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VANISHING VISCOSITY LIMIT FOR INCOMPRESSIBLE FLUIDS WITH A SLIP BOUNDARY CONDITION*

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Abstract. We study the the regularity and vanishing viscosity limit of the 3-D Navier-Stokes system in a class of bounded domains with a slip boundary condition. We derive the convergence in $H^{2k+1}(\Omega)$, for any $k \geq 1$, if the initial date holds some sufficient conditions.

Key words. Navier-Stokes equations, slip boundary condition, vanishing viscosity limit.

AMS subject classifications. 35Q30, 76D05, 76D09.

1. Introduction and results.

Let Ω be an open bounded domain in R^3. We consider the initial and boundary value problem for the system of viscous Navier-Stokes equations

$$\begin{cases} \partial_t \mathbf{u}^\nu - \nu \triangle \mathbf{u}^\nu + (\mathbf{u}^\nu \cdot \nabla)\mathbf{u}^\nu + \nabla \mathbf{p}^\nu = \mathbf{0} \text{ in } \boldsymbol{\Omega}, \\ \nabla \cdot \mathbf{u}^\nu = \mathbf{0} \text{ in } \boldsymbol{\Omega}, \\ \mathbf{u}^\nu = \mathbf{u_0}, \text{ at } \mathbf{t} = \mathbf{0}, \end{cases} \tag{1}$$

with the following slip without friction boundary conditions

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{0}, \nabla \times \mathbf{u} \cdot \tau = \mathbf{0} \text{ on } \partial\boldsymbol{\Omega}, \tag{2}$$

where $\nabla \cdot$ and $\nabla \times$ denote the div and curl operators, $\mathbf{n}$ the outward normal vector and τ any unit tangential vector of $\partial\Omega$.

The corresponding Euler system is usually equipped with the slip boundary condition, namely

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla \mathbf{p} = \mathbf{0} \text{ in } \boldsymbol{\Omega}, \\ \nabla \cdot \mathbf{u} = \mathbf{0} \text{ in } \boldsymbol{\Omega}, \\ \mathbf{u} = \mathbf{u_0}, \text{ at } \mathbf{t} = \mathbf{0}, \end{cases} \tag{3}$$

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{0} \text{ on } \partial\boldsymbol{\Omega}. \tag{4}$$

Our aim is to investigate strong convergence, up to the boundary, of the solution $\mathbf{u}^\nu$ of the Navier-Stokes system (1) to the solution $\mathbf{u}$ of the Euler system (3), as $\nu \to 0$.

Existence of classical solution to Euler equations (3) in local time under the boundary condition (4) can be founded in [1] and [2]. The interested readers may consult [3] and [4] for the mathematical theories of the Navier-Stokes equations.

The issue of vanishing viscosity limits of the Navier-Stokes equations is classical and fundamental importance in fluid dynamics and turbulence theory (see e.g. [5] [6], [7], [8], [9], [10]). An interesting result about a complete asymptotic expansion to viscosity under an non characteristic boundary case was derived in [11].

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In flat boundary case, the 3-D inviscid limit for solution $\mathbf{u}^\nu$ to the slip boundary problem (1) and (2) has been considered in [12] and [13]. In [12], they state the convergence ($\mathbf{u}^\nu \to \mathbf{u}$) in $L^p(0,T;H^3(\Omega)) \cap C([0,T];H^2(\Omega))$ as the viscosity $\nu \to 0$. And in [13], the convergence was developed in $C([0,T];W^{s,p}(\Omega))$, where $p > \frac{3}{2}$ and $s < 3$.

It should be noted that the approach encounters great difficulties for general domains as pointed out in [13]. Thus, following [13], we restrict the problem to a cubic domain $Q = [0,1]^2_{per} \times (0,1)$ with the boundary conditions on two opposite faces $z = 0$ and $z = 1$, and others be assumed periodic, which was called flat boundary case.

We prove the following results.

THEOREM 1.1. *Let the initial data* $\mathbf{u_0} \in \mathbf{V^{2k-1}} \cap \mathbf{H^{2k+1}}$, $k \geq 1$. *Then there exist strong solution of the Navier-Stokes equations (1)-(2) in the "cubic domain" (flat boundary case) on some time interval* $[0,T]$, *s.t.*

$$\|\mathbf{u}^\nu\|_{\mathbf{L^\infty(0,T;H^{2k+1})}} \leq \mathbf{C}, \|\partial_\mathbf{t}\mathbf{u}^\nu\|_{\mathbf{L^2(0,T;H^{2k})}} \leq \mathbf{C}. \tag{5}$$

And

$$\mathbf{u}^\nu \to \mathbf{u} \ \textit{in} \ \mathbf{C([0,T];H^{2k})}, \ \textit{as} \ \nu \to \mathbf{0}, \tag{6}$$

where $\mathbf{u}$ *is the unique solution of the Euler equations (3) and (4).*

THEOREM 1.2. *Let the conditions of Theorem 1.1 be satisfied, then*

$$\|\mathbf{u}^\nu - \mathbf{u}\|_{\mathbf{L^\infty(0,T;H^{2k-1})}} \leq \mathbf{C}\nu. \tag{7}$$

