METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 15, No. 3, pp. 377–390, September 2008

ON SOLUTIONS WITH POINT RUPTURES FOR A SEMILINEAR ELLIPTIC PROBLEM WITH SINGULARITY*

ZONGMING GUO† AND JUNCHENG WEI‡

Abstract. We consider the following semilinear elliptic equation with singular nonlinearity:

$$\Delta u - \frac{\lambda}{u^\nu} = 0 \text{ in } B,\ u = \psi \text{ on } \partial B$$

where $\lambda > 0, \nu > 0$ and $\psi \in C^{2,\alpha}(\partial B)$ and B is the unit ball in $\mathbb{R}^N$. Under various conditions on λ, ν and ψ, we construct solutions with one isolated zero in B.

Key words. Point ruptures, singularity.

AMS subject classifications. Primary 35B35, 35B40; Secondary 35J60

1. Introduction. Let B be the unit ball of $\mathbb{R}^N$ $(N \geq 2)$. The main purpose of this paper is to construct nonnegative solutions with one isolated zero point of the semilinear elliptic Dirichlet problem

$$\Delta u - \lambda u^{-\nu} = 0 \quad \text{in } B, \quad u = \psi \ \text{ on } \partial B, \tag{1.1}$$

where $\lambda,\ \nu > 0$, $\psi \in C^{2,\alpha}(\partial B)$ with $\psi(\theta) > 0$ for $\theta \in S^{N-1} = \partial B$.

Problem (1.1) appears in several applications in mechanics and physics, and in particular can be used to model the electrostatic Micro-Electromechanic System (MEMS) devices. See [FMP], [GG1], [GG2], [GG3], [GPW] and the references therein. In particular, in [GG1], [GG2] and [GG3], Ghoussoub and Guo have given a thorough study on the following problem

$$\begin{cases} u_t = \Delta u - \frac{\lambda f(x)}{u^2}, & x \in \Omega,\, t > 0, \\ u(x,0) = 1 \text{ for } \ x \in \Omega, & u(x,t) = 1 \text{ for } x \in \partial\Omega \end{cases} \tag{1.2}$$

where $\lambda > 0, f(x)$ is a positive function and Ω is a bounded smooth domain in $\mathbb{R}^N$. (1.1) is just the steady state of (1.2) with $f(x) \equiv 1$ and $\nu = 2$. The set $\{x|u(x) = 0\}$ is called **touch town** set and plays an important role in MEMS.

Problem (1.1) can also be considered as steady state problem of thin films problems. Equations of the type

$$u_t = -\nabla \cdot (f(u)\nabla\Delta u) - \nabla \cdot (g(u)\nabla u) \tag{1.3}$$

have been used to model the dynamics of thin films of viscous fluids, where $z = u(x,t)$ is the height of the air/liquid interface. The zero set $\mathbf{\Sigma}_u = \{u = 0\}$ is the liquid/solid interface and is sometimes called set of **ruptures**. Ruptures play a very important role in the study of thin films. The coefficient $f(u)$ reflects surface tension effects- a typical choice is $f(u) = u^3$. The coefficient of the second-order term can reflect additional forces such as gravity $g(u) = u^3$, van der Waals interactions $g(u) = u^m, m < 0$. For more background on thin films, we refer to [BBD, BP1, BP2, LP1, LP2, LP3, WB,

*Received February 10, 2008; accepted for publication October 21, 2008.
†Department of Mathematics, Henan Normal University, Xinxiang, 453002, P. R. China (guozm@public.xxptt.ha.cn).
‡Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong (wei@math.cuhk.edu.hk).

YD, YH] and the references therein. By choosing $f(u) = u^p, g(u) = u^{-m}$, (1.3) is equivalent to a fourth order equation

$$u_t = -\nabla \cdot (u^p \nabla (\Delta u - u^{-\nu}) \tag{1.4}$$

with $\nu = p + m - 1$. Again, solutions to (1.1) are steady-states of (1.4).

