Big)-\Big(\dot{g}(u)\ddot{h}(u)-\ddot{g}(u)\dot{h}(u)\Big)\Big)^2 \\
& \quad +\Big(g(u)\ddot{h}(u)-\ddot{g}(u)h(u)\Big)^2\Big\}\Big],
\end{aligned}
$$

for $u \in U$*. Here the dot* ($\dot{}$) *denotes the derivative with respect to* u*. Then the curvature functions* k_1 *and* k_2 *of* C *satisfy*

$$k_1(u) = \alpha\{(k_1(u))^2 + (k_2(u))^2\}$$

on each point $\mathbf{x}(u)$ *of* C*.*

Let C be a generalized Mannheim curve in E^4, and let $\mathbf{e}_2$ be the second vector field in the Frenet frame field along C. We can consider a ruled surface S in E^4 by the curve C and the first normal lines of C. Let S be a surface in E^4 defined by $\mathbf{X} : L \times \mathbf{R} \to E^4$, where $\mathbf{X}(s,t) = \mathbf{x}(s) + t\mathbf{e}_2(s)$. The ruled surface S is generated by the motion of a straight line (first normal line) ℓ along the curve C. The curve C is called the directrix (or base curve) of S, and a straight line ℓ is called a generating line (or generator, ruling) of S. We remark that the ruled surface S is, as a submanifold in E^4, not a totally geodesic submanifold in E^4. The striction curve of S is defined by

$$\mathbf{y}(s) = \mathbf{x}(s) - \frac{< \mathbf{x}'(s), \mathbf{e}_2'(s) >}{\|\mathbf{e}_2'(s)\|}\mathbf{e}_2(s)$$

for $s \in L$. The geometric meaning of the striction curve of a ruled surface is referred to [4], [7] and [8]. Since C is a genealized Mannheim curve in E^4, we have

$$-\frac{< \mathbf{x}'(s), \mathbf{e}_2'(s) >}{\|\mathbf{e}_2'(s)\|} = -\frac{-k_1(s)}{(k_1(s))^2 + (k_2(s))^2} = \alpha.$$

Thus, by the definition of Mannheim mate curve, we easily prove the following theorem:

Theorem 4.5 *Let S be a ruled surface in E^4 such that its directrix is a generalized Mannheim curve C and its generating line is the first normal line of C. Then the Mannheim mate $\hat{C}$ is the striction curve of S.*

5. Proofs of Theorems

Let C be a special Frenet curve in E^4 whose Frenet frame field is denoted by $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$. Let k_1, k_2, k_3 be curvature functions of C. The arc-length parametrization of C is given by $\mathbf{x} : L \to E^4$ $(s \mapsto \mathbf{x}(s))$.

5.1. Proof of Theorem 4.1

Let C be a generalized Mannheim curve in E^4 and let $\hat{C}$ be the generalized Mannheim mate curve of C. The curve $\hat{C}$ is reparametrized by $\hat{\mathbf{x}}(s) = \mathbf{x}(s) + \alpha(s) \cdot \mathbf{e}_2(s)$, where $\alpha : L \ni s \mapsto \alpha(s) \in \mathbf{R}$ is a smooth function. A smooth function $f : L \ni s \mapsto f(s) = \hat{s} \in \hat{L}$ is defined by $f(s) = \displaystyle\int_0^s \left\| \frac{d\hat{\mathbf{x}}(s)}{ds} \right\| ds = \hat{s}$. We remark that $\hat{s}$ is the arc-length parameter of $\hat{C}$, and the bijection $\phi : C \to \bar{C}$ is defined by $\phi(\mathbf{x}(s)) = \hat{\mathbf{x}}(f(s))$. Since the first normal line at each point of C is included in the plane generated the second normal line and the third normal line of $\hat{C}$ at corresponding point under a bijection ϕ, for each $s \in L$, the vector $\mathbf{e}_2(s)$ is given by linear combination of $\hat{\mathbf{e}}_3(f(s))$ and $\hat{\mathbf{e}}_4(f(s))$, that is, we can set $\mathbf{e}_2(s) = g(s) \cdot \hat{\mathbf{e}}_3(f(s)) + h(s) \cdot \hat{\mathbf{e}}_4(f(s))$ for some smooth functions g and h on L.

We differentiate both sides of equality: $\hat{\mathbf{x}}(f(s)) = \mathbf{x}(s) + \alpha(s) \cdot \mathbf{e}_2(s)$ with respect to s. Then we have

$$
\begin{aligned}
& f'(s) \cdot \hat{\mathbf{e}}_1(f(s)) \\
& \quad = \mathbf{e}_1(s) + \alpha'(s) \cdot \mathbf{e}_2(s) \\
& \quad + \alpha(s)(-k_1(s) \cdot \mathbf{e}_1(s) + k_2(s) \cdot \mathbf{e}_3(s) \\
& \quad = (1 - \alpha(s)k_1(s)) \cdot \mathbf{e}_1(s) + \alpha'(s) \cdot \mathbf{e}_2(s) \\
& \quad + \alpha(s)k_2(s) \cdot \mathbf{e}_3(s),
\end{aligned}
$$

for $s \in L$. Let $<\ ,\ >$ denote the inner product of vectors. By the fact:

$$< \hat{\mathbf{e}}_1(f(s)), g(s) \cdot \hat{\mathbf{e}}_3(f(s)) + h(s) \cdot \hat{\mathbf{e}}_4(f(s)) >= 0,$$

we have $\alpha'(s) = 0$ for any $s \in L$ so that the function α is of constant, say α. Thus we have

$$f'(s) \cdot \hat{\mathbf{e}}_1(f(s)) = (1 - \alpha k_1(s)) \cdot \mathbf{e}_1(s) + \alpha k_2(s) \cdot \mathbf{e}_3(s),$$

that is,

$$\hat{\mathbf{e}}_1(f(s)) = \frac{1 - \alpha k_1(s)}{f'(s)} \cdot \mathbf{e}_1(s) + \frac{\alpha k_2(s)}{f'(s)} \cdot \mathbf{e}_3(s),$$

where $f'(s) = \sqrt{(1-\alpha k_1(s))^2 + (\alpha k_2(s))^2}$ for $s \in L$. By differentiation of both sides of the above equality with respect to s, we have

$$\begin{aligned}
&f'(s)\hat{k}_1(f(s)) \cdot \hat{\mathbf{e}}_2(f(s)) \\
&= \left(\frac{1-\alpha k_1(s)}{f'(s)}\right)' \cdot \mathbf{e}_1(s) \\
&\quad + \left(\frac{(1-\alpha k_1(s))k_1(s) - \alpha(k_2(s))^2}{f'(s)}\right) \cdot \mathbf{e}_2(s) \\
&\quad + \left(\frac{\alpha k_2(s)}{f'(s)}\right)' \cdot \mathbf{e}_3(s) \\
&\quad + \left(\frac{\alpha k_2(s) k_3(s)}{f'(s)}\right) \cdot \mathbf{e}_4(s)
\end{aligned}$$

for $s \in L$. By the fact:

$$< \hat{\mathbf{e}}_2(f(s)), g(s) \cdot \hat{\mathbf{e}}_3(f(s)) + h(s) \cdot \hat{\mathbf{e}}_4(f(s)) >= 0\,, s \in L,$$

we have that coefficient of $\mathbf{e}_2$ in the above equation is zero, that is,

$$(1-\alpha k_1(s))k_1(s) - \alpha(k_2(s))^2 = 0\,, s \in L.$$

Thus we have $k_1(s) = \alpha\{(k_1(s))^2 + (k_2(s))^2\}$ for $s \in L$. This completes the proof.