Further, if $\|\partial_n^{2k}\omega_\tau\|_{L^\infty(0,T;C^2(\partial\Omega))} \leq C$, $\|\partial_t\partial_n^{2k}\omega_\tau\|_{L^2(0,T;C^1(\partial\Omega))} \leq C$, *then*

$$\|\mathbf{u}^\nu - \mathbf{u}\|_{\mathbf{L^\infty(0,T;H^{2k+1})}} \leq \mathbf{C}\nu^{\frac{1}{4}}. \tag{8}$$

REMARK 1.1. We are grateful to the referee for the information about [13], which unfortunately was not known to us. In [13] the authors consider the case $W^{k,p}$, where k is an arbitrary integer satisfying $k \geq 3$, and the exponent p may be arbitrarily large. In the present paper we assume that $p = 2$, moreover the integer k is replaced by, say, $2k+1$. On the other hand, we carefully and deeply study, and show explicitly, the necessary and sufficient compatibility conditions that the initial data must verify. In [13], the authors assume that the compatibility conditions hold but they do not investigate this significant problem. Furthermore, our general approach is quite different from that followed in [13]. Finally, we present some additional results concerning the dependence on ν of the speed of convergence.

It is worth noting that during the preparation of a couple of revised versions of our manuscript, [15] appears. In this reference, by appealing to a reflection technique, stronger results than that shown here and in [13] are proved.

The paper is organized as follows. Some tools are drawn in Sect. 2. A priori estimates to the Navier-Stokes systems are given in Sect. 3. The results of vanishing viscosity limit and the convergence rate are presented in Sect. 4. The proof of result (8) is given in Sect. 5.

2. Notations and preliminaries. Throughout the rest of this paper, denote by $\mathbf{v}_\tau = \mathbf{v}\cdot\tau$ and $\mathbf{v_n} = \mathbf{v}\cdot\mathbf{n}$ on the boundary $\partial\Omega$. For the flat boundary case, $\mathbf{v}\cdot\mathbf{n} = \mathbf{0}$ and $\nabla\times\mathbf{v} = \mathbf{0}$ are equivalent to $\mathbf{v_n} = \mathbf{0}$ and $\partial_n\mathbf{v}_\tau = \mathbf{0}$ on ∂Q. And $\partial Q = \{(x,y,z); z=0, z=1\}\cap\overline{Q}$. For convenience, Ω and Q may be omitted when we write these spaces without confusion.

We begin our analysis with a formula of integration by parts.

Let Ω be a regular open, bounded set in R^3. Then, for sufficiently regular vector fields $\mathbf{v}$,

$$-\int_\Omega \triangle\mathbf{v}\cdot\mathbf{v}\mathbf{dx} = \|\nabla\mathbf{v}\|^{\mathbf{2}}_{\mathbf{L^2}} - \int_{\partial\mathbf{\Omega}} \partial_{\mathbf{n}}\mathbf{v}\cdot\mathbf{v}\mathbf{d}\sigma. \tag{9}$$

It is easily shown that if $\mathbf{v}$ is sufficiently regular vector fields in a flat boundary domain then

$$\partial_{\mathbf{n}}\mathbf{v}\cdot\mathbf{v} = \partial_{\mathbf{n}}\mathbf{v}_\tau\cdot\mathbf{v}_\tau + \partial_{\mathbf{n}}\mathbf{v}_{\mathbf{n}}\cdot\mathbf{v}_{\mathbf{n}}. \tag{10}$$

It follows that $\partial_{\mathbf{n}}\mathbf{v}\cdot\mathbf{v}$ vanishes on the boundary if either of the following conditions is satisfied,

$$\begin{aligned} &(a)\quad \mathbf{v}\cdot\mathbf{n} = \mathbf{0}, \nabla\times\mathbf{v}\times\mathbf{n} = \mathbf{0} \text{ on } \partial\mathbf{\Omega},\\ &(b)\quad \mathbf{v}\times\mathbf{n} = \mathbf{0} \text{ on } \partial\mathbf{\Omega}, \nabla\cdot\mathbf{u} = \mathbf{0} \text{ in } \mathbf{\Omega}. \end{aligned} \tag{11}$$

To study functions with either of above boundary conditions, we introduce series of function sets.

Let

$$H = \{\mathbf{v}\in\mathbf{H^1}; \nabla\cdot\mathbf{v} = \mathbf{0} \text{ in } \mathbf{\Omega}\},$$

$$V^{-1} = \{\mathbf{v}\in\mathbf{H}; \mathbf{v_n} = \mathbf{0} \text{ on } \partial\mathbf{\Omega}\},$$

$$V^0 = \{\mathbf{v}\in\mathbf{H}; \mathbf{v}_\tau = \mathbf{0} \text{ on } \partial\mathbf{\Omega}\},$$

$$V^{2k} = \{\mathbf{v}\in\mathbf{H^{2k+1}}; \partial_{\mathbf{n}}^{\mathbf{2j}}\mathbf{v}\in\mathbf{V^0}, \mathbf{j} = \mathbf{0},\mathbf{1},\cdots,\mathbf{k}\},$$

$$V^{2k+1} = \{\mathbf{v}\in\mathbf{H^{2k+2}}; \mathbf{v}\in\mathbf{V^{-1}}, \partial_{\mathbf{n}}^{\mathbf{2j+1}}\mathbf{v}\in\mathbf{V^0}, \mathbf{j} = \mathbf{0},\mathbf{1},\cdots,\mathbf{k}\}.$$

Then, the following propositions are easily obtained

PROPOSITION 2.1. *If* $\mathbf{v}\in\mathbf{V^{2k+1}}$*, then* $\partial_n^{2j}\mathbf{v}\in\mathbf{V^{-1}}$*,* $j = 0,1,\cdots,k+1$.