In [GW1], we computed the Hansdorff dimension of rupture sets for

$$\Delta u - \frac{\lambda}{u^\nu} + h(x) = 0 \text{ in } \Omega \tag{1.5}$$

We showed that if u is a nonnegative **stationary** solution of (1.5) such that $u \in H^1(\Omega)$ and $\int_\Omega u^{1-\nu} dx < \infty$, then the zero set of u has locally finite Hausdorff $[(N-2)\nu + (N+2)]/(\nu+1)$-dimensional measure. However, it is a difficult question to construct solutions to (1.1) exhibiting point ruptures. If $\nu > 0$, it is easy to see that there exists a radial solution $u_0(x) = |x|^{2/(\nu+1)}$ of the problem

$$\Delta u - \lambda_0 u^{-\nu} = 0 \text{ in } B, \quad u = 1 \text{ on } \partial B, \tag{1.6}$$

where $\lambda_0 = \frac{2(N+(N-2)\nu)}{(\nu+1)^2} > 0$. On the other hand, if $\Omega \subset \mathbb{R}^2$ is convex and has two symmetries, a solution with a point rupture was proved in [GW2].

The purpose of this paper is to construct nonnegative solutions of (1.1) with one isolated zero point, under various conditions on ψ and ν. Our main idea is to study the surjectivity properties of the linearized operator associated with the known rupture solution $|x|^{\frac{2}{\nu+1}}$ in some weighted Hölder spaces. The weighted Hölder space has been introduced and used by Mazzeo and Pacard [MP], Mazzeo-Pacard-Uhlenbeck [MPU] in constructing singular solutions to Yamabe type problems. It is also used by Rebai [R1], [R2] to construct solutions singular on submanifolds.

The corresponding Neumann problem

$$\Delta u - \frac{1}{u^\nu} + h(|x|) = 0 \text{ in } B, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial B \tag{1.7}$$

has been studied by del Pino and Hernandez [DH] for $\nu > 1$. They showed that (1.7) has at least one nonnegative radial solution $u = u(r)$ satisfying $a_1 r^{2/(\nu+1)} \le u(r) \le a_2$, $a_1, a_2 > 0$.

A different kind of problem

$$\Delta u + k(x)\frac{1}{u^\alpha} = 0 \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{1.8}$$

was studied in [CR, De, GHW, Go, GL] and the references therein, where $k(x) > 0$. The regularity of ∇u is obtained. Problem (1.8) is fundamentally different from (1.1): the sign of nonlinearity makes the Maximum Principle applicable to (1.8) which allow the use of e.g. a super-sub solutions scheme. In fact the following problem

$$\Delta u + \frac{1}{u^\alpha} - h(x) = 0 \text{ in } \Omega, u = \psi \text{ on } \partial\Omega,$$

possesses a (unique) positive solution in case that h is, for example, positive.

Acknowledgments. Part of this paper was done while the first author was visiting the Department of Mathematics, Chinese University of Hong Kong, he would like to thank the Department for its hospitality. The research of the first author is supported by NSF of China (10871060). The research of the second author is supported by an Earmarked Grant of RGC of Hong Kong.

2. Preliminary computation. Let (λ_0, u_0) be the radial solution of (1.6). We define the linearized operator $\mathcal{L} : w \longmapsto \Delta w + \lambda_0 \nu u_0^{-(\nu+1)} w$. Clearly we have

$$\lambda_0 \nu u_0^{-(\nu+1)} = \frac{c_0}{r^2}, \tag{2.1}$$

where c_0 is a positive constant. Precisely,

$$c_0 = \frac{2\nu(N + (N-2)\nu)}{(\nu+1)^2}.$$

It is known that the eigenvalues of the problem

$$-\Delta_\theta v = \sigma v, \quad \theta \in S^{N-1} \tag{2.2}$$

are $\sigma_k = k(N+k-2)$, $k \geq 0$ with multiplicity $m_k = \frac{(N-3+k)!(N-2+2k)}{k!(N-2)!}$. In particular, we denote that $\sigma_0 = 0$, $\sigma_1 = N-1$, $\sigma_2 = N-1$, $\ldots$, $\sigma_N = N-1$, $\sigma_{N+1} = 2N$ and $\varphi_j(\theta)$ $(j = 0, 1, \ldots)$ the eigenfunction corresponding to σ_j which is normalized in such a way that

$$\int_{S^{N-1}} \varphi_j^2(\theta) d\theta = 1.$$

Note that $\varphi_0(\theta) \equiv \text{Const}$.

