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SUBDIFFERENTIAL INVERSE PROBLEMS FOR MAGNETOHYDRODYNAMICS*

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Abstract. The theory of solvability of an abstract evolution inequality in a Hilbert space for the operators with the quadratic nonlinearity is presented. It is then applied for the study of an inverse problem for MHD flows. For the three-dimensional flows the global in time existence of the weak solutions to the inverse problem is proved. For the two-dimensional flows existence and uniqueness of the strong solutions are proved.

Key words. Equations of magnetohydrodynamics, variational inequalities, inverse problems.

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1. Inverse Problem for MHD. The flow of a homogeneous viscous incompressible conducting fluid in a bounded domain $\Omega \subset \mathbb{R}^d$, where $d = 2$ or 3, with connected boundary $\Gamma = \partial\Omega$ is described by the magnetohydrodynamic (MHD) equations in dimensionless variables:

$$\partial u/\partial t - \nu\Delta u + (u\nabla)u = -\nabla p + S \cdot \operatorname{rot} B \times B,\ x \in \Omega,\ t > 0, \tag{1}$$

$$\partial B/\partial t + \operatorname{rot} E = 0,\ j = \operatorname{rot} B = 1/\nu_m(E + u \times B + \sum_{i=1}^{m} \alpha_i(t)E_i), \tag{2}$$

$$\operatorname{div} u = 0,\ \ \operatorname{div} B = 0. \tag{3}$$

Here u, B, E and j are vector fields of velocity, magnetic induction, electric intensity and current density respectively; p is a flow pressure, $\nu = 1/Re$. $\nu_m = 1/R_m$, $S = M^2/Re\,R_m$, where Re,Re_m and M are the Reynolds number, Reynolds magnetic number and Hartman number. $E_i = E_i(x)$ – the given external electric fields. The functions $\alpha_i = \alpha_i(t)$, $i = 1, ..., m$ are considered as a controls.

In the two-dimensional case, the current density, electric field, and the expressions $\operatorname{rot} B$ and $u \times B$ are scalar quantities; in addition,

$$\operatorname{rot} B = \partial B_2/\partial x_1 - \partial B_1/\partial x_2,\ \ u \times B = Z(u) \cdot B,$$

$$\operatorname{rot} B \times v = \operatorname{rot} B\ Z(v),\ \operatorname{rot} E = -Z(\nabla E).$$

Here $Z(v) = \{-v_2, v_1\}$ is the rotation of the vector $\{v_1, v_2\}$ by $\pi/2$.

We supplement equations (1)–(3) with the initial conditions

$$u|_{t=0} = u_0(x),\ B|_{t=0} = B_0(x),\ x \in \Omega \tag{4}$$

and the conditions on the boundary Γ of the flow domain,

$$u = 0,\ B \cdot n = 0,\ n \times E = 0\ \ (x, t) \in \Gamma \times (0, T), \tag{5}$$

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where n is the unit outward normal on the Γ.

Let us consider the following inverse problem for the model (1)–(5).

Find the functions $\alpha_i = \alpha_i(t)$, $t \in (0,T)$, $i = 1,...,m$ *and the corresponding solution* $y = \{u, B\}$ *of the (1)–(5) under additional conditions*

$$\text{(6)}\quad \alpha_i(t) \geq 0, \int_\Omega \operatorname{rot} B \cdot E_i\, dx \geq q_i(t), \quad \alpha_i(t)(\int_\Omega \operatorname{rot} B \cdot E_i\, dx - q_i(t)) = 0, \ t \in (0,T).$$

Here the functions q_i, E_i *and the initial conditions* u_0, B_0 *are given.*

Note that the quantity $\int_\Omega \operatorname{rot} B \cdot E_i\, dx$ is proportional to the work of the external electric field E_i on conduction currents $j = \operatorname{rot} B$ per unit time. In fact, the non-local conditions (6) describe the control of electric field power by dynamic change of current amplitudes.

Classical boundary value problems for system (1)–(3) were considered in [1]. Subdifferential boundary value problems for hydrodynamic equations and Maxwell equations were investigated in [2]–[4]. The existence and uniqueness of the solution of the Problem (1)–(6) will be proved on the basis of the development of the theory of abstract evolution equations and Navier – Stokes inequalities [5]–[7].

The main results of this paper are the global in time existence theorem for the three-dimensional inverse problem and the existence and uniqueness of the strong solutions in the two-dimensional case.

The outline of the paper is as follows. In Section 2 the subdifferential inverse problem for the abstract Navier – Stokes system is stated. In Section 3 we give the functional setting for MHD equations and prove the existence and uniqueness theorems. In Section 4 the sketch of abstract theorems proving is presented.