PROPOSITION 2.2. *If* $\mathbf{v}\in\mathbf{V^{2k}}$*, then* $\partial_n^{2j+1}\mathbf{v}\in\mathbf{V^{-1}}$*,* $j = 0,1,\cdots,k$.

Rewrite (11) with the new notations,

LEMMA 2.1. *If* $\mathbf{v}\in\mathbf{V^0}$ *or* $\mathbf{v}\in\mathbf{V^1}$*, then* $\partial_n\mathbf{v}\cdot\mathbf{v} = \mathbf{0}$ *on* $\partial\Omega$.

This result together with the following two are main tools in our proof.

LEMMA 2.2. $(\mathbf{u}\cdot\nabla)\mathbf{v}$ *is normal to the boundary, if either of the following conditions holds*

$$\begin{aligned} &(a)\qquad \mathbf{u}\in\mathbf{V^0}, \partial_{\mathbf{n}}\mathbf{v}\in\mathbf{V^0},\\ &(b)\qquad \mathbf{u}\in\mathbf{V^{-1}}, \mathbf{v}\in\mathbf{V^0}. \end{aligned} \tag{12}$$

The proof is left to the reader.

It should be considered that when $\mathbf{v}$ is not in V^0 or V^1. For the energy estimates, we construct a boundary layer to fill the gap.

LEMMA 2.3. *In the flat boundary case, assume* $\|\mathbf{h}_\tau\|_{\mathbf{C^1(\partial\Omega)}} \leq \mathbf{C}$ *for* $k \geq 1$. *Then, for any* $\nu << 1$, *there are* $\mathbf{v}^\nu \in \mathbf{V^{2k-1}}$, $\chi^\nu = \nabla \times \mathbf{v}^\nu$, *such that* $\nabla \times \mathbf{v}^\nu \equiv \mathbf{0}$ *as* $\nu^{\frac{1}{2}} \leq z \leq 1-\nu^{\frac{1}{2}}$, *furthermore,*

$$\begin{aligned}
&\chi_\tau^\nu \in \mathbf{C^{2k+1}}(\overline{\mathbf{\Omega}}), \chi_\mathbf{n}^\nu \in \mathbf{C^{2k}}(\overline{\mathbf{\Omega}}),\\
&\partial_n^{2k}\chi_\tau^\nu = \mathbf{h}_\tau, \partial_\mathbf{n}^{\mathbf{2k}}\chi_\mathbf{n}^\nu = \mathbf{0} \text{ on } \partial\mathbf{\Omega},\\
&\|z^i(1-z)^i\partial_n^{2k+1}\chi_\tau^\nu\|_{\mathbf{L^p}} \leq \mathbf{C}\nu^{\frac{1}{2p}+\frac{i-1}{2}},\\
&\|z^i(1-z)^i\partial_n^{2k}\chi_\mathbf{n}^\nu\|_{\mathbf{L^p}} \leq \mathbf{C}\nu^{\frac{1}{2p}+\frac{i+1}{2}},\\
&\|\partial_n^{2k}\chi^\nu\|_{\mathbf{L^p}} \leq \mathbf{C}\nu^{\frac{1}{2p}+\frac{i}{2}},
\end{aligned} \tag{13}$$

for $i \in R^+$, $1 \leq p < +\infty$.

Proof. It's trivial to find a function $\varphi(z) \in C^1[0,\infty)$, s.t.

$$\begin{cases}
\varphi(z) = 1 \text{ at } z = 0,\\
\varphi(z) = 0 \text{ at } z \geq 1,\\
\displaystyle\int_0^1 F^j(\varphi)(s)ds = 0, j = 0, 1, \cdots, 2k-1,
\end{cases} \tag{14}$$

where F is an integrate operator from $C[0,\infty)$ to $C^1[0,\infty)$, s.t. $F(f)(z) = \int_0^z f(s)ds$. $F^0(f) = f$, $F^j = F(F^{j-1})$, for $j \geq 1$.