We define the indicial roots of $\mathcal{L}$ by

$$\gamma_j^{\pm} = \frac{2-N}{2} \pm \Big(\Big(\frac{N-2}{2}\Big)^2 + \sigma_j - c_0 \Big)^{1/2}, \;\; j \geq 0. \tag{2.3}$$

We deduce the following proposition by simple computations.

Proposition 2.1. *The following inequalities hold:*

(1) If $N = 2$, then $\Re(\gamma_0^{\pm}) = 0$, $\Im(\gamma_0^{\pm}) \neq 0$ for $\nu > 0$.

(2) If $N = 3$, then $\gamma_0^- \leq (2-N)/2 \leq \gamma_0^+ < 0$ are real numbers for $0 < \nu \leq (2^{1/2}8 - 11)/7$; $\Re(\gamma_0^{\pm}) = (2-N)/2$, $\Im(\gamma_0^{\pm}) \neq 0$ for $\nu > (2^{1/2}8 - 11)/7$.

(3) If $N = 4$, then $\gamma_0^- \leq (2-N)/2 \leq \gamma_0^+ < 0$ are real numbers for $0 < \nu \leq (3^{1/2}2 - 3)/3$; $\Re(\gamma_0^{\pm}) = (2-N)/2$, $\Im(\gamma_0^{\pm}) \neq 0$ for $\nu > (3^{1/2}2 - 3)/3$.

(4) If $N = 5$, then $\gamma_0^- \leq (2-N)/2 \leq \gamma_0^+ < 0$ are real numbers for $0 < \nu \leq 1/3$; $\Re(\gamma_0^{\pm}) = (2-N)/2$, $\Im(\gamma_0^{\pm}) \neq 0$ for $\nu > 1/3$.

(5) If $N = 6$, then $\gamma_0^- \leq (2-N)/2 \leq \gamma_0^+ < 0$ are real numbers for $0 < \nu \leq (5^{1/2} - 1)/2$, $\Re(\gamma_0^{\pm}) = (2-N)/2$, $\Im(\gamma_0^{\pm}) \neq 0$ for $\nu > (5^{1/2} - 1)/2$.

(6) If $N = 7$, then $\gamma_0^- \leq (2-N)/2 \leq \gamma_0^+ < 0$ are real numbers for $0 < \nu \leq (6^{1/2}8 - 3)/15$, $\Re(\gamma_0^{\pm}) = (2-N)/2$, $\Im(\gamma_0^{\pm}) \neq 0$ for $\nu > (6^{1/2}8 - 3)/15$.

(7) If $N = 8$, then $\gamma_0^- \leq (2-N)/2 \leq \gamma_0^+ < 0$ are real numbers for $0 < \nu \leq (7^{1/2}2 + 1)/3$, $\Re(\gamma_0^{\pm}) = (2-N)/2$, $\Im(\gamma_0^{\pm}) \neq 0$ for $\nu > (7^{1/2}2 + 1)/3$.

(8) If $N = 9$, then $\gamma_0^- \leq (2-N)/2 \leq \gamma_0^+ < 0$ are real numbers for $0 < \nu \leq (8^{1/2}8 + 13)/7$, $\Re(\gamma_0^{\pm}) = (2-N)/2$, $\Im(\gamma_0^{\pm}) \neq 0$ for $\nu > (8^{1/2}8 + 13)/7$.

(9) If $N \geq 10$, then $\gamma_0^- \leq (2-N)/2 \leq \gamma_0^+ < 0$ are real numbers for $\nu > 0$.