2. Subdifferential inverse problem for Navier–Stokes system. Let V and H be real separable Hilbert spaces with the norms denoted by $\|\cdot\|$ and $|\cdot|$. V is dense in H, embedding of V in H is compact and

$$V \subset H = H' \subset V',$$

where H' and V' are dual spaces of H and V. $(\cdot,\cdot)$ denotes the pairing between V and V' and the scalar product in H.

Consider a linear continuous operator $A : V \to V'$ and a bilinear operator $\mathcal{B}(u,v)$: $V \times V \to V'$ such as

$$\text{(7)}\qquad (Av, v) \geq \alpha\|v\|^2,\ \alpha > 0, \ \ (Av, w) = (Aw, v) \ \ \forall v, w \in V;$$

$$\text{(8)}\qquad \mathcal{B}[y] = \mathcal{B}(y,y), \quad (\mathcal{B}(u,v), v) = 0 \ \forall u, v \in V;$$

Let $\{Q_i\}$, $i = \overline{1,m}$ be a linearly independent system in V'. Consider an evolution equation

$$\text{(9)}\qquad y' + Ay + \mathcal{B}[y] = f + \sum_{i=1}^m \alpha_i(t) Q_i, \qquad t \in (0,T)$$

under initial condition

$$\text{(10)}\qquad y(0) = y_0.$$

Here $y' = dy/dt$.

Problem P. *Find the functions* $\alpha_i = \alpha_i(t)$, $t \in (0,T)$, $i = 1,...,m$ *and the corresponding solution* y *of the (9)–(10) under additional conditions*

(11) $\alpha_i(t) \geq 0$, $(Q_i, y(t)) \geq q_i(t)$, $\alpha_i(t)((Q_i, y(t)) - q_i(t)) = 0$, $t \in (0,T)$, $i = 1,...,m$.

Here $f \in V'$, *the functions* q_i *and the initial value* y_0 *are given.*

2.1. Transformation of Problem P. Let $\{z_i\}$, $i = \overline{1,m}$ be an appropriate biorthogonal system in the space V, $(Q_i, z_k) = \delta_{ik}$. Now we set

$$r(t) = \sum_{i=1}^{m} q_i(t) z_i, \quad K = \{z \in V : (Q_i, z) \geq 0, \quad i = \overline{1,m}\,\}.$$

Denote by Φ the indicator function of K,

$$\Phi(y) = \begin{cases} 0, & \text{if } y \in K, \\ +\infty, & \text{otherwise}\ . \end{cases}$$

Note that Φ is a convex on V and weakly lower semicontinuous.

Let $z - r(t) \in K$, $t \in (0,T)$. We multiply equation (9) by $(y - z)$ and use the conditions (11) to obtain after some calculation that

$$(y' + Ay + \mathcal{B}[y] - f, y - z) \leq 0.$$

Thus, if y is a solution of Problem P then

$$(y' + Ay + \mathcal{B}[y] - f, y - z) + \Phi(y - r) - \Phi(z - r) \leq 0$$

and we get the Cauchy Problem for evolution equation with multivalued operator (variational inequality),

$$f \in y' + Ay + \mathcal{B}[y] + \partial\Phi(y - r), \quad y(0) = y_0. \tag{12}$$

Conversely, if for some $\xi \in V'$ we have $(-\xi) \in \partial\Phi(y - r)$ then this implies [8],[9]

$$\xi = \sum_{1}^{m} \alpha_i(t) Q_i, \quad \alpha_i(t) = (\xi, z_i),$$

where $\alpha_i \geq 0$, $(Q_i, y) \geq q_i$, $\alpha_i(Q_i, y) - q_i) = 0$, $i = 1,...,m$.

2.2. Solvability of Problem P. Let $L^s(0,T;X)$, $1 \leq s \leq \infty$ (respectively $C(0,T;X)$) denote the space of s – summable (respectively continuos) functions from $[0,T]$ to X. We denote the space of distributions on $(0,T)$ by $\mathcal{D}'(0,T)$ and the usual Sobolev space by W_s^l.

Define the functional

$$G(y) = \begin{cases} \int_0^T \Phi(y(t))dt, & \text{if } \ \Phi(y(\cdot)) \in L^1(0,T), \\ +\infty & \text{else.} \end{cases}$$

Definition 1. The set of functions $\alpha_i \in \mathcal{D}'(0,T)$, $i = 1,...,m$ and $y \in L^2(0,T;V)$ is called *weak solution* to the Problem P, if

$$G(y - r) < +\infty, \ \alpha_i = (f - y' - Ay - \mathcal{B}[y], z_i)$$

and following inequality holds

$$\int_0^T (z' + Ay + \mathcal{B}[y] - f, y - z)dt + G(y - r) - G(z - r) \le \frac{|y_0 - z(0)|^2}{2} \tag{13}$$

for all z such that $z \in L^2(0, T; V),\ z' \in L^2(0, T; V')$.