Denote by $\varphi^\nu(z) = \varphi(\frac{z}{\nu^{\frac{1}{2}}})$. Then,

$$\|z^i\partial_z^j\varphi^\nu\|_{L^p} \leq C\nu^{\frac{1}{2p}+\frac{i-j}{2}} \text{ for } i \in R^+, j = 0, 1 \text{ and } 1 \leq p \leq +\infty.$$

Set $\psi^\nu(z)_\tau = \mathbf{h}_\tau(\mathbf{0})\varphi^\nu(\mathbf{z}) + \mathbf{h}_\tau(\mathbf{1})\varphi^\nu(\mathbf{1-z})$, and $\psi_3^\nu = -\int_0^z \nabla_\tau \cdot \psi_\tau(x,y,s)ds$. It follows that

$$\nabla \cdot \psi^\nu = 0 \text{ in } \Omega, \quad \psi_\tau^\nu = \mathbf{h}_\tau \text{ on } \partial\mathbf{\Omega}.$$

Next, set

$$\chi^\nu = F^{2k}(\psi^\nu).$$

Since $F^j(\varphi^\nu) = 0$ on $\partial\Omega$, for $j = 1, 2, \cdots, 2k$, it follows that $\partial_z^j\chi_\tau^\nu = 0$, $\partial_z^{j+1}\chi_3 = 0$ on $\partial\Omega$, for $j = 0, 1, \cdots, 2k-1$. Furthermore, $\nabla \cdot \chi^\nu = 0$ in Ω. In other words, $\chi^\nu \in V^{2k-2}$. Therefore, $\int \chi_3 = 0$.

Finally, let ζ^ν satisfy the following equations

$$\begin{cases}
-\triangle\zeta^\nu = \chi^\nu \text{ in } \Omega,\\
\zeta_\tau^\nu = 0, \partial_z\zeta_3^\nu = 0 \text{ on } \partial\Omega.
\end{cases} \tag{15}$$

The necessary condition $\int \chi_3 = 0$ of existence holds from classical elliptic theories. Applying div to equation (15), together with $\mathrm{div}\zeta^\nu = 0$ on $\partial\Omega$, then $\nabla \cdot \zeta^\nu = 0$ in Ω.

Set $\mathbf{v}^\nu = \nabla \times \zeta^\nu$ and notice that $\nabla \times \zeta^\nu = -\triangle\chi^\nu$, then the proof is completed after a simple calculation. □

REMARK 2.1. Replacing the C^1 regularity of $\mathbf{h}_\tau$ with C^s, then the $W^{2k+s,p}$ estimates of $\mathbf{v}^\nu$ can be obtained.

3. A priori estimates. In this section, we derive formal energy estimates assuming that $\mathbf{u_0}$ and $\mathbf{u}^\nu$ are sufficiently regular. As pointed out in [12] and [13], the key in studying the vanishing viscosity limit is to control the vorticity created on the boundary.

Set

$$\omega^\nu = \nabla \times \mathbf{u}^\nu.$$

Recalling boundary conditions (2) together with the notations introduced in section 2,

$$\mathbf{u}^\nu \in \mathbf{V^1}, \quad \omega^\nu \in \mathbf{V^0}. \tag{16}$$

By applying the operator curl to both sides of equation (1) one gets,

$$\partial_t \omega^\nu - \nu \triangle \omega^\nu + (\mathbf{u}^\nu \cdot \nabla)\omega^\nu - (\omega^\nu \cdot \nabla)\mathbf{u}^\nu = \mathbf{0} \text{ in } \mathbf{\Omega}. \tag{17}$$

By appealing to Lemma 2.2 and (16), one obtains

Lemma 3.1. *Let $\mathbf{u}^\nu$ be a vector field in Ω, and $\omega^\nu = \nabla \times \mathbf{u}^\nu$. Assume $u_3 = \omega_1 = \omega_2 = 0$ on $\partial\Omega$. Then the vector fields $(\mathbf{u}^\nu \cdot \nabla)\omega^\nu$ and $(\omega^\nu \cdot \nabla)\mathbf{u}^\nu$ are normal to $\partial\Omega$.*

It was inspired in [12].

Since $\omega^\nu \in V^0$, by appealing to equation (17), it follows that $\partial_n^2 \omega^\nu \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$. Recalling the definition of V^2, one obtains $\omega^\nu \in V^2$. Then, $\mathbf{u}^\nu \in \mathbf{V^3}$.

Lemma 3.2. *Let $\mathbf{u}^\nu$ and ω^ν be as above, with sufficient regularity, $\mathbf{u}^\nu \in \mathbf{H^{2k+2}}$. Then, for $k \in N$,*

$$\mathbf{u}^\nu \in \mathbf{V^{2k+1}}, \omega^\nu \in \mathbf{V^{2k}}. \tag{18}$$

Proof. The mathematical inductive method is used to prove the result.

When $k = 1$, the claim follows from the above analysis.

Next, we assume $\mathbf{u}^\nu \in \mathbf{V^{2j+1}}, \omega^\nu \in \mathbf{V^{2j}}$ for $j = 0, 1, \cdots, k-1$.