5.2. Proof of Theorem 4.2

Let $\hat{s}$ be the arc-length of $\hat{C}$. That is, $\hat{s}$ is defined by

$$\hat{s} = \int_0^s \left\| \frac{d\hat{\mathbf{x}}(s)}{ds} \right\| ds$$

for $s \in L$. We can consider a smooth function $f : L \ni s \mapsto f(s) = \hat{s} \in \hat{L}$. By the assumption of the curvature functions k_1 and k_2, we have

$$\begin{aligned}
f'(s) &= \sqrt{(1-\alpha \cdot k_1(s))^2 + (\alpha \cdot k_2(s))^2} \\
&= \sqrt{1-\alpha \cdot k_1(s)}
\end{aligned}$$

for $s \in L$. The representation of $\hat{C}$ by arc-length parameter $\hat{s}$ is denoted by $\hat{\mathbf{x}}(\hat{s})$, here we use the same letter "$\hat{\mathbf{x}}$" for simplicity. Then we can simply write

$$\begin{aligned}
\hat{\mathbf{x}}(\hat{s}) &= \hat{\mathbf{x}}(f(s)) \\
&= \mathbf{x}(s) + \alpha \cdot \mathbf{e}_2(s).
\end{aligned}$$

for curve $\hat{C}$. This notation is used in subsections 5.2 and 5.3. Thus we have

$$\begin{aligned}
\frac{d\hat{\mathbf{x}}(\hat{s})}{ds} &= \left.\frac{d\hat{\mathbf{x}}(\hat{s})}{d\hat{s}}\right|_{\hat{s}=f(s)} f'(s) \\
&= f'(s) \cdot \hat{\mathbf{e}}_1(f(s))
\end{aligned}$$

and

$$\begin{aligned}
f'(s)\cdot\hat{\mathbf{e}}_1(f(s)) &= \mathbf{e}_1(s) \\
&\quad +\alpha\{-k_1(s)\cdot\mathbf{e}_1(s)+k_2(s)\cdot\mathbf{e}_3(s)\} \\
&= (1-\alpha k_1(s))\cdot\mathbf{e}_1(s)+\alpha k_2(s)\cdot\mathbf{e}_3(s).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\hat{\mathbf{e}}_1(f(s)) &= \sqrt{1-\alpha k_1(s)}\cdot\mathbf{e}_1(s) \\
&\quad +\frac{\alpha k_2(s)}{\sqrt{1-\alpha k_1(s)}}\cdot\mathbf{e}_3(s)
\end{aligned}$$

for $s\in L$. We differentiate both sides of the above equality with respect to s, then we have

$$\begin{aligned}
&f'(s)\cdot\left.\frac{d\hat{\mathbf{e}}_1(\hat{s})}{d\hat{s}}\right|_{\hat{s}=f(s)} \\
&= \left(\sqrt{1-\alpha k_1(s)}\right)'\cdot\mathbf{e}_1(s) \\
&\quad +\sqrt{1-\alpha k_1(s)}k_1(s)\cdot\mathbf{e}_2(s) \\
&\quad +\left(\frac{\alpha k_2(s)}{\sqrt{1-\alpha\cdot k_1(s)}}\right)'\cdot\mathbf{e}_3(s) \\
&\quad +\frac{\alpha k_2(s)}{\sqrt{1-\alpha\cdot k_1(s)}}(-k_2(s)\cdot\mathbf{e}_2(s)+k_3(s)\cdot\mathbf{e}_4(s)).
\end{aligned}$$

Since $\hat{C}$ is a special Frenet curve, we have

$$\left.\frac{d\hat{\mathbf{e}}_1(\hat{s})}{d\hat{s}}\right|_{\hat{s}=f(s)}=\hat{k}_1(f(s))\cdot\hat{\mathbf{e}}_2(f(s)).$$

Thus we have

$$\begin{aligned}
&f'(s)\hat{k}_1(f(s))\cdot\hat{\mathbf{e}}_2(f(s)) \\
&= \left(\sqrt{1-\alpha k_1(s)}\right)'\cdot\mathbf{e}_1(s) \\
&\quad +\left(\sqrt{1-\alpha k_1(s)}k_1(s)-\frac{\alpha(k_2(s))^2}{\sqrt{1-\alpha k_1(s)}}\right)\cdot\mathbf{e}_2(s) \\
&\quad +\left(\frac{\alpha k_2(s)}{\sqrt{1-\alpha k_1(s)}}\right)'\cdot\mathbf{e}_3(s) \\
&\quad +\left(\frac{\alpha k_2(s)k_3(s)}{\sqrt{1-\alpha k_1(s)}}\right)\cdot\mathbf{e}_4(s).
\end{aligned}$$

Under the our assumption, it holds

$$\begin{aligned}
&\sqrt{1-\alpha k_1(s)}k_1(s) + \frac{\alpha k_2(s)}{\sqrt{1-\alpha k_1(s)}}(-k_2(s)) \\
&= \frac{1}{\sqrt{1-\alpha k_1(s)}}(k_1(s) - \alpha(k_1(s))^2 - \alpha(k_2(s))^2) \\
&= 0.
\end{aligned}$$

We have that the coefficient of $\mathbf{e}_2(s)$ in the above equality vanishes. Thus, for each $s \in L$, the vector $\hat{\mathbf{e}}_2(f(s))$ is given by linear combination of $\mathbf{e}_1(s)$, $\mathbf{e}_3(s)$ and $\mathbf{e}_4(s)$. And, as above, the vector $\hat{\mathbf{e}}_1(f(s))$ is given by linear combination of $\mathbf{e}_1(s)$ and $\mathbf{e}_3(s)$. Since the curve $\hat{C}$ is a special Frennet curve in E^4, the vector $\mathbf{e}_2(s)$ is given by linear combination of $\hat{\mathbf{e}}_3(f(s))$ and $\hat{\mathbf{e}}_4(f(s))$.

Therefore, the first normal line at each point of C is included in the plane generated the second normal line and the third normal line of $\bar{C}$ at corresponding point under ϕ. Here the bijection $\phi : C \to \hat{C}$ is defined by $\phi(\mathbf{x}(s)) = \hat{\mathbf{x}}(f(s))$. This completes the proof.