Definition 2. The set of functions $\alpha_i \in L^2(0, T),\ i = 1, ..., m$ and $y \in C([0, T]; V)$ is called *strong solution* to the Problem P, if $y(0) = y_0$,

$$y' \in L^2(0, T; V) \cap L^\infty(0, T; H),\ \ \alpha_i = (f - y' - Ay - \mathcal{B}[y], z_i),$$

and

$$f(t) - (y'(t) + Ay(t) + \mathcal{B}[y(t)]) \in \partial\Phi(y(t) - r(t)) \text{ a.e. on } (0, T). \tag{14}$$

We have the following results on the solvability of Problem P.

Theorem 1. Let

$$r \in L^2(0, T; V);\ f,\ r' \in L^2(0, T; V'); \tag{15}$$

$$y_0 - r(0) \in \overline{K}^H = \text{ closure of } K \text{ in the norm of } H, \tag{16}$$

and

$$|(\mathcal{B}(w, v), w)| \le k_1 \|w\|^{1+\theta} \cdot |w|^{1-\theta} \cdot \|v\|, \tag{17}$$

where $\theta \in [0, 1), k_1 > 0$ are constants independent of $v, w \in V$. Then there exists a weak solution of Problem P.

Let U and H_0 be real separable Hilbert spaces, let U be continuously and densely embedded in V , let $H \subset H_0$, let the norm in H_0 be equivalent to the norm in H, and in addition, let $Az + \mathcal{B}[z] \in H_0$ whenever $z \in U$,

$$|Az + \mathcal{B}[z]| \le k_2(1 + \|z\|_U^2), \tag{18}$$

where $k_2 > 0$ is independent of $z \in U$.

Theorem 2. Let $g = f - r' - Ar - \mathcal{B}[r],\ f_1 = f - r' - Ar$, and

$$\begin{aligned} &y_0 - r(0) \in U \cap K,\ g(0) \in H_0,\ f, f' \in L^2(0, T; V'), \\ &r' \in L^2(0, T; V) \cap L^\infty(0, T; H),\ f_1' \in L^2(0, T; V'); \end{aligned} \tag{19}$$

$$|(B(w, v), w)| \le k_3 \|w\|^{1+\theta} \cdot |w|^{1-\theta} \cdot \|v\|^\gamma \cdot |v|^{1-\gamma}, \tag{20}$$

where $\theta, \gamma \in [0, 1/2]$ and $k_3 > 0$ are constants independent of $v, w \in V$. Then problem P has exactly one strong solution.

The solvability of variational inequalities associated with nonlinear boundary value problems for equations of magnetohydrodynamics was proved in [6],[7]. In the study of inverse problems, convex set restrictions on function y depends on time. Theorems 1 and 2 above improve the results in [6] and [7] for the case of $r \ne 0$. The sketch of the proofs of the two theorems will be given in Section 4.

3. Solvability of Inverse MHD Problem. In the sequel, without loss of generality, we set $S = 1$ in the equations (1). Otherwise we can reduce to the case by introducing new functions $B := \sqrt{S}B$, $E := \sqrt{S}E$, and $E_i := \sqrt{S}E_i$.

3.1. Spaces and operators for MHD. Consider the following spaces of vector functions defined in a bounded domain $\Omega \in \mathbb{R}^d$ with connected boundary $\Gamma \in C^2$, $d = 2, 3$:

$$\mathcal{U}_1 = \{v \in C^\infty(\bar{\Omega}) : \text{div } v = 0,\ x \in \Omega,\ v = 0,\ x \in \Gamma\},$$

$$\mathcal{U}_2 = \{v \in C^\infty(\bar{\Omega}) : \text{div } v = 0,\ x \in \Omega,\ n \cdot v = 0,\ x \in \Gamma\}.$$

The Hilbert spaces V_1 and V_2 are defined as the closures of the spaces $\mathcal{U}_1$ and $\mathcal{U}_2$ in the norm of $W_2^1(\Omega)$, and the spaces H_1 and H_2 are defined as the closures of $\mathcal{U}_1$ and $\mathcal{U}_2$ in the norm of $L^2(\Omega)$. In fact, $H_1 = H_2$. The inner products in the spaces H_1 and H_2 and in the spaces V_1 and V_2 are given by the relations

$$(u, v)_0 = \int\limits_\Omega (u \cdot v)dx,\ ((u, v)) = (\text{rot}\, u, \text{rot}\, v)_0 = \int\limits_\Omega (\text{rot}\, u \cdot \text{rot}\, v)dx \quad \forall u, v \in V_1, V_2$$

respectively. The norm of the spaces V_1 and V_2 given by the inner product $((u, v))$ is equivalent to the norm of the space $W_2^1(\Omega)$. Let