By applying operator ∂_n^{2k-2} to both sides of equation (17), one gets

$$\begin{aligned} &\partial_t \partial_n^{2k-2} \omega^\nu - \nu \triangle \partial_n^{2k-2} \omega^\nu \\ &+ \sum_{j=0}^{2k-2} [(\partial_n^j \mathbf{u}^\nu \cdot \nabla)\partial_\mathbf{n}^{\mathbf{2k-2-j}} \omega^\nu - (\partial_\mathbf{n}^{\mathbf{2k-2-j}} \omega^\nu \cdot \nabla)\partial_\mathbf{n}^\mathbf{j} \mathbf{u}^\nu] = \mathbf{0} \text{ in } \mathbf{\Omega}. \end{aligned} \tag{19}$$

From the inductive hypothesis, $\mathbf{u}^\nu \in \mathbf{V^{2k-1}}$, $\omega^\nu \in V^{2k-2}$.

Case 1: j is even. By Proposition 2.1, $\partial_n^j \mathbf{u}^\nu \in \mathbf{V^{-1}}$, $\partial_n^{j+1} \mathbf{u}^\nu \in \mathbf{V^0}$, $\partial_n^{2k-2-j}\omega^\nu \in V^0$. It follows from Lemma 2.2 that $(\partial_n^j \mathbf{u}^\nu \cdot \nabla)\partial_\mathbf{n}^{\mathbf{2k-2-j}}\omega^\nu$ and $(\partial_n^{2k-2-j}\omega^\nu \cdot \nabla)\partial_n^j \mathbf{u}^\nu$ are normal to the boundary $\partial\Omega$.

Case 2: j is odd. By Proposition 2.2, $\partial_n^j \mathbf{u}^\nu \in \mathbf{V^0}$, $\partial_n^{2k-2-j}\omega^\nu \in V^{-1}$, $\partial_n^{2k-1-j}\omega^\nu \in V^0$. It follows from Lemma 2.2 that $(\partial_n^j \mathbf{u}^\nu \cdot \nabla)\partial_\mathbf{n}^{\mathbf{2k-2-j}}\omega^\nu$ and $(\partial_n^{2k-2-j}\omega^\nu \cdot \nabla)\partial_n^j \mathbf{u}^\nu$ are normal to the boundary $\partial\Omega$.

Taking them into equation (19), one obtains $\partial_n^{2k}\omega^\nu \in V^0$.

Finally, according to the definition of V^{2k} and V^{2k+1}, the proof is completed. ∎

Applying the operator $\partial^{\alpha}_{x,y,z}$, where α is a multi-index and $|\alpha| \le 2k$, to both sides of equation (17), one gets

$$\begin{aligned} &\partial_t \partial^{\alpha}_{x,y,z}\omega^{\nu} - \nu\triangle\partial^{\alpha}_{x,y,z}\omega^{\nu} + (\mathbf{u}^{\nu}\cdot\nabla)\partial^{\alpha}_{\mathbf{x},\mathbf{y},\mathbf{z}}\omega^{\nu} - (\partial^{\alpha}_{\mathbf{x},\mathbf{y},\mathbf{z}}\omega^{\nu}\cdot\nabla)\mathbf{u}^{\nu} \\ &+ \sum_{\beta+\gamma=\alpha,|\beta|=1} (\partial^{\beta}_{x,y,z}\mathbf{u}^{\nu}\cdot\nabla)\partial^{\gamma}_{\mathbf{x},\mathbf{y},\mathbf{z}}\omega^{\nu} + \sum_{\beta+\gamma=\alpha,|\beta|\ge\mathbf{2}} (\partial^{\beta}_{\mathbf{x},\mathbf{y},\mathbf{z}}\mathbf{u}^{\nu}\cdot\nabla)\partial^{\gamma}_{\mathbf{x},\mathbf{y},\mathbf{z}}\omega^{\nu} \\ &- \sum_{\beta+\gamma=\alpha,|\beta|\ge 1} (\partial^{\gamma}_{x,y,z}\omega^{\nu}\cdot\nabla)\partial^{\beta}_{x,y,z}\mathbf{u}^{\nu} = \mathbf{0}. \end{aligned} \tag{20}$$

Next, multiply both sides of the above equation by $\partial^{\alpha}_{x,y,z}\omega^{\nu}$ and integrate in Ω. Note that $\partial^{\alpha}_{x,y,z}\omega^{\nu} \in V^0$ or $\partial^{\alpha}_{x,y,z}\omega^{\nu} \in V^1$ for $|\alpha| \le 2k$,

$$\|\nabla\mathbf{u}^{\nu}\|_{\mathbf{L}^{\infty}} \le \mathbf{C}\|\mathbf{u}^{\nu}\|_{\mathbf{H}^3} \le \mathbf{C}\|\omega^{\nu}\|_{\mathbf{H}^{2k}},$$

and

$$\|\mathbf{u}^{\nu}\|_{\mathbf{W}^{\mathbf{j},4}} \le \mathbf{C}\|\omega^{\nu}\|_{\mathbf{W}^{\mathbf{j}-1,4}} \le \mathbf{C}\|\omega^{\nu}\|_{\mathbf{H}^{\mathbf{j}}}, \mathbf{2} \le \mathbf{j} \le \mathbf{2k}.$$

By Lemma 2.1 and summing up for all $|\alpha| \le 2k$, one obtains

$$\frac{1}{2}\frac{d}{dt}\|\omega^{\nu}\|^2_{H^{2k}} + \nu\|\nabla\omega^{\nu}\|^2_{H^{2k}} \le C\|\omega^{\nu}\|^3_{H^{2k}}. \tag{21}$$