5.3. Proof of Theorem 4.3

(i) Let C be a special Frenet curve in E^4 with the Frenet frame field $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ and curvature functions k_1, k_2 and k_3. The first curvature function and the second curvature function of C are of positive constant, say k_1 and k_2, respectively. Thus $\dfrac{k_1}{(k_1)^2 + (k_2)^2}$ is a positive constant number, say α. We define a regular smooth curve $\hat{C}$ by

$$\hat{\mathbf{x}} : L \mapsto E^4$$

and

$$\hat{\mathbf{x}}(s) = \mathbf{x}(s) + \alpha \cdot \mathbf{e}_2(s), \quad s \in L.$$

Let $\hat{s}$ denote the arc-length parameter of $\hat{C}$, and let $f : L \mapsto \hat{L}$ be a function defined by

$$\begin{aligned}
\hat{s} = f(s) &= \sqrt{(1-\alpha k_1)^2 + (\alpha k_2)^2}\, s \\
&= \sqrt{1-\alpha k_1}\, s.
\end{aligned}$$

Then we have $f'(s) = \sqrt{1-\alpha k_1}$, and

$$\begin{aligned}
f'(s) \cdot \hat{\mathbf{e}}_1(f(s)) &= \mathbf{e}_1(s) + \alpha \cdot \mathbf{e}_2'(s) \\
&= (1-\alpha k_1) \cdot \mathbf{e}_1(s) + \alpha k_2 \cdot \mathbf{e}_3(s),
\end{aligned}$$

that is,

$$\hat{\mathbf{e}}_1(f(s)) = \sqrt{1-\alpha k_1} \cdot \mathbf{e}_1(s) + \frac{\alpha k_2}{\sqrt{1-\alpha k_1}} \cdot \mathbf{e}_3(s).$$

Defferentiating both sides of the above equality with respect to s, we have

$$\begin{aligned}
f'(s)\cdot\left.\frac{d\hat{\mathbf{e}}_1(\hat{s})}{d\hat{s}}\right|_{\hat{s}=f(s)} &= \sqrt{1-\alpha k_1}\cdot\mathbf{e}_1'(s)+\frac{\alpha k_2}{\sqrt{1-\alpha k_1}}\cdot\mathbf{e}_3'(s)\\
&= \left(k_1\sqrt{1-\alpha k_1}-\frac{\alpha(k_2)^2}{\sqrt{1-\alpha k_1}}\right)\cdot\mathbf{e}_2(s)\\
&\quad+\frac{\alpha k_2 k_3(s)}{\sqrt{1-\alpha k_1}}\cdot\mathbf{e}_4(s)\\
&= \frac{\alpha k_2 k_3(s)}{\sqrt{1-\alpha k_1}}\cdot\mathbf{e}_4(s)
\end{aligned}$$

Thus, since k_3 doesn't vanish, we have

$$\hat{k}_1(f(s)) := \left\|\left.\frac{d\hat{\mathbf{e}}_1(\hat{s})}{d\hat{s}}\right|_{\hat{s}=f(s)}\right\| = sign(k_3)\frac{\alpha k_2 k_3(s)}{1-\alpha k_1}>0.$$

Here $sign(k_3)$ denotes the sign of the function k_3, that is, if k_3 is positive valued function(resp. negative valued function) then $sign(k_3) = +1$ (resp. -1) so that $sign(k_3)k_3(s)$ is positive for any $s \in L$. We can put

$$\hat{\mathbf{e}}_2(\hat{s}) := \frac{1}{\hat{k}_1(\hat{s})}\cdot\frac{d\hat{\mathbf{e}}_1(\hat{s})}{d\hat{s}}\,, s\in L.$$

Then we have

$$\hat{\mathbf{e}}_2(f(s)) = sign(k_3)\cdot\mathbf{e}_4(s).$$

By differentiation of the above with respect to s, we have

$$f'(s)\cdot\left.\frac{d\hat{\mathbf{e}}_2(\hat{s})}{d\hat{s}}\right|_{\hat{s}=f(s)} = -sign(k_3)k_3(s)\cdot\mathbf{e}_3(s)$$

and we have

$$\begin{aligned}
&\left.\frac{d\hat{\mathbf{e}}_2(\hat{s})}{d\hat{s}}\right|_{\hat{s}=f(s)}+\hat{k}_1(f(s))\cdot\hat{\mathbf{e}}_1(f(s))\\
&= sign(k_3)\frac{\alpha k_2 k_3(s)}{\sqrt{1-\alpha k_1}}\cdot\mathbf{e}_1(s)-sign(k_3)k_3(s)\sqrt{1-\alpha k_1}\cdot\mathbf{e}_3(s).
\end{aligned}$$

Since $sign(k_3)k_3(s)$ is positive for $s \in L$,

$$\begin{aligned}
&\hat{k}_2(f(s))\\
&= \left\|\left.\frac{d\hat{\mathbf{e}}_2(\hat{s})}{d\hat{s}}\right|_{\hat{s}=f(s)}+\hat{k}_1(f(s))\cdot\hat{\mathbf{e}}_1(f(s))\right\|\\
&= \sqrt{\frac{\alpha^2 {k_2}^2(k_3(s))^2}{1-\alpha k_1}+(1-\alpha k_1)(k_3(s))^2}\\
&= \sqrt{(k_3(s))^2}\\
&= sign(k_3)k_3(s)>0.
\end{aligned}$$

Thus we can put

$$\begin{aligned}
\mathbf{e}_3(f(s)) &= \frac{1}{\hat{k}_2(f(s))} \cdot \left(\left.\frac{d\hat{\mathbf{e}}_2(\hat{s})}{d\hat{s}}\right|_{\hat{s}=f(s)} + \hat{k}_1(f(s)) \cdot \hat{\mathbf{e}}_1(f(s)) \right) \\
&= \frac{\alpha k_2}{\sqrt{1-\alpha k_1}} \cdot \mathbf{e}_1(s) - \sqrt{1-\alpha k_1} \cdot \mathbf{e}_3(s)
\end{aligned}$$

for $s \in L$. By differentiation of the above with respect to s, we have

$$f'(s) \cdot \left.\frac{d\hat{\mathbf{e}}_3(\hat{s})}{d\hat{s}}\right|_{\hat{s}=f(s)} = \frac{k_2}{\sqrt{1-\alpha k_1}} \cdot \mathbf{e}_2(s) - \sqrt{1-\alpha k_1} k_3(s) \cdot \mathbf{e}_4(s).$$

Since $f'(s) = \sqrt{1-\alpha k_1}$ and $\hat{k}_2(f(s)) \cdot \hat{\mathbf{e}}_2(f(s)) = k_3(s) \cdot \mathbf{e}_4(s)$, we have

$$\left.\frac{d\hat{\mathbf{e}}_3(\hat{s})}{d\hat{s}}\right|_{\hat{s}=f(s)} + \hat{k}_2(f(s)) \cdot \hat{\mathbf{e}}_2(f(s)) = \frac{k_2}{1-\alpha k_1} \cdot \mathbf{e}_2(s).$$

Thus, taking note of (4th step) in Remark 2.1, we have $\hat{\mathbf{e}}_4(f(s)) = \varepsilon \mathbf{e}_2(s)$ for $s \in L$, where $\varepsilon = \pm 1$. We must determine whether ε is 1 or -1 under the condition that the frame field $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3, \hat{\mathbf{e}}_4\}$ is of positive orentation.