(10)

$$\gamma_1^{\pm} = \gamma_2^{\pm} = \ldots \gamma_N^{\pm} = \frac{(2-N)}{2} \pm \Big| \frac{(N + (N-4)\nu)}{2(\nu+1)} \Big|.$$

Thus, for $N = 2$,

$$\gamma_1^{\pm} = \gamma_2^{\pm} = \begin{cases} \pm\frac{(1-\nu)}{(1+\nu)} & \textit{if } 0 < \nu \leq 1 \\ \pm\frac{(\nu-1)}{(1+\nu)} & \textit{if } \nu > 1, \end{cases}$$

which implies that

$$\gamma_1^{+} = \gamma_2^{+} = \begin{cases} \frac{(1-\nu)}{(1+\nu)} & \textit{if } 0 < \nu \leq 1 \\ \frac{(\nu-1)}{(1+\nu)} & \textit{if } \nu > 1, \end{cases}$$

$$\gamma_1^{-} = \gamma_2^{-} = \begin{cases} \frac{(\nu-1)}{(1+\nu)} & \textit{if } 0 < \nu \leq 1 \\ \frac{(1-\nu)}{(1+\nu)} & \textit{if } \nu > 1. \end{cases}$$

For $N = 3$,

$$\gamma_1^{\pm} = \gamma_2^{\pm} = \gamma_3^{\pm} = \begin{cases} -\frac{1}{2} \pm \frac{(3-\nu)}{2(1+\nu)} & \textit{if } 0 < \nu \leq 3 \\ -\frac{1}{2} \pm \frac{(\nu-3)}{2(1+\nu)} & \textit{if } \nu > 3, \end{cases}$$

which implies that

$$\gamma_1^{+} = \gamma_2^{+} = \gamma_3^{+} = \begin{cases} \frac{(1-\nu)}{(1+\nu)} & \textit{if } 0 < \nu \leq 3 \\ -\frac{2}{(1+\nu)} & \textit{if } \nu > 3, \end{cases}$$

$$\gamma_1^{-} = \gamma_2^{-} = \gamma_3^{-} = \begin{cases} -\frac{2}{(1+\nu)} & \textit{if } 0 < \nu \leq 3 \\ \frac{(1-\nu)}{(1+\nu)} & \textit{if } \nu > 3. \end{cases}$$

For $N \geq 4$,

$$\gamma_1^{+} = \gamma_2^{+} = \ldots = \gamma_N^{+} = \frac{(1-\nu)}{(1+\nu)},$$

$$\gamma_1^{-} = \gamma_2^{-} = \ldots = \gamma_N^{-} = \frac{[(3-N)\nu + (1-N)]}{(1+\nu)}.$$

(11)

$$\begin{aligned}
\gamma_{N+1}^{+} &= \frac{(2-N)}{2} + \Big(\frac{(N-2)^2}{4} + 2N - \frac{2\nu(N+(N-2)\nu)}{(\nu+1)^2}\Big)^{1/2} \\
&= \frac{(2-N)}{2} + \Big(\frac{(N^2-4N+20)\nu^2 + 2(N^2+4)\nu + (N+2)^2}{2(\nu+1)}\Big)^{1/2} \\
&> \frac{(2-N)}{2} + \Big(\frac{(N-2)^2\nu^2 + 2(N-2)(N+2)\nu + (N+2)^2}{2(\nu+1)}\Big)^{1/2} \\
&= \frac{(2-N)}{2} + \frac{(N-2)\nu + (N+2)}{2(\nu+1)} \\
&= \frac{2}{(\nu+1)}.
\end{aligned}$$