Comparing with the ordinary differential equation

$$\begin{cases} y'(t) = Cy^{\frac{3}{2}}, \\ y(0) = \|\omega_0\|^2_{H^{2k}}, \end{cases} \tag{22}$$

where $\omega_0 = \nabla\times\mathbf{u_0}$, and denoting by T^* the blow up time, it follows that a priori estimates hold, for $T < T^*$,

$$\|\omega^{\nu}\|_{L^{\infty}(0,T;H^{2k})} \le C, \|\nabla\omega^{\nu}\|_{L^2(0,T;H^{2k})} \le C\nu^{-\frac{1}{2}}. \tag{23}$$

Thus, we have the following result.

THEOREM 3.1. *Let* $\mathbf{u_0} \in \mathbf{V^{2k-1}} \cap \mathbf{H^{2k+1}}$, $k \ge 1$. *Then there exist* T *and* $C(\|\mathbf{u_0}\|_{\mathbf{H^{2k+1}}}, \mathbf{T})$*, s.t.*

$$\|\mathbf{u}^{\nu}\|_{\mathbf{L}^{\infty}(\mathbf{0},\mathbf{T};\mathbf{H}^{2\mathbf{k}+1})} \le \mathbf{C}, \|\nabla\mathbf{u}^{\nu}\|_{\mathbf{L}^2(\mathbf{0},\mathbf{T};\mathbf{H}^{2\mathbf{k}+1})} \le \mathbf{C}\nu^{-\frac{1}{2}}. \tag{24}$$

Taking the inner product $((17), \partial_t\omega^{\nu})_{H^{2k-1}}$, it follows that $\|\partial_t\omega^{\nu}\|_{L^2(0,T;H^{2k-1})} \le C$, and $\|\partial_t\mathbf{u}^{\nu}\|_{\mathbf{L^2(0,T;H^{2k})}} \le \mathbf{C}$.

According to $\|\omega_0\|_{H^{2k}} \le C$, then by equation (17), $\|\partial_t\omega|_{t=0}\|_{H^{2k-2}} \le C$.

Similarly, applying operator $\partial_t\partial^{\alpha}_{x,y,z}$ to both sides of equation (17), for $|\alpha| \le 2k-2$, and multiplying $\partial_t\partial^{\alpha}_{x,y,z}\omega^{\nu}$, we have

$$\|\partial_t\mathbf{u}^{\nu}\|_{\mathbf{L}^{\infty}(\mathbf{0},\mathbf{T};\mathbf{H}^{2\mathbf{k}-1})} \le \mathbf{C}, \|\nabla\partial_{\mathbf{t}}\mathbf{u}^{\nu}\|_{\mathbf{L}^2(\mathbf{0},\mathbf{T};\mathbf{H}^{2\mathbf{k}-1})} \le \mathbf{C}\nu^{-\frac{1}{2}}. \tag{25}$$

Thus, we can conclude

THEOREM 3.2. *Let the conditions of Theorem 3.1 be satisfied, then for $s \le k$*

$$\begin{aligned} &\|\partial_t^s \mathbf{u}^\nu\|_{\mathbf{L}^\infty(\mathbf{0},\mathbf{T};\mathbf{H}^{\mathbf{2k+1-2s}})} \le \mathbf{C}, \\ &\|\nabla \partial_t^s \mathbf{u}^\nu\|_{\mathbf{L}^2(\mathbf{0},\mathbf{T};\mathbf{H}^{\mathbf{2k+1-2s}})} \le \mathbf{C}\nu^{-\frac{1}{2}}, \\ &\|\partial_t^{s+1} \mathbf{u}^\nu\|_{\mathbf{L}^2(\mathbf{0},\mathbf{T};\mathbf{H}^{\mathbf{2k-2s}})} \le \mathbf{C}. \end{aligned} \tag{26}$$

where $C = C(\|\mathbf{u_0}\|_{\mathbf{H}^{\mathbf{2k+1}}}, \mathbf{T})$.

Then, the regularity of the solution of the Navier-Stokes equations (1) and (2) is investigated,

THEOREM 3.3. *Let the conditions of Theorem 3.1 be satisfied. Then, for $s \le k$, there exist a time T depending on the initial date and unique classical solution of Navier-Stokes equations (1) with boundary conditions (2). In addition,*

$$\begin{aligned} &\|\partial_t^s \mathbf{u}^\nu\|_{\mathbf{L}^\infty(\mathbf{0},\mathbf{T};\mathbf{H}^{\mathbf{2k+1-2s}})} \le \mathbf{C}, \\ &\|\nabla \partial_t^s \mathbf{u}^\nu\|_{\mathbf{L}^2(\mathbf{0},\mathbf{T};\mathbf{H}^{\mathbf{2k+1-2s}})} \le \mathbf{C}\nu^{-\frac{1}{2}}, \\ &\|\partial_t^{s+1} \mathbf{u}^\nu\|_{\mathbf{L}^2(\mathbf{0},\mathbf{T};\mathbf{H}^{\mathbf{2k-2s}})} \le \mathbf{C}, \end{aligned} \tag{27}$$

where $C = C(\|\mathbf{u_0}\|_{\mathbf{H}^{\mathbf{2k+1}}}, \mathbf{T})$.