We have, by $\det[\mathbf{e}_1(s), \mathbf{e}_2(s), \mathbf{e}_3(s), \mathbf{e}_4(s)] = 1$ for $s \in L$.

$$\begin{aligned}
&\det[\hat{\mathbf{e}}_1(f(s)), \hat{\mathbf{e}}_2(f(s)), \hat{\mathbf{e}}_3(f(s)), \hat{\mathbf{e}}_4(f(s))] \\
&= \det\left[\sqrt{1-\alpha k_1} \cdot \mathbf{e}_1(s), sign(k_3)\mathbf{e}_4(s), -\sqrt{1-\alpha k_1} \cdot \mathbf{e}_3(s), \varepsilon \mathbf{e}_2(s)\right] \\
&\quad + \det\left[\frac{\alpha k_2}{\sqrt{1-\alpha k_1}} \cdot \mathbf{e}_3(s), sign(k_3)\mathbf{e}_4(s), \frac{\alpha k_2}{\sqrt{1-\alpha k_1}} \cdot \mathbf{e}_1(s), \varepsilon \mathbf{e}_2(s)\right] \\
&= sign(k_3)\varepsilon \left((1-\alpha k_1) + \frac{\alpha^2 {k_2}^2}{1-\alpha k_1} \right) \\
&= sign(k_3)\varepsilon
\end{aligned}$$

and $\det[\hat{\mathbf{e}}_1(f(s)), \hat{\mathbf{e}}_2(f(s)), \hat{\mathbf{e}}_3(f(s)), \hat{\mathbf{e}}_4(f(s))] = 1$ for any $s \in L$. Thus we have $\varepsilon = sign(k_3)$. Therefore, we have

$$\hat{\mathbf{e}}_4(f(s)) = sign(k_3)\mathbf{e}_2(s)$$

and

$$\hat{k}_3(f(s)) = \left\langle \left.\frac{d\hat{\mathbf{e}}_3(\hat{s})}{d\hat{s}}\right|_{\hat{s}=f(s)}, \hat{\mathbf{e}}_4(f(s)) \right\rangle = sign(k_3)\frac{k_2}{1-\alpha k_1}$$

for $s \in L$.

By the above facts, $\hat{C}$ is a spacial Frenet curve in E^4 and the first normal line at each point of C is the third normal line of $\hat{C}$ at corresponding each point under the bijection $\phi : C \ni \mathbf{x}(s) \mapsto \phi(\mathbf{x}(s)) = \hat{\mathbf{x}}(f(s)) \in \hat{C}$.

(ii) Let C be a special Frenet curve in E^4 with the Frenet frame field $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ and curvature functions k_1, k_2 and k_3. Let $\hat{C}$ be a special Frenet curve in E^4 with the Frenet frame field $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3, \hat{\mathbf{e}}_4\}$ and curvature functions $\hat{k}_1$, $\hat{k}_2$ and $\hat{k}_3$. We assume that the first normal line at each point of C is the third normal line of $\hat{C}$ at corresponding each point under a bijection $\phi : C \to \hat{C}$. Then the curve $\hat{C}$ is parametrized by

$$\hat{\mathbf{x}}(s) = \mathbf{x}(s) + \alpha(s) \cdot \mathbf{e}_2(s)\,, s \in L.$$

Let $\hat{s}$ be the arc-length parameter of $\hat{C}$ given by

$$\begin{aligned}\hat{s} &= f(s) \\ &= \int_0^s \sqrt{(1-\alpha(s)k_1(s))^2 + (\alpha'(s))^2 + (\alpha(s)k_2(s))^2}\, ds.\end{aligned}$$

Here $f : L \ni s \mapsto f(s) = \hat{s} \in \hat{L}$. A bijection $\phi : C \to \hat{C}$ is given by $\phi(\mathbf{x}(s)) = \hat{\mathbf{x}}(f(s))$. Thus our assumption implies that $\hat{\mathbf{e}}_4(f(s)) = \pm\mathbf{e}_2(s)$ for $s \in L$.

Now, we have

$$\begin{aligned}f'(s) \cdot \hat{\mathbf{e}}_1(f(s)) &= (1-\alpha(s)k_1(s)) \cdot \mathbf{e}_1(s) \\ &\quad + \alpha'(s) \cdot \mathbf{e}_2(s) + \alpha(s)k_2(s) \cdot \mathbf{e}_3(s).\end{aligned}$$

Taking the inner product of $f'(s) \cdot \hat{\mathbf{e}}_1(f(s))$ and $\hat{\mathbf{e}}_4(f(s))$, we have

$$< f'(s) \cdot \hat{\mathbf{e}}_1(f(s)), \hat{\mathbf{e}}_4(f(s)) >= 0.$$

On the other hand, we have

$$\begin{aligned}&< f'(s) \cdot \hat{\mathbf{e}}_1(f(s)), \hat{\mathbf{e}}_4(f(s)) > \\ &= \; < (1-\alpha(s)k_1(s)) \cdot \mathbf{e}_1(s) + \alpha'(s) \cdot \mathbf{e}_2(s) + \alpha(s)k_2(s) \cdot \mathbf{e}_3(s), \pm\mathbf{e}_2(s) > \\ &= \; \pm\alpha'(s).\end{aligned}$$

Thus $\alpha(s) = \alpha$ for $s \in L$, where α denotes a constant number. From this fact, we have

$$f'(s) = \sqrt{(1-\alpha k_1(s))^2 + (\alpha k_2(s))^2} > 0$$

and

$$\hat{\mathbf{e}}_1(f(s)) = \left(\frac{1-\alpha k_1(s)}{f'(s)}\right) \cdot \mathbf{e}_1(s) + \left(\frac{\alpha k_2(s)}{f'(s)}\right) \cdot \mathbf{e}_3(s).$$

We differentiate the above equality with respect to s, then we have

$$\begin{aligned}&f'(s)\hat{k}_1(f(s)) \cdot \hat{\mathbf{e}}_2(f(s)) \\ &= \left(\frac{1-\alpha k_1(s)}{f'(s)}\right)' \cdot \mathbf{e}_1(s) + \left(\frac{k_1(s) - \alpha\{(k_1(s))^2 + (k_2(s))^2\}}{f'(s)}\right) \cdot \mathbf{e}_2(s) \\ &\quad + \left(\frac{\alpha k_2(s)}{f'(s)}\right)' \cdot \mathbf{e}_3(s) + \left(\frac{\alpha k_2(s)k_3(s)}{f'(s)}\right) \cdot \mathbf{e}_4(s).\end{aligned}$$