$$V = V_1 \times V_2,\ \ H = H_1 \times H_2,\ \ V \subset H = H' \subset V'.$$

These embeddings are dense and continuous. The norms of the spaces V and H are denoted by $\|\cdot\|$ and $|\cdot|$, respectively; $(\cdot, \cdot)$ is the duality between V' and V and the inner product in H. If $y = \{u, B\}$ and $z = \{v, w\}$, then

$$(y, z) = (u, v)_0 + (B, w)_0,\ (y, z)_V = ((u, v)) + ((B, w)).$$

Navier–Stokes operators. We define mappings $A : V \to V'$ and $\mathcal{B} : V \times V \to V'$ by the relations

$$(Ay, z) = \nu((u, v)) + \nu_m((B, w)),$$

$$(\mathcal{B}(y_1, y_2), z) = ((u_1 \cdot \nabla)u_2 - \text{rot}\, B_2 \times B_1, v)_0 - (u_2 \times B_1, \text{rot}\, w)_0,$$

which are valid for arbitrary $y = \{u, B\}$, $y_1 = \{u_1, B_1\}$, $y_2 = \{u_2, B_2\}$, $z = \{v, w\}$ in the space V.

Note that the operator A satisfies the conditions (7). The mappings $\mathcal{B}(y, z)$ and $\mathcal{B}[y] = \mathcal{B}(y, y)$ satisfy the relations $(\mathcal{B}(y, z), z) = 0$,

$$(\mathcal{B}[y], z) = (\text{rot}\, u \times u, v)_0 - (\text{rot}\, B \times B, v)_0 - (u \times B, \text{rot}\, w)_0.$$

As a consequence of the multiplicative inequality

$$\|f\|_{L^4(\Omega)} \leq K\|f\|^{d/4}_{W_2^1(\Omega)} \cdot \|f\|^{1-d/4}_{L^2(\Omega)},$$

in the domain $\Omega \subset \mathbb{R}^d$, we have the estimate

$$(\mathcal{B}(y, z), y) \leq C\|z\| \cdot \|y\|^{1+d/4} \cdot |y|^{1-d/4}, \tag{21}$$

where $C > 0$ is independent of $y, z \in V$. If $d = 2$, then we have the stronger inequality

$$(\mathcal{B}(y,z),y) \le C\|z\|^{1/2} \cdot |z|^{1/2} \cdot \|y\|^{3/2} \cdot |y|^{1/2}. \tag{22}$$

Thus the defined mapping $\mathcal{B}$ satisfies the conditions (17) and, if $d = 2$, the condition (20).

Let us consider the given vector – functions $E_i \in W_2^1(\Omega)$, where $n \times E_i = 0$ on Γ, $i = 1, 2, ..., m$. We define the functionals $Q_i \in V'$ by the relations

$$(Q_i, z) = (\operatorname{rot} E_i, w)_0 = (E_i, \operatorname{rot} w)_0,$$

if $z = \{v, w\} \in V$. Now we denote by $\Phi(y)$ the indicator function of the set K, where $K = \{z \in V : (Q_i, z) \le 0, \quad i = \overline{1, m}\ \}$.

3.2. An analysis of the problem (1)-(6). Let $y = \{u, B\}$ be a sufficiently smooth solution of nonlocal unilateral problem (1) – (6), and let $y_0 = \{u^0, B^0\}$. Let the system of functions $\{\operatorname{rot} E_i,\ i = \overline{1,m}\}$ be linearly independent in the space H_2. We choose an arbitrary element $z = \{v, w\} \in V$, multiply equation (1) by $(v - u)$ and equation (2) by $(w - B)$, and integrate by parts over the domain Ω with the use of boundary conditions for the velocity, electric and magnetic fields, and test functions v and w. By adding the resulting relations and by taking into account the condition (6), we obtain the inequality

$$(y' + Ay + \mathcal{B}[y], z - y) + \Phi(z - r) - \Phi(y - r) \ge 0, \tag{23}$$

where $r(t) \in V$ given by the relation $r = \{0, \sum_{i=1}^m q_i(t) w_i\}$. Here the system of functions $\{w_i,\ i = \overline{1,m}\}$ is biorthogonal to the system $\{-\operatorname{rot} E_i,\ i = \overline{1,m}\}$ in the space $L^2(\Omega)$.

Conversely, consider an element $y = \{u, B\}$ that is a sufficiently smooth solution of the variational inequality (23). We set $z = \{u \pm v, B\}$, where $v \in C_0^\infty(\Omega)$ and $\operatorname{div} v = 0$. Then it follows from (23) that

$$(u', v)_0 + \nu(\operatorname{rot} u, \operatorname{rot} v)_0 + ((u \cdot \nabla)u - \operatorname{rot} B \times B, v)_0 = 0. \tag{24}$$

Relation (24), together with the condition $\operatorname{div} u = 0$, implies that

$$u' + \nu\Delta u + (u \cdot \nabla)u - \operatorname{rot} B \times B = -\nabla p, \tag{25}$$

for some function p. The boundary conditions for u follow from the inclusion $u(\cdot, t) \in V_1$.