3. A right inverse for $\mathcal{L}$. We introduce the weighted Hölder spaces as in [MP, MPU, Re1, Re2]. For any $k \geq 0$, $\alpha \in (0,1)$ and $\mu \in \mathbb{R}$, we define some weighted Hölder spaces $C^{k,\alpha}_\mu$ as follows

$$C^{k,\alpha}_\mu = \{u \in C^{k,\alpha}_{loc}(B\backslash\{0\}) : \|u\|_{C^{k,\alpha}_\mu} = \sup_{r \leq 1/2}(r^{-\mu}|u|_{k,\alpha,[r,2r]}) < +\infty\},$$

where, by definition

$$|u|_{k,\alpha,[r,2r]} = \sup_{r\leq|x|\leq 2r}(\Sigma^k_{j=0} r^j|\nabla^j u|) + r^{k+\alpha}\sup_{r\leq|x|,|y|\leq 2r; x\neq y}\frac{|\nabla^k u(y) - \nabla^k u(x)|}{|y-x|^\alpha}.$$

In addition, for all $j \geq 0$, we define

$$C^{2,\alpha}_{\mu,j} = \{v \in C^{2,\alpha}_\mu : v|_{\partial B} \in \operatorname{span}(\varphi_0(\theta), \ldots, \varphi_j(\theta))\}. \tag{3.1}$$

It follows from (2.1) that the linear operator $\mathcal{L}$ is well defined from $C^{2,\alpha}_\mu$ into $C^{0,\alpha}_{\mu-2}$. The proof of the following proposition is a little variant of the proof of Proposition 3 of [Re2].

Proposition 3.1. *Assume that $N \geq 3$ and $\nu > 0$, or $N = 2$ and $0 < \nu \leq 3$, and $0 < 2/(\nu+1) < \mu < \gamma^+_{N+1}$. Then for any $g \in C^{0,\alpha}_{\mu-2}$ there exists a unique solution of $\mathcal{L}w = g$ in $B\backslash\{0\}$ which belongs to the space $C^{2,\alpha}_{\mu,N}$. In addition, the mapping $g \in C^{0,\alpha}_{\mu-2} \to w \in C^{2,\alpha}_{\mu,N}$ is bounded.*

Proof. By our assumptions, we know from Proposition 2.1 that for $N \geq 3$ and $\nu > 0$,

$$\Re(\gamma^+_0) < \mu \ \text{ and } \ \Re(\gamma^+_0) + \mu + N - 3 > -1,$$

$$\gamma^+_1 = \ldots = \gamma^+_N < \mu < \gamma^+_{N+1} \ \text{ and } \ \mu + \gamma^+_1 > -1.$$

For $N = 2$ and $0 < \nu \leq 3$,

$$\Re(\gamma^+_0) < \mu \ \text{ and } \ \Re(\gamma^+_0) + \mu + N - 3 > -1,$$

$$\gamma^+_1 = \gamma^+_2 < \mu < \gamma^+_3 \ \text{ and } \ \mu + \gamma^+_1 > 0.$$

Choosing

$$w_i(r) = -r^{\gamma^+_i}\int_r^1 \kappa^{1-N-2\gamma^+_i}\int_0^\kappa s^{N-1+\gamma^+_i} g_i(s)\,ds\,d\kappa$$

for $i = N+1$, $N+2$, ..., and

$$w_i(r) = \Re(r^{\gamma^+_i}\int_0^r \kappa^{1-N-2\gamma^+_i}\int_0^\kappa s^{N-1+\gamma^+_i} g_i(s)\,ds\,d\kappa)$$

for $i = 0, 1, \ldots, N$ as those in the proof of Proposition 3 of [Re2], we easily know that $w_i(r)$ exists for each i and that there exists some constant $c_i > 0$ such that for all $r \in (0,1]$, $r^{-\mu}|w_i(r)| \leq c_i\|g\|_{C^{2,\alpha}_{\mu-2}}$. Thus, this proposition can be easily obtained from Proposition 3 of [Re2] by choosing $j = N+1$. ∎

Let us define

$$(C^{2,\alpha}_\mu \oplus r^{\gamma_1^+}\operatorname{span}\{\varphi_1(\theta),\varphi_2(\theta),\dots,\varphi_N(\theta)\})_0 = \Big\{w \in C^{2,\alpha}_\mu \oplus r^{\gamma_1^+}\operatorname{span}\{\varphi_1(\theta),\varphi_2(\theta),\dots,\varphi_N(\theta)\} : \ w|_{\partial B} \in \operatorname{span}\{1\}\Big\}.$$

We easily obtain the following corollaries from the previous propositions.