By the fact:
$$< f'(s)\hat{k}_1(f(s)) \cdot \hat{\mathbf{e}}_2(f(s)), \hat{\mathbf{e}}_4(f(s)) >= 0\,, s \in L,$$
it holds that
$$k_1(s) = \alpha\{(k_1(s))^2 + (k_2(s))^2\}\,, s \in L$$
so that α is a "positive" constant number. Thus we have
$$\begin{aligned}
&\hat{\mathbf{e}}_2(f(s)) \\
&= \frac{1}{f'(s)K(s)}\left(\frac{1-\alpha k_1(s)}{f'(s)}\right)' \cdot \mathbf{e}_1(s) \\
&\quad + \frac{1}{f'(s)K(s)}\left(\frac{\alpha k_2(s)}{f'(s)}\right)' \cdot \mathbf{e}_3(s) \\
&\quad + \frac{1}{f'(s)K(s)}\left(\frac{\alpha k_2(s)k_3(s)}{f'(s)}\right) \cdot \mathbf{e}_4(s)
\end{aligned}$$
for $s \in L$. Here we set $K(s) = \hat{k}_1(f(s))\,, s \in L$. By differentiation of the above equality with respect to s, we have
$$\begin{aligned}
&f'(s)\{-\hat{k}_1(f(s)) \cdot \hat{\mathbf{e}}_1(f(s)) + \hat{k}_2(f(s)) \cdot \hat{\mathbf{e}}_3(f(s))\} \\
&= \left\{\frac{1}{f'(s)K(s)}\left(\frac{1-\alpha k_1(s)}{f'(s)}\right)'\right\}' \cdot \mathbf{e}_1(s) \\
&\quad + \left\{\frac{k_1(s)}{f'(s)K(s)}\left(\frac{1-\alpha k_1(s)}{f'(s)}\right)' - \frac{k_2(s)}{f'(s)K(s)}\left(\frac{\alpha k_2(s)}{f'(s)}\right)'\right\} \cdot \mathbf{e}_2(s) \\
&\quad + \left\{\left(\frac{1}{f'(s)K(s)}\left(\frac{\alpha k_2(s)}{f'(s)}\right)'\right)' - \frac{k_3(s)}{f'(s)K(s)}\left(\frac{\alpha k_2(s)k_3(s)}{f'(s)}\right)\right\} \cdot \mathbf{e}_3(s) \\
&\quad + \left\{\left(\frac{1}{f'(s)K(s)}\left(\frac{\alpha k_2(s)k_3(s)}{f'(s)}\right)\right)' + \frac{k_3(s)}{f'(s)K(s)}\left(\frac{\alpha k_2(s)}{f'(s)}\right)'\right\} \cdot \mathbf{e}_4(s).
\end{aligned}$$
By the facts
$$< f'(s)\{-\hat{k}_1(f(s)) \cdot \hat{\mathbf{e}}_1(f(s)) + \hat{k}_2(f(s)) \cdot \hat{\mathbf{e}}_3(f(s))\}, \hat{\mathbf{e}}_4(f(s)) >= 0$$
and
$$\hat{\mathbf{e}}_4(f(s)) = \pm\mathbf{e}_2(s),$$
we have
$$k_1(s)\left(\frac{1-\alpha k_1(s)}{f'(s)}\right)' - k_2(s)\left(\frac{\alpha k_2(s)}{f'(s)}\right)' = 0,$$
that is,
$$\begin{aligned}
&\alpha k_1(s)k_1'(s)f'(s) + k_1(s)(1-\alpha k_1(s))f''(s) \\
&\quad + \alpha k_2(s)k_2'(s)f'(s) - \alpha(k_2(s))^2 f''(s) = 0.
\end{aligned}$$

We remark that

$$f'(s) = \sqrt{1 - \alpha k_1(s)}, \qquad f''(s) = -\frac{\alpha k_1'(s)}{2\sqrt{1 - \alpha k_1(s)}},$$

so that we have

$$2\alpha k_1'(s)(1 - \alpha k_1(s)) = 0.$$

The above equality yields $k_1'(s) = 0$ for any $s \in L$. Thus the first curvature function k_1 is of constant(that is, of positive constant). By the relation $k_1 = \alpha\{(k_1)^2 + (k_2)^2\}$, the second curvature function k_2 is of positive constant too. This completes the proof.

5.4. Proof of Theorem 4.4

Let C be a curve defined by

$$\mathbf{x}(u) = \begin{bmatrix} \alpha \displaystyle\int f(u) \sin u \, du \\ \alpha \displaystyle\int f(u) \cos u \, du \\ \alpha \displaystyle\int f(u) g(u) \, du \\ \alpha \displaystyle\int f(u) h(u) \, du \end{bmatrix}, \quad u \in U \subset \mathbf{R}.$$

Here α is a positive constant number, g and h are any smooth functions : $U \to \mathbf{R}$, and $f : U \to \mathbf{R}$ is a positive valued smooth function. Then we have

$$\dot{\mathbf{x}}(u) = \begin{bmatrix} \alpha f(u) \sin u \\ \alpha f(u) \cos u \\ \alpha f(u) g(u) \\ \alpha f(u) h(u) \end{bmatrix}, \quad u \in U.$$

Here the dot ($\dot{}$) denotes differentiation with respect to u. The arc-length parameter s of C is given by $s = \psi(u) = \displaystyle\int_{u_0}^{u} \|\dot{\mathbf{x}}(u)\| \, du$, where

$$\|\dot{\mathbf{x}}(u)\| = \alpha f(u)\{1 + (g(u))^2 + (h(u))^2\}^{1/2}.$$

Let φ denote the inverse function of $\psi : U \to L \subset \mathbf{R}$. Thus $u = \varphi(s)$, and we have

$$\varphi'(s) = \left\| \left. \frac{d\mathbf{x}(u)}{du} \right|_{u=\varphi(s)} \right\|^{-1}, \; s \in L.$$

Here the prime ($'$) denotes differentiation with respect to s.

Thus the unit tangent vector $\mathbf{e}_1(s)$ to the curve C at each point $\mathbf{x}(\varphi(s))$ is given by

$$\mathbf{e}_1(s) = \begin{bmatrix} \{1+(g(\varphi(s)))^2+(h(\varphi(s)))^2\}^{-1/2}\sin(\varphi(s)) \\ \{1+(g(\varphi(s)))^2+(h(\varphi(s)))^2\}^{-1/2}\cos(\varphi(s)) \\ \{1+(g(\varphi(s)))^2+(h(\varphi(s)))^2\}^{-1/2}g(\varphi(s)) \\ \{1+(g(\varphi(s)))^2+(h(\varphi(s)))^2\}^{-1/2}h(\varphi(s)) \end{bmatrix}, s \in L.$$

Hereafter, we use the following abbreviations for simple expression:

$$\sin := \sin(\varphi(s)), \quad \cos := \cos(\varphi(s)),$$

$$f := f(\varphi(s)), \quad g := g(\varphi(s)), \quad h := h(\varphi(s))$$

$$\dot{g} := \dot{g}(\varphi(s)) = \left.\frac{dg(u)}{du}\right|_{u=\varphi(s)}, \quad \dot{h} := \dot{h}(\varphi(s)) = \left.\frac{dh(u)}{du}\right|_{u=\varphi(s)},$$

$$\ddot{g} := \ddot{g}(\varphi(s)) = \left.\frac{d^2g(u)}{du^2}\right|_{u=\varphi(s)}, \quad \ddot{h} := \ddot{h}(\varphi(s)) = \left.\frac{d^2h(u)}{du^2}\right|_{u=\varphi(s)},$$

$$\varphi' := \varphi'(s) = \left.\frac{d\varphi(s)}{ds}\right|_{s},$$

$$A := 1+g^2+h^2, \quad B := g\dot{g}+h\dot{h}, \quad C := \dot{g}^2+\dot{h}^2$$

$$D := g\ddot{g}+h\ddot{h}, \quad E := \dot{g}\ddot{g}+\dot{h}\ddot{h}, \quad F := \ddot{g}^2+\ddot{h}^2.$$