Next we set $z = \{u, \widetilde{w}\}$ in (23), where function $\widetilde{w}(\cdot, t) \in V_2$ satisfy the conditions $(\operatorname{rot} E_i, \widetilde{w})_0 \ge q_i(t), \quad i = 1, ..., m$. By the structure of functional Φ we obtain the inequalities

$$(\operatorname{rot} E_i, B)_0 \ge q_i(t), \ \ i = 1, ..., m,$$

$$(B', \widetilde{w} - B)_0 + (\nu_m \operatorname{rot} B - u \times B, \operatorname{rot}(\widetilde{w} - B))_0 \ge 0 \tag{26}$$

Then we obtain from variational inequality (26) the relation

$$(B', w)_0 + (\nu_m \operatorname{rot} B - u \times B, \operatorname{rot} w)_0 = \sum_1^m \alpha_i(t)(\operatorname{rot} E_i, w)_0 \ \ \forall w \in V_2. \tag{27}$$

Here $\alpha_i \geq 0$ and $(\operatorname{rot} E_i, B)_0 - q_i(t))\alpha_i(t) = 0$.

Now, we show that equality (27) still hold if $w \in C_0^\infty(\Omega)$. Indeed, if $\operatorname{div} w \neq 0$, we consider the scalar function ϕ such that

$$\Delta\phi = \operatorname{div} w \text{ in } \Omega, \ \frac{\partial \phi}{\partial n} = 0 \text{ on } \Gamma.$$

Then $\widehat{w} = w - \nabla\phi \in V_2$ and $\operatorname{rot} w = \operatorname{rot} \widehat{w}$. The condition $\operatorname{div} B = 0$ imply that $(B, w)_0 = (B, \widehat{w})_0$. Hence, for each $w \in C_0^\infty(\Omega)$ we have the equality (27). Setting

$$E = \nu_m \operatorname{rot} B - u \times B - \sum_1^m \alpha_i(t) E_i,$$

integrating by parts in (27) we get the equations (2). It follows from the first equation (2) and (27) that $n \times E = 0$ on Γ.

Thus, the Problem (1)-(6) is reduced to an abstract variational inequality (12) which is equivalent of Problem P. Therefore, a *weak* (respectively, *strong*) solution of Problem (1)-(6) is defined as a *weak* (respectively, *strong*) solution of the Problem P, where spaces and operators defined in the Section 3.1.

As a consequence of the theorems 1,2, we have a following result.

THEOREM 3. Let

$$u_0 \in H_1, \ B_0 \in H_2, \ E_i \in W_2^1(\Omega), \ n \times E_i|_\Gamma = 0, \ i = 1, ..., m,$$

and let the system of vortices $\{\operatorname{rot} E_i, \ i = \overline{1, m}\}$ be linearly independent in the space H_2,

$$q_i \in W_2^1(0, T), \int_\Omega \operatorname{rot} E_i \cdot B_0 \, dx \geq q_i(0), \ i = 1, ..., m.$$

Then there exists a weak solution of Problem (1)-(6). If d = 2 and, in addition,

$$u_0 \in W_2^2(\Omega) \cap V_1, \ B_0 \in W_2^2(\Omega) \cap V_2, \ (n \times \operatorname{rot} B_0)|_\Gamma = 0, \ q_i \in W_2^2(0, T), \tag{28}$$

then the weak solution is strong and unique.

Proof. Let us verify the validity of the assumptions of Theorems 1 and 2 for Problem (1)-(6). Just now we note that the operators A and $\mathcal{B}$ defined in Section 3.1 satisfy conditions (7), (8), f = 0, and the estimates (21) and (22) imply that conditions (17) and (20) hold. In addition, to prove the existence of a unique strong solution, we set $U = W_2^2(\Omega) \cap V$. Then $\mathcal{B}[g] \in H_0 = L^2(\Omega) \times L^2(\Omega)$ for all $g \in U$. If $z = \{v, w\} \in H$ and $y_0 = \{u_0, B_0\}$ satisfies condition (28), then

$$(Ay_0, z) = -\nu(\Delta u_0, v)_0 - \nu_m(\Delta B_0, w)_0 - \nu_m \int_\Gamma (n \times \operatorname{rot} B_0) w d\Gamma.$$

Therefore, it follows from (28) that condition (19) is valid for Theorem 2.

4. Proof of Theorems 1 and 2. In this section we will prove two solvability theorems. Note that the proof is valid for the variational inequality (12) with arbitrary convex lower semicontinuous functional Φ, $\Phi \not\equiv +\infty$, with an effective domain K on which Φ is continuous.