4. The vanishing viscosity limit. This section focuses on the vanishing viscosity limit of the Navier-Stokes system for the flat boundary case.

THEOREM 4.1. *Let the conditions of Theorem 3.1 be satisfied for $k \ge 1$. Then as $\nu \to 0$, $\mathbf{u}^\nu$ converges to the unique solution $\mathbf{u}$ of the Euler system with the same initial date in the sense*

$$\begin{aligned} &\mathbf{u}^\nu \rightharpoonup \mathbf{u} \ \textit{in}\ \mathbf{L}^\infty(\mathbf{0},\mathbf{T};\mathbf{H}^{\mathbf{2k+1}}) \ \textit{weak-}*, \\ &\mathbf{u}^\nu \to \mathbf{u} \ \textit{in}\ \mathbf{C}(\mathbf{0},\mathbf{T};\mathbf{H}^{\mathbf{2k}}). \end{aligned} \tag{28}$$

Proof. It follows from Theorem 3.3 that

$$\mathbf{u}^\nu \text{ is uniformly bounded in } \mathbf{L}^\infty(\mathbf{0},\mathbf{T};\mathbf{H}^{\mathbf{2k+1}}), \tag{29}$$

and

$$\partial_t \mathbf{u}^\nu \text{ is uniformly bounded in } \mathbf{L}^2(\mathbf{0},\mathbf{T};\mathbf{H}^{\mathbf{2k}}), \tag{30}$$

for all $\nu > 0$. By the standard compactness result, there exist a subsequence ν_k of ν and vector function $\mathbf{u}$, such that

$$\mathbf{u}^{\nu_k} \to \mathbf{u} \text{ in } \mathbf{C}(\mathbf{0},\mathbf{T};\mathbf{H}^{\mathbf{2k}}), \tag{31}$$

as $\nu \to 0$. Passing to the limit, we can find $\mathbf{u}$ solves the Euler equations (3) and (4). Together with the uniqueness of the strong solution of the Euler systems, we then show the convergence of the whole sequence. □

Now, we present the convergence rate.

Set $\omega = \nabla \times \mathbf{u}$. Recalling Lemma 3.2 and Theorem 4.1, one obtains

$$\mathbf{u} \in \mathbf{V}^{\mathbf{2k-1}}, \omega \in \mathbf{V}^{\mathbf{2k-2}}, \mathbf{u} \in \mathbf{L}^\infty(\mathbf{0},\mathbf{T};\mathbf{H}^{\mathbf{2k+1}}). \tag{32}$$

Set $\widetilde{\mathbf{u}}^\nu = \mathbf{u}^\nu - \mathbf{u}$, $\widetilde{\omega}^\nu = \nabla \times \widetilde{\mathbf{u}}^\nu$. We can find

$$\partial_t \widetilde{\omega}^\nu - \nu \triangle \widetilde{\omega}^\nu + (\mathbf{u}^\nu \cdot \nabla)\widetilde{\omega}^\nu + (\widetilde{\mathbf{u}}^\nu \cdot \nabla)\omega - (\widetilde{\omega}^\nu \cdot \nabla)\mathbf{u}^\nu - (\omega \cdot \nabla)\widetilde{\mathbf{u}}^\nu = \nu \triangle \omega. \tag{33}$$

Noting that $\widetilde{\mathbf{u}}^\nu \in V^{2k-1}, \widetilde{\omega}^\nu \in V^{2k-2}$, and $\|\triangle \omega\|_{H^{2k-2}} \le C$, the same argument in proof of Theorem 3.1 can be followed. Taking the inner products $((33), \widetilde{\omega}^\nu)_{H^{2k-2}}$, we have

$$\frac{1}{2}\frac{d}{dt}\|\omega^\nu - \omega\|^2_{H^{2k-2}} + \nu \|\nabla(\omega^\nu - \omega)\|^2_{H^{2k-2}} \le C\|\omega^\nu\|^2_{H^{2k-2}} + C\nu^2. \tag{34}$$

Then, the desired result (7) in Theorem 1.2 is obtained by Gronwall inequality.

5. Some additional results. There is a gap between $\partial_n^{2k}\omega_\tau$ and 0. In other words, ω is not in V^{2k}. Assuming $\|\partial_n^{2k}\omega_\tau\|_{L^\infty(0,T;C^2(\partial\Omega))} \le C$, $\|\partial_t \partial_n^{2k}\omega_\tau\|_{L^2(0,T;C^1(\partial\Omega))} \le C$, then by Lemma 2.3, there are $\mathbf{v}^\nu \in \mathbf{V^{2k-1}}$, $\chi^\nu = \nabla \times \mathbf{v}^\nu$, s.t. $\partial_n^{2k}\chi^\nu_\tau = -\partial_n^{2k}\omega_\tau$ on $\partial\Omega$, $\|z^j(1-z)^j\partial_n^{2k+1}\chi^\nu\|_{L^\infty(0,T;L^2)} \le C\nu^{\frac{2j-1}{4}}$, $j = 0, 1$, and further $\|\partial_t \chi^\nu\|_{L^2(0,T;H^{2k})} \le C\nu^{\frac{1}{4}}$, $\|\partial_\tau \chi^\nu\|_{L^\infty(0,T;H^{2k})} \le C\nu^{\frac{1}{4}}$.