Then we have

$$\dot{A} = 2B, \ \dot{B} = C+D, \ \dot{C} = 2E, \ \varphi' = \alpha^{-1}f^{-1}A^{-1/2},$$

and

$$\mathbf{e}_1 = \mathbf{e}_1(s) = \begin{bmatrix} A^{-1/2}\sin \\ A^{-1/2}\cos \\ A^{-1/2}g \\ A^{-1/2}h \end{bmatrix}.$$

In order to get the curvature functions k_1 and k_2, we go on with calculation. Now we have

$$\mathbf{e}_1' = \varphi' \begin{bmatrix} -\frac{1}{2}A^{-3/2}\dot{A}\sin + A^{-1/2}\cos \\ -\frac{1}{2}A^{-3/2}\dot{A}\cos - A^{-1/2}\sin \\ -\frac{1}{2}A^{-3/2}\dot{A}g + A^{-1/2}\dot{g} \\ -\frac{1}{2}A^{-3/2}\dot{A}h + A^{-1/2}\dot{h} \end{bmatrix} = \varphi' \begin{bmatrix} -A^{-3/2}B\sin + A^{-1/2}\cos \\ -A^{-3/2}B\cos - A^{-1/2}\sin \\ -A^{-3/2}Bg + A^{-1/2}\dot{g} \\ -A^{-3/2}Bh + A^{-1/2}\dot{h} \end{bmatrix}.$$

Thus we have

$$k_1 := k_1(s) = \|\mathbf{e}_1'\| = \varphi' A^{-1}(A + AC - B^2)^{1/2}.$$

Next, Since $\mathbf{e}_2 = (k_1)^{-1}\mathbf{e}_1'$, we have

$$\mathbf{e}_2 = \begin{bmatrix} -A^{-1/2}B(A+AC-B^2)^{-1/2}\sin + A^{1/2}(A+AC-B^2)^{-1/2}\cos \\ -A^{-1/2}B(A+AC-B^2)^{-1/2}\cos - A^{1/2}(A+AC-B^2)^{-1/2}\sin \\ -A^{-1/2}B(A+AC-B^2)^{-1/2}g + A^{1/2}(A+AC-B^2)^{-1/2}\dot{g} \\ -A^{-1/2}B(A+AC-B^2)^{-1/2}h + A^{1/2}(A+AC-B^2)^{-1/2}\dot{h} \end{bmatrix}.$$

After long process of calculation, we have

$$\mathbf{e}_2' + k_1 \cdot \mathbf{e}_1 = \varphi' A^{-3/2}(A+AC-B^2)^{-3/2} \begin{bmatrix} (P-Q)\sin - R\cos \\ (P-Q)\cos + R\sin \\ Pg - R\dot{g} + Q\ddot{g} \\ Ph - R\dot{h} + Q\ddot{h} \end{bmatrix},$$

where

$$\begin{aligned} P &= (A+AC-B^2)^2 + (A+AC-B^2)(B^2-AC-AD) \\ &\qquad + AB(B+AE-BD) \\ Q &= A^2(A+AC-B^2) \\ R &= A^2(B+AE-BD). \end{aligned}$$

Now, we have $P = A^2(1 + C + BE - D - CD)$. Thus we have

$$\mathbf{e}_2' + k_1 \cdot \mathbf{e}_1 = \varphi' A^{1/2}(A+AC-B^2)^{-3/2} \begin{bmatrix} (\tilde{P}-\tilde{Q})\sin - \tilde{R}\cos \\ (\tilde{P}-\tilde{Q})\cos + \tilde{R}\sin \\ \tilde{P}g - \tilde{R}\dot{g} + \tilde{Q}\ddot{g} \\ \tilde{P}h - \tilde{R}\dot{h} + \tilde{Q}\ddot{h} \end{bmatrix},$$

where

$$\begin{aligned} \tilde{P} &= 1 + C + BE - D - CD \\ \tilde{Q} &= A + AC - B^2 \\ \tilde{R} &= B + AE - BD. \end{aligned}$$

Consequently, we have

$$
\begin{aligned}
&\|\mathbf{e}_2' + k_1 \cdot \mathbf{e}_1\|^2 \\
&\quad = (\varphi')^2 A(A+AC-B^2)^{-3} \Big\{ \tilde{P}^2 - 2\tilde{P}\tilde{Q} + \tilde{Q}^2 + \tilde{R}^2 \\
&\qquad + \tilde{P}^2(g^2+h^2) + \tilde{R}^2(\dot{g}^2+\dot{h}^2) + \tilde{Q}^2(\ddot{g}^2+\ddot{h}^2) \\
&\qquad - 2\tilde{P}\tilde{R}(g\dot{g}+h\dot{h}) - 2\tilde{R}\tilde{Q}(\dot{g}\ddot{g}+\dot{h}\ddot{h}) + 2\tilde{P}\tilde{Q}(g\ddot{g}+h\ddot{h}) \Big\} \\
&\quad = (\varphi')^2 A(A+AC-B^2)^{-3} \{ A(1+C+BE-D-CD)^2 \\
&\qquad -2(1+C+BE-D-CD)(A+AC-B^2) + (A+AC-B^2)^2 \\
&\qquad +(B+AE-BD)^2 + C(B+AE-BD)^2 + F(A+AC-B^2)^2 \\
&\qquad -2B(1+C+BE-D-CD)(B+AE-BD) \\
&\qquad -2E(B+AE-BD)(A+AC-B^2) \\
&\qquad +2D(1+C+BE-D-CD)(A+AC-B^2) \} \\
&\quad = (\varphi')^2 A(A+AC-B^2)^{-3} \\
&\qquad \times \{ (A+AC-B^2)^2(1+F) \\
&\qquad\quad +2(A+AC-B^2)(D-1)(1+C+BE-D-CD) \\
&\qquad\quad -2(A+AC-B^2)E(B+AE-BD) \\
&\qquad\quad +A(1+C+BE-D-CD)^2 + (B+AE-BD)^2(1+C) \\
&\qquad\quad -2B(1+C+BD-D-CD)(B+AE-BD) \} .
\end{aligned}
$$

Now, calculating the last three terms of the above, we have

$$
\begin{aligned}
&A(1+C+BE-D-CD)^2 + (B+AE-BD)^2(1+C) \\
&\quad -2B(1+C+BD-D-CD)(B+AE-BD) \\
&\quad = (A+AC-B^2)(1+C-2D-2CD+D^2+CD^2 \\
&\qquad\qquad +2BE-2BDE+AE^2).
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
(k_2)^2 &= \|\mathbf{e}_2' + k_1 \cdot \mathbf{e}_1\|^2 \\
&= (\varphi')^2 A(A+AC-B^2)^{-2} \\
&\quad \times \{ (A+AC-B^2)(1+F) \\
&\qquad +2(D-1)(1+C+BE-D-CD) \\
&\qquad -2E(B+AE-BD) \\
&\qquad +1+C-2D-2CD+D^2+CD^2 \\
&\qquad +2BE-2BDE+AE^2 \} . \\
&= (\varphi')^2 A(A+AC-B^2)^{-2} \\
&\quad \times \{ (A+AC-B^2)(1+F) \\
&\qquad -1-C+2D+2CD-2BE \\
&\qquad -AE^2-D^2-CD^2+2BDE \} .
\end{aligned}
$$