Proof of Theorem 1. Let

$$\Phi_\lambda(u) = \inf\{\frac{\|u-v\|^2}{2\lambda} + \Phi(v); v \in V\}, u \in V, \lambda > 0.$$

The Fréchet derivative of Φ_λ coincides with the Yosida approximation to the multimapping $u \to \partial\Phi(u)$,

$$\nabla\Phi_\lambda = \frac{1}{\lambda}J(I - J_\lambda); \quad J_\lambda = (I + \lambda J^{-1}\partial\Phi)^{-1}.$$

Here I is the identity operator, $J : V \to V'$ is the duality mapping, and $v^* = Jv$, if $(v^*, v) = \|v\|^2$. In addition, we have the relations [9]

(29)

$$\Phi_\lambda(w) = \frac{1}{2\lambda}\|w - J_\lambda w\|^2 + \Phi(J_\lambda w);\ \Phi(J_\lambda w) \le \Phi_\lambda(w) \le \Phi(w);\ \lim_{\lambda\to 0}\Phi_\lambda(w) = \Phi(w).$$

Throughout the following, without loss of generality, we assume that $w_0 = y_0 - r(0) \in K$ and

$$\Phi(w) \ge \Phi(w_0) \quad \forall w \in V. \tag{30}$$

Indeed, in this case, if inequality (30) fails, then one can always replace the functional Φ by the functional $\Phi_1(w) = \Phi(w) - (\chi, w - w_0)$, $\chi \in \partial\Phi(w_0)$, by adding the subgradient χ to the right-hand side of the inclusion (12).

In V we choose a complete system of elements $\{v_1, v_2, ...\}$, $V = \overline{\bigcup V_m}$. Here V_m is the subspace spanned by the system $\{v_1, ..., v_m\}$. For now, we suppose that

$$w_0 \in V_{m_0}, w_0 = \sum_1^{m_0} g_j^0 v_j, \quad r(t) \in V_{m_0}, r(t) = \sum_1^{m_0} h_j(t) v_j. \tag{31}$$

Consider the Galerkin approximation $w_m(t)$ to the function $w = y - r$, where y is a solution of inequality (12),

$$w_m(t) = \sum_1^m g_{jm}(t) v_j, \quad m = 1, 2, \dots ,$$

$$(w_m' + Aw_m + \mathcal{B}(w_m, r) + \mathcal{B}(w_m + r, w_m) + \nabla\Phi_\lambda(w_m) - g, v_j) = 0, \tag{32}$$

$$j = \overline{1, m}, \quad w_m(0) = w_0.$$

Here $g = f - r' - Ar - \mathcal{B}[r]$.

We obtain estimates for a solution of the system of ordinary differential equations (32), which permits one to obtain the variational inequality (12) from (32) in the limit as $m \to +\infty$ and $\lambda \to 0$. We multiply (32) by $(g_{jm} - g_j^0)$ and sum the resulting relation with respect to j from 1 to $m > m_0$. Then

$$\frac{1}{2} \cdot \frac{d}{dt}|w_m - w_0|^2 + (Aw_m + \mathcal{B}(w_m, r) + \mathcal{B}(w_m + r, w_m) +$$

$$+\nabla\Phi_\lambda(w_m) - g, w_m - w_0) = 0. \tag{33}$$

By taking into account relations (7), (8), and (17), the monotonicity of the gradient $\nabla\Phi_\lambda$ and condition (30), from (33), one can readily obtain the inequality

$$\frac{d}{dt}|w_m - w_0|^2 + \nu\|w_m - w_0\|^2 \le C_1(1 + |w_m - w_0|^2). \tag{34}$$

Here and throughout the following, $C_1, C_2, ...$ are positive constants independent of m and λ. The estimates

$$|w_m|^2 \le C_2, \ \int_0^T \|w_m(t)\|^2 dt \le C_3. \tag{35}$$

are a consequence of inequality (34) and the Gronwall inequality. These estimates, together with (33) and the relation

$$\Phi_\lambda(w_0) - \Phi_\lambda(w_m) \ge (\nabla\Phi_\lambda(w_m), w_0 - w_m),$$

imply that

$$\int_0^T (\Phi_\lambda(w_m) - \Phi(w_0))dt \le \int_0^T (\Phi_\lambda(w_m) - \Phi_\lambda(w_0))dt \le C_4.$$

Then, on the basis of the regularization properties (32), we have the estimates

$$\int_0^T \|w_m - J_\lambda w_m\|^2 dt \le C_5\lambda, \quad \int_0^T \Phi_\lambda(w_m)dt \le C_6, \quad \int_0^T \Phi(J_\lambda w_m)dt \le C_7. \tag{36}$$