Set $\widehat{\mathbf{u}}^\nu = \mathbf{u}^\nu - \mathbf{u} - \mathbf{v}^\nu$, $\widehat{\omega}^\nu = \nabla \times \widehat{\mathbf{u}}^\nu = \omega^\nu - \omega - \chi^\nu$.

From equation (33), one obtains,

$$\begin{aligned} &\partial_t \widehat{\omega}^\nu - \nu \triangle \widehat{\omega}^\nu + (\mathbf{u}^\nu \cdot \nabla)\widehat{\omega}^\nu + (\widehat{\mathbf{u}}^\nu \cdot \nabla)\omega - (\widehat{\omega}^\nu \cdot \nabla)\mathbf{u}^\nu - (\omega \cdot \nabla)\widehat{\mathbf{u}}^\nu \\ &= -\partial_t \chi^\nu - \nu \triangle \chi^\nu + \nu \triangle \omega - (\mathbf{u}^\nu \cdot \nabla)\chi^\nu - (\mathbf{v}^\nu \cdot \nabla)\omega + (\chi^\nu \cdot \nabla)\mathbf{u}^\nu + (\omega \cdot \nabla)\mathbf{v}^\nu. \end{aligned} \tag{35}$$

Then, taking the inner products $((35), \widehat{\omega}^\nu)_{H^{2k}}$, Note that

$$\partial_n^{2k-1}\widehat{\omega}^\nu_n = 0, \partial_n^{2k}\widehat{\omega}^\nu_\tau = 0 \text{ on } \partial\Omega,$$

$$|\nu \int \triangle \partial_n^{2k}\chi^\nu \cdot \partial_n^{2k}\widehat{\omega}^\nu| = |\nu \int \nabla \partial_n^{2k}\chi^\nu \cdot \nabla \partial_n^{2k}\widehat{\omega}^\nu| \le \frac{\nu}{4}\|\nabla \partial_n^{2k}\widehat{\omega}^\nu\|^2_{L^2} + C\nu^{\frac{1}{2}},$$

$$|\nu \int \triangle \partial_n^{2k}\omega \cdot \partial_n^{2k}\widehat{\omega}^\nu| = |\nu \int \nabla \partial_n^{2k}\omega \cdot \nabla \partial_n^{2k}\widehat{\omega}^\nu| \le \frac{\nu}{4}\|\nabla \partial_n^{2k}\widehat{\omega}^\nu\|^2_{L^2} + C\nu,$$

$$\|(\mathbf{u}^\nu \cdot \nabla)\partial_\mathbf{n}^{\mathbf{2k}}\chi^\nu\|_{\mathbf{L^2}} \le \|(\mathbf{u}^\nu_\tau \cdot \nabla_\tau)\partial_\mathbf{n}^{\mathbf{2k}}\chi^\nu\|_{\mathbf{L^2}} + \|\frac{\mathbf{u_3^\nu}}{\mathbf{z(1-z)}}\|_{\mathbf{L^\infty}}\|\mathbf{z(1-z)}\partial_\mathbf{n}^{\mathbf{2k+1}}\chi^\nu\|_{\mathbf{L^2}} \le \mathbf{C}\nu^{\frac{1}{4}}.$$

And it follows in the same manner,

$$\frac{1}{2}\frac{d}{dt}\|\widehat{\omega}^\nu\|^2_{H^{2k}} + \nu\|\nabla \widehat{\omega}^\nu\|^2_{H^{2k}} \le C\|\widehat{\omega}^\nu\|^2_{H^{2k}} + C\nu^{\frac{1}{2}} + \|\partial_t \chi^\nu\|^2_{H^{2k}}. \tag{36}$$

Therefore,

$$\|\widehat{\omega}^\nu\|_{L^\infty(0,T;H^{2k})} \le C\nu^{\frac{1}{4}}.$$

And the result (8) in Theorem 1.2 is concluded.

Finally, we give three remarks.

Remark 5.1. Let the conditions of Theorem 1.1 be satisfied, then for $s \le k-1$,

$$\|\partial_t^s \mathbf{u}^\nu - \partial_\mathbf{s}^\mathbf{t}\mathbf{u}\|_{\mathbf{L^\infty(0,T;H^{2k-1-2s})}} \le \mathbf{C}\nu. \tag{37}$$

Remark 5.2. Let the conditions of Theorem 1.2 be satisfied, then for $s \le k$,

$$\|\partial_t^s \mathbf{u}^\nu - \partial_\mathbf{t}^\mathbf{s}\mathbf{u}\|_{\mathbf{L^\infty(0,T;H^{2k+1-2s})}} \le \mathbf{C}\nu^{\frac{1}{4}}. \tag{38}$$

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