Since we have $(k_1)^2 = (\varphi')^2 A^{-2}(A + AC - B^2)$, we have

$$
\begin{aligned}
&(k_1)^2 + (k_2)^2 \\
&\quad = (\varphi')^2 A^{-2}(A + AC - B^2)^{-2} \\
&\quad \times \big\{(A + AC - B^2)^3 \\
&\qquad +A^3(A + AC - B^2 + AF + ACF - B^2F - 1 - C + 2D \\
&\qquad\qquad +2CD - 2BE - AE^2 - D^2 - CD^2 + 2BDE)\big\}.
\end{aligned}
$$

By the fact: $\varphi' = \alpha^{-1} f^{-1} A^{-1/2}$, we have

$$
\begin{aligned}
&(k_1)^2 + (k_2)^2 \\
&\quad = \alpha^{-2} f^{-2} A^{-3}(A + AC - B^2)^{-2} \\
&\quad \times \big\{(A + AC - B^2)^3 \\
&\qquad +A^3(A + AC - B^2 + AF + ACF - B^2F - 1 - C + 2D \\
&\qquad\qquad +2CD - 2BE - AE^2 - D^2 - CD^2 + 2BDE)\big\}
\end{aligned}
$$

and

$$k_1 = \alpha^{-1} f^{-1} A^{-3/2}(A + AC - B^2)^{1/2}.$$

Thus, by setting

$$
\begin{aligned}
f \;=\; & A^{-3/2}(A + AC - B^2)^{-5/2} \\
& \times \big\{(A + AC - B^2)^3 \\
& \quad +A^3(A + AC - B^2 + AF + ACF - B^2F - 1 - C + 2D \\
& \qquad +2CD - 2BE - AE^2 - D^2 - CD^2 + 2BDE)\big\}.
\end{aligned}
$$

we have

$$k_1 = \alpha\{(k_1)^2 + (k_2)^2\}.$$

We can show a clear expression of the function f with g, h, $\dot{g}$, $\dot{h}$, $\cdots$, that is, we have

$$
\begin{aligned}
&A = 1 + g^2 + h^2, \\
&A + AC - B^2 = 1 + g^2 + h^2 + \dot{g}^2 + \dot{h}^2 + (g\dot{h} - \dot{g}h)^2
\end{aligned}
$$

and

$$
\begin{aligned}
& A + AC - B^2 + AF + ACF - B^2F - 1 - C \\
& \qquad + 2D + 2CD - 2BE - AE^2 - D^2 - CD^2 + 2BDE \\
& = (g + \ddot{g})^2 + (h + \ddot{h})^2 + \{(g\dot{h} - \dot{g}h) - (\dot{g}\ddot{h} - \ddot{g}\dot{h})\}^2 + (g\ddot{h} - \ddot{g}h)^2.
\end{aligned}
$$

Therefore, we have

$$\begin{aligned} f &= (1+g^2+h^2)^{-\frac{3}{2}} \times \Big(1+g^2+h^2+\dot{g}^2+\dot{h}^2+(g\dot{h}-\dot{g}h)^2\Big)^{-\frac{5}{2}} \\ &\quad \times\Bigg[\Big(1+g^2+h^2+\dot{g}^2+\dot{h}^2+(g\dot{h}-\dot{g}h)^2\Big)^3 \\ &\quad +(1+g^2+h^2)^3 \times \Big\{(g+\ddot{g})^2+(h+\ddot{h})^2 \\ &\qquad +\Big((g\dot{h}-\dot{g}h)-(\dot{g}\ddot{h}-\ddot{g}\dot{h})\Big)^2+(g\ddot{h}-\ddot{g}h)^2\Big\}\Bigg]. \end{aligned}$$

This completes the proof.

6. Examples of generalized Mannheim curves

(1) In Theorem 4.4, we set that $g(u)=\sinh(u)$, $h(u)=\cosh(u)$ for $u \in U \subset \mathbf{R}$ and α is a positive constant number. Then we have

$$f(u)=\frac{1+\cosh^2 u}{\sqrt{2}\cosh^2 u}.$$

Thus a curve C in E^4 is given by

$$\mathbf{x}: U \to E^4$$

and

$$\mathbf{x}(u)=\begin{bmatrix} \dfrac{\alpha}{\sqrt{2}}\displaystyle\int \frac{1+\cosh^2 u}{\cosh^2 u}\sin u\,du \\ \dfrac{\alpha}{\sqrt{2}}\displaystyle\int \frac{1+\cosh^2 u}{\cosh^2 u}\cos u\,du \\ \dfrac{\alpha}{\sqrt{2}}\displaystyle\int \frac{1+\cosh^2 u}{\cosh^2 u}\sinh u\,du \\ \dfrac{\alpha}{\sqrt{2}}\displaystyle\int \frac{1+\cosh^2 u}{\cosh^2 u}\cosh u\,du \end{bmatrix}, u \in U.$$

Let s be the arc-length parameter of the curve C. Then u is given by $u=\varphi(s)$, and we have

$$\left.\frac{du}{ds}\right|_s=\varphi'(s)=\frac{\cosh\varphi(s)}{\alpha(1+\cosh^2\varphi(s))},\ s \in L.$$

The parametric representation of C with the arc-length parameter s is given by $\mathbf{x}(s) = \mathbf{x}(\varphi(s))$, that is,

$$\mathbf{x}(s) = \begin{bmatrix} \dfrac{1}{\sqrt{2}} \displaystyle\int \dfrac{1}{\cosh g(s)} \sin g(s)\, ds \\ \dfrac{1}{\sqrt{2}} \displaystyle\int \dfrac{1}{\cosh g(s)} \cos g(s)\, ds \\ \dfrac{1}{\sqrt{2}} \displaystyle\int \dfrac{\sinh g(s)}{\cosh g(s)}\, ds \\ \dfrac{1}{\sqrt{2}} \displaystyle\int 1\, ds \end{bmatrix}, s \in L.$$

Then the curve C is a special Frenet curve in E^4 and

$$\begin{aligned} k_1(s) &= \frac{1}{\alpha(1+\cosh^2 g(s))}, \\ k_2(s) &= \frac{\cosh g(s)}{\alpha(1+\cosh^2 g(s))}, \\ k_3(s) &= \frac{1}{\alpha(1+\cosh^2 g(s))} \end{aligned}$$

for $s \in L$. We obtain $k_1(s) = \alpha\{(k_1)^2(s) + (k_2)^2(s)\}$, $s \in L$. We consider a regular smooth curve $\hat{C}$ defined by $\hat{\mathbf{x}}(s) = \mathbf{x}(s) + \alpha \cdot \mathbf{e}_2(s)$, $s \in L$, that is,

$$\hat{\mathbf{x}}(s) = \frac{1}{\sqrt{2}} \begin{bmatrix} \displaystyle\int \frac{1}{\cosh g(s)} \sin g(s)\, ds - \alpha \frac{\sinh g(s)}{\cosh g(s)} \sin g(s) + \alpha \cos g(s) \\ \displaystyle\int \frac{1}{\cosh g(s)} \cos g(s)\, ds - \alpha \frac{\sinh g(s)}{\cosh g(s)} \cos g(s) - \alpha \sin g(s) \\ \displaystyle\int \frac{\sinh g(s)}{\cosh g(s)}\, ds + \alpha \frac{1}{\cosh g(s)} \\ \displaystyle\int 1\, ds \end{bmatrix}.$$

We can prove that $\hat{C}$ is a special Frenet curve in E^4. Therefore, the curve C is a generalized Mannheim curve in E^4 and the curve $\hat{C}$ is a generalized Mannheim mate curve of C.