Corollary 3.2. *Assume that $N \geq 3$ and $\nu > 0$, or $N = 2$ and $0 < \nu \leq 3$, and $0 < 2/(\nu+1) < \mu < \gamma^+_{N+1}$. Then for any $g \in C^{0,\alpha}_{\mu-2}$ there exists a unique solution of $\mathcal{L}w = g$ in $B\backslash\{0\}$ which belongs to the space $(C^{2,\alpha}_\mu \oplus r^{\gamma_1^+} span\{\varphi_1(\theta),\dots,\varphi_N(\theta)\})_0$. In addition, the mapping $g \in C^{0,\alpha}_{\mu-2} \to w \in (C^{2,\alpha}_\mu \oplus r^{\gamma_1^+} span\{\varphi_1(\theta),\dots,\varphi_N(\theta)\})_0$ is bounded.*

The same results hold for $N = 2$ and $1 < \nu \leq 3$; $N = 3$ and $\nu > 3$, if the space

$$(C^{2,\alpha}_\mu \oplus r^{\gamma_1^+} span\{\varphi_1(\theta),\dots,\varphi_N(\theta)\})_0$$

is replaced by $(C^{2,\alpha}_\mu \oplus r^{\gamma_1^-} span\{\varphi_1(\theta),\dots,\varphi_N(\theta)\})_0$.

Proof. Choosing

$$w_i(r) = -r^{\gamma_i^+}\int_r^1 \kappa^{1-N-2\gamma_i^+}\int_0^\kappa s^{N-1+\gamma_i^+} g_i(s)dsd\kappa$$

for $i = 1, 2, \dots$, and

$$w_0(r) = \Re(r^{\gamma_0^+}\int_0^r \kappa^{1-N-2\gamma_0^+}\int_0^\kappa s^{N-1+\gamma_0^+} g_0(s)dsd\kappa)$$

as those in the proof of Proposition 3 of [Re2], we easily know that $w_i(r)$ exists for each i. We know that for $i = 0, N+1, N+2, \dots$, there exist constants $c_i > 0$ such that for all $r \in (0,1]$, $r^{-\mu}|w_i(r)| \leq c_i\|g\|_{C^{2,\alpha}_{\mu-2}}$. Thus, the first part of this corollary can be easily obtained from Corollary 2 of [Re2].

To show the second part, we choose

$$w_i(r) = -r^{\gamma_i^+}\int_r^1 \kappa^{1-N-2\gamma_i^+}\int_0^\kappa s^{N-1+\gamma_i^+} g_i(s)dsd\kappa$$

for $i = N+1, N+2, \dots$;

$$w_i(r) = -r^{\gamma_i^-}\int_r^1 \kappa^{1-N-2\gamma_i^-}\int_0^\kappa s^{N-1+\gamma_i^-} g_i(s)dsd\kappa$$

for $i = 1, 2, \dots, N$; and

$$w_0(r) = \Re(r^{\gamma_0^+}\int_0^r \kappa^{1-N-2\gamma_0^+}\int_0^\kappa s^{N-1+\gamma_0^+} g_0(s)dsd\kappa)$$

as those in the proof of Proposition 3 of [Re2]. It is easily known that, for $N = 2$ and $1 < \nu \leq 3$; $N = 3$ and $\nu > 3$, $w_i(r)$ exists for each i and, for $i = 0, N+1, N+2, \dots$, there exist constants $c_i > 0$ such that for all $r \in (0,1]$, $r^{-\mu}|w_i(r)| \leq c_i\|g\|_{C^{2,\alpha}_{\mu-2}}$. Moreover, if $0 < 2/(\nu+1) < \mu < \gamma^+_{N+1}$, then $\gamma_1^- < \mu$. Thus, the second part of this corollary can also be easily obtained from Corollary 2 of [Re2]. ∎