Let us show that w_m is compact in $L^2(0, T; H)$. By multiplying (32) by $(g_{jm}(t) - g_{jm}(s)),\ s \in (0, T)$ and by summing the resulting relation with respect to $j = 1, ..., m$, we obtain

$$\frac{1}{2} \cdot \frac{d}{dt}|w_m(t) - w_m(s)|^2 + (Aw_m(t) + B(w_m(t), z(t)) +$$

$$+B(w_m(t) + z(t), w_m(t)) - g(t), w_m(t) - w_m(s)) =$$

$$= (\nabla\Phi_\lambda(w_m(t)), w_m(s) - w_m(t)) \le \Phi_\lambda(w_m(s)) - \Phi_\lambda(w_m(t)) \le$$

$$\le \Phi_\lambda(w_m(s)) - \Phi(J_\lambda w_m(s)) \le \Phi_\lambda(w_m(s)) - \Phi(w_0).$$

By integrating the last inequality with respect to t on the interval $(s, s+h)$ and with respect to s on $(0, T-h)$ and by using the estimates (35) and (36) and condition (30), we estimate the equicontinuity of the sequence $w_m(t)$ as

$$\int_0^{T-h} |w_m(s+h) - w_m(s)|^2 ds \le C_8 h^{\frac{1-\theta}{2}}. \tag{37}$$

It follows from the estimates (35)-(37) that there exists an element $w \in L^2(0, T; V) \cap L^\infty(0, T; H)$ and a subsequence $w_{m'}, \lambda' \to 0$, such that $w_{m'} \to w$ weakly in

$L^2(0,T;V)$, $*$ - weakly in $L^\infty(0,T;H)$, strongly in $L^2(0,T;H)$ as $m' \to \infty, \lambda' \to 0$. From the last, we find that $J_{\lambda'} w_{m'} \to w$ weakly in $L^2(0,T;V)$; therefore, $G(w) < +\infty$.

We take $z(t) = \sum_1^M c_j(t) v_j$, where $M > 0$ is a fixed number, and $c_j(t) \in C^1[0,T]$. By multiplying (32) by $(g_{jm}(t) + h_j(t) - c_j(t))$ and by integrating the resulting relation by parts on $(0,T)$, we obtain the inequality

$$\int_0^T \{(z' + Ay_m + \mathcal{B}[y_m] - f, y_m - z) + \Phi_\lambda(w_m) - \Phi_\lambda(z - r)\}dt \le \frac{|y_0 - z(0)|^2}{2}. \tag{38}$$

Here $y_m = w_m + r$. Let $y = w + r$. Results of convergence for sequence w_m and properties (29) permit one to obtain from (38) the variational inequality (13), which is valid for an arbitrary function z such that $z \in L^2(0,T;V), z' \in L^2(0,T;V')$ since the system $\{\sum_1^M c_j(t) v_j, M \in \mathbb{N}\}$ is dense in the above-mentioned space. For an arbitrary element $w_0 \in \overline{K}^H$ and for function $r \in L^2(0,T;V)$, $r' \in L^2(0,T;V')$ one can consider their approximations by elements $w_0^l \in K$ and by functions $r_l(t) = \sum_1^{m_0} h_j^l(t) v_j$. In this case, condition (31) is valid, for example, if the abovementioned element w_0^l is chosen as v_1. Having obtained solutions y_l of inequality (13) for the data thus regularized, we pass to the limit as $l \to \infty$ on the basis of estimates of the form (35),(37) for y_l. Then we obtain the assertion of the theorem.

Proof of Theorem 2. First, let us prove the uniqueness of a strong solution of the problem. Let y_1 and y_2 be solutions of inclusion (14), and let $y = y_1 - y_2$ and $y(0) = 0$. Then

$$(y_i'(t) + Ay_i + \mathcal{B}[y_i] - f, y_i(t) - z) + \Phi(y_i - r) - \Phi(z - r) \le 0 \quad \forall z \in L^2(0,T;V), i = 1,2.$$

We set $z = y_2$ in the inequality for y_1 and $z = y_1$ in the inequality for y_2. By adding these inequalities, by integrating the resulting relation with respect to time from 0 to t, and by taking into account condition (20), we obtain

$$|y(t)|^2 + 2\nu \int_0^t \|y(\tau)\|^2 d\tau \le 2K_3 \int_0^t \|y\|^{1+\theta} \cdot |y|^{1-\theta} \cdot \|y_2\|^\gamma \cdot |y_2|^{1-\gamma} d\tau. \tag{39}$$

Note that $y_2 \in L^\infty(0,T;H)$, and therefore,

$$|y(t)|^2 + 2\nu \int_0^t \|y(s)\|^2 ds \le \varepsilon \int_0^t \|y(s)\|^2 ds + C_\varepsilon \int_0^t \|y_2\|^{\frac{2\gamma}{(1-\theta)}} \cdot |y|^2 ds. \tag{40}$$

The function $t \to \|y_2(t)\|^{\frac{2\gamma}{1-\theta}}$ is integrable if $\theta, \gamma \in [0, \frac{1}{2}]$. Therefore, by the Gronwall inequality, we obtain $y(t) = 0$, $t \in (0,T)$.