(2) In Theorem 4.4, we set $g(u) = a\sin(bu)$ and $h(u) = a\cos(bu)$ for $u \in U \subset \mathbf{R}$, where a is a positive constant number, and b is a constant number greater than one. Then the function f is of constant, that is,

$$f = \frac{1 + a^2 b^4}{(1 + a^2 b^2)^{3/2}}.$$

Thus a curve C in E^4 is defined by

$$\mathbf{x}(u) = \begin{bmatrix} \dfrac{\alpha(1+a^2b^4)}{(1+a^2b^2)^{3/2}} \displaystyle\int \sin(u)\,du \\ \dfrac{\alpha(1+a^2b^4)}{(1+a^2b^2)^{3/2}} \displaystyle\int \cos(u)\,du \\ \dfrac{\alpha a(1+a^2b^4)}{(1+a^2b^2)^{3/2}} \displaystyle\int \sin(bu)\,du \\ \dfrac{\alpha a(1+a^2b^4)}{(1+a^2b^2)^{3/2}} \displaystyle\int \cos(bu)\,du \end{bmatrix} = \begin{bmatrix} -\dfrac{\alpha(1+a^2b^4)}{(1+a^2b^2)^{3/2}} \cos(u) \\ \dfrac{\alpha(1+a^2b^4)}{(1+a^2b^2)^{3/2}} \sin(u) \\ -\dfrac{\alpha a(1+a^2b^4)}{b(1+a^2b^2)^{3/2}} \cos(bu) \\ \dfrac{\alpha a(1+a^2b^4)}{b(1+a^2b^2)^{3/2}} \sin(bu) \end{bmatrix}$$

for $u \in U$. Let s be the arc-length parameter of C. Since we have

$$u = \frac{(a+a^2b^2)^{3/2}}{\alpha(1+a^2)^{1/2}(1+a^2b^4)} s,$$

the arc-length parametrization of C is given by

$$\mathbf{x}(s) = \begin{bmatrix} -\dfrac{\alpha(1+a^2b^4)}{(1+a^2b^2)^{3/2}} \cos\left(\dfrac{(a+a^2b^2)^{3/2}}{\alpha(1+a^2)^{1/2}(1+a^2b^4)} s\right) \\ \dfrac{\alpha(1+a^2b^4)}{(1+a^2b^2)^{3/2}} \sin\left(\dfrac{(a+a^2b^2)^{3/2}}{\alpha(1+a^2)^{1/2}(1+a^2b^4)} s\right) \\ -\dfrac{\alpha a(1+a^2b^4)}{b(1+a^2b^2)^{3/2}} \cos\left(\dfrac{b(a+a^2b^2)^{3/2}}{\alpha(1+a^2)^{1/2}(1+a^2b^4)} s\right) \\ \dfrac{\alpha a(1+a^2b^4)}{b(1+a^2b^2)^{3/2}} \sin\left(\dfrac{b(a+a^2b^2)^{3/2}}{\alpha(1+a^2)^{1/2}(1+a^2b^4)} s\right) \end{bmatrix}$$

for $s \in L$. Then C is a special Frenet curve and

$$\begin{aligned} k_1(s) &= \frac{(1+a^2b^2)^2}{\alpha(1+a^2)(1+a^2b^4)}, \\ k_2(s) &= \frac{a(b^2-1)(1+a^2b^2)}{\alpha(1+a^2)(1+a^2b^4)}, \\ k_3(s) &= \frac{b(1+a^2b^2)}{\alpha(1+a^2b^4)} \end{aligned}$$

for $s \in L$. The curvature functions k_1, k_2, k_3 are constant functions, and it holds that $k_1 = \alpha\{(k_1)^2 + (k_2)^2\}$. We define a smooth curve $\hat{C}$ by $\hat{\mathbf{x}}(s) = \mathbf{x}(s) + \alpha \cdot \mathbf{e}_2(s)$. Let $\hat{s}$ be arc-length parameter of the curve $\hat{C}$. Here we notice that

$$s = \frac{(1+a^2)^{1/2}(1+a^2b^4)^{1/2}}{a(b^2-1)} \hat{s}.$$

The parametric representation of the curve $\hat{C}$ is given by

$$\hat{\mathbf{x}} : \hat{L} \to E^4$$

and

$$\hat{\mathbf{x}}(\hat{s}) = \begin{bmatrix} -\dfrac{\alpha a^2 b^2 (b^2-1)}{(1+a^2b^2)^{3/2}} \cos\left(\dfrac{(1+a^2b^2)^{3/2}}{\alpha a (b^2-1)(1+a^2b^4)^{1/2}}\hat{s}\right) \\ \dfrac{\alpha a^2 b^2 (b^2-1)}{(1+a^2b^2)^{3/2}} \sin\left(\dfrac{(1+a^2b^2)^{3/2}}{\alpha a (b^2-1)(1+a^2b^4)^{1/2}}\hat{s}\right) \\ \dfrac{\alpha a (b^2-1)}{b(1+a^2b^2)^{3/2}} \cos\left(\dfrac{b(1+a^2b^2)^{3/2}}{\alpha a (b^2-1)(1+a^2b^4)^{1/2}}\hat{s}\right) \\ -\dfrac{\alpha a (b^2-1)}{b(1+a^2b^2)^{3/2}} \sin\left(\dfrac{b(1+a^2b^2)^{3/2}}{\alpha a (b^2-1)(1+a^2b^4)^{1/2}}\hat{s}\right) \end{bmatrix}$$

for $\hat{s} \in \hat{L}$. Consequently, the curve $\hat{C}$ is a special Frenet curve in E^4 and there exists a bijection $\phi : C \to \hat{C}$ such that $\phi(\mathbf{x}(s)) = \hat{\mathbf{x}}(\hat{s})$, where $\hat{s} = \dfrac{a(b^2-1)}{(1+a^2)^{1/2}(1+a^2b^4)^{1/2}}\, s$. And the first normal line at each point of C coincides the third normal line of $\hat{C}$ at corresponding point under ϕ. Therefore the curve C is a generalized Mannheim curve in E^4.

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