To prove the existence, we use the fact that the space U is dense in the space V. Therefore, we suppose that the basis elements v_j belong to the space U. We obtain additional a priori estimates for the approximate solution y_m, which provide the desired regularity of the limit element y. By multiplying (32) by $g_{jm}'(t)$ and by summing the resulting relation over $j = 1, ..., m$, we obtain

$$|w_m'(t)|^2 + (Aw_m + \mathcal{B}(r + w_m, w_m) + \mathcal{B}(w_m, r), w_m') + (\nabla\Phi_\lambda(w_m), w_m') = (g, w_m'). \tag{41}$$

Condition (19) describing the coordination and regularity of the original data permits one to find from (41) that

$$\{w_m'(0)\} \text{ is a bounded sequence in the space } H_0. \tag{42}$$

We differentiate relation (32) with respect to t, multiply the resulting relation by $g'_{jm}(t)$, and sum with respect to $j = 1, ..., m$. Since $\nabla\Phi_\lambda$ is monotone, we have

$$\frac{1}{2}\frac{d}{dt}|w'_m(t)|^2 + (Aw'_m, w'_m) + (\mathcal{B}(y'_m, y_m), w'_m) + (\mathcal{B}(y_m, r'), w'_m) \leq (f'_1, w'_m), \tag{43}$$

where $y_m = r + w_m$, $f_1 = f - r' - Ar$.

From the estimate (35), condition (20), and the Holder inequality, we obtain

$$\begin{aligned} &|B(y'_m, y_m), w'_m)| \leq k_3\|w'_m\|^{1+\theta} \cdot |w'_m|^{1-\theta} \cdot \|y_m\|^\gamma \cdot |y_m|^{1-\gamma} + \\ &+C_5\|z'\| \cdot \|y_m\| \cdot \|w'_m\| \leq \tfrac{\nu}{2}\|w'_m\|^2 + C_6\|y_m\|^{\frac{2\gamma}{(1-\theta)}}|w'_m|^2 + C_7\|y'_m\|^2. \end{aligned} \tag{44}$$

By substituting this estimate into (43), we obtain

$$\frac{1}{2}\frac{d}{dt}|w'_m(t)|^2 + \frac{\alpha}{2}\|w'_m\|^2 \leq C_8(\|y_m\|^2 + \|y_m\|^{\frac{2\gamma}{(1-\theta)}}|w'_m|^2 + \|f'_1\|_*). \tag{45}$$

Note also that

$$\int_0^T \|y_m\|^{\frac{2\gamma}{(1-\theta)}} dt \leq C_9(\int_0^T \|y_m\|^2 dt)^{\frac{\gamma}{1-\theta}}. \tag{46}$$

By virtue of the estimates (35), (42), (44), and (46) and the Gronwall inequality, we find that $\{w'_m\}$ is bounded in $L^2(0,T;V) \cap L^\infty(0,T;H)$. This, together with the estimates obtained in the proof of Theorem 1, is sufficient to pass to the limit in system (32) and obtain conditions imposed on the function $y(t) = w(t) + r$ so as to provide the existence of a strong solution.

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REFERENCES

[1] M. Sermange and R. Temam, *Some mathematical questions related to the MHD equations*, Comm. on Pure and Applied Math., 36 (1983), pp. 635–664.
[2] G. Duvaut and J. L. Lions, *Inequalities In Mechanics and Physics*, Springer-Verlag. 1976.
[3] T. V. Bespalova and A. Yu Chebotarev, *Variational inequalities and inverse subdifferential problems for the Maxwell equations in a harmonic regime*, Translation in Differ. Equ., 36:6 (2000), pp. 825–832.
[4] A. Yu Chebotarev, *Subdifferential inverse problems for evolution Navier-Stokes systems*, J. Inverse and Ill Posed Problems, 8:3 (2000), pp. 275–287.
[5] A. Yu Chebotarev, *Variational inequalities for an operator of Navier-Stokes type, and one-sided problems for equations of a viscous heat-conducting fluid*, Translation in Math. Notes, 70:1-2 (2001), pp. 264–274.
[6] A. Yu Chebotarev and A. S. Savenkova, *Variational inequalities in the magnetohydrodynamics*, Mat. Notes, 82:1 (2007), pp. 135–149.
[7] A. Yu Chebotarev, *Subdifferential Boundary Value Problems of Magnetohydrodynamics*, Differential Equations, 43:12 (2007), pp. 1742–1752.
[8] V. Barbu, *Analysis and control of nonlinear infinite dimensional systems*, Academic Press, 1993.
[9] J. P. Aubin, *Optimal and Equilibria*, Springer-Verlag, 1993.