Corollary 3.3. *Assume that* $N = 2$ *and* $\nu > 3$, *and* $0 < 2/(\nu+1) < \mu < \min\{\gamma_1^+, \gamma_3^+\}$. *Then for any* $g \in C^{0,\alpha}_{\mu-2}$ *there exists a unique solution of* $\mathcal{L}w = g$ *in* $B\backslash\{0\}$ *which belongs to the space* $C^{2,\alpha}_{\mu,0}$. *In addition, the mapping* $g \in C^{0,\alpha}_{\mu-2} \to w \in C^{2,\alpha}_{\mu,0}$ *is bounded.*

Proof. It is easily known from Proposition 2.1 that $\gamma_1^+ = \gamma_2^+ > 2/(1+\nu)$ for $N = 2$ and $\nu > 3$. Therefore, choosing

$$w_i(r) = -r^{\gamma_i^+} \int_r^1 \kappa^{1-N-2\gamma_i^+} \int_0^\kappa s^{N-1+\gamma_i^+} g_i(s) ds d\kappa$$

for $i = 1, 2, \ldots$, and

$$w_0(r) = \Re(r^{\gamma_0^+} \int_0^r \kappa^{1-N-2\gamma_0^+} \int_0^\kappa s^{N-1+\gamma_0^+} g_0(s) ds d\kappa)$$

as those in the proof of Proposition 3 of [Re2], we easily know that $w_i(r)$ exists for each i and that there exists some constant $c_i > 0$ such that for all $r \in (0, 1]$, $r^{-\mu}|w_i(r)| \le c_i \|g\|_{C^{2,\alpha}_{\mu-2}}$. Thus, this corollary can be easily obtained from Proposition 3 of [Re2]. □

4. The case of $\psi(\theta) = 1 + \zeta(\theta)$**.** In this section we will find nonnegative solutions u of (1.1) with $\psi(\theta) = 1 + \zeta(\theta)$ and $\|\zeta\|_{C^{2,\alpha}(S^{N-1})}$ being sufficiently small. Moreover, u has a nonremovable zero point. We first obtain the following theorem.

Theorem 4.1. *Given* $N \ge 3$ *and* $\nu > 0$, *or* $N = 2$ *and* $0 < \nu \le 3$, *there exists* $\epsilon > 0$, *such that, for any* $\eta \in C^{2,\alpha}(S^{N-1})$, *if* $\|\eta\|_{C^{2,\alpha}(S^{N-1})} < \epsilon$, *there exist* $\zeta_\eta \in C^{2,\alpha}(S^{N-1})$ *satisfying* $\|\zeta_\eta\|_{C^{2,\alpha}(S^{N-1})} \le \Lambda\epsilon < 1/4$ *(*$\Lambda > 0$ *independent of* ϵ*) and a nonnegative solution* u_η *of the problem*

$$\Delta u = \lambda_0 u^{-\nu} \ \text{ in } B\backslash\{0\}, \quad u = 1 + \eta + \zeta_\eta \ \text{ on } S^{N-1} \tag{4.1}$$

with a nonremovable zero at 0.

Proof. Choosing $0 < 2/(\nu+1) < \mu < \gamma_{N+1}^+$, we have from Proposition 2.1 that for $N \ge 3$ and $\nu > 0$ or $N = 2$ and $0 < \nu \le 3$,

$$\gamma_1^+ = \ldots = \gamma_N^+ < \mu < \gamma_{N+1}^+.$$

For any $\eta \in C^{2,\alpha}(S^{N-1})$ we define $w_\eta(x) = \chi(r)\eta(\theta)$ where χ is some fixed regular function which equals to 0 in $B_{1/2}$ and equals to 1 outside $B_{3/4}$.

We are going to find a solution $v \in C^{2,\alpha}_{\mu,N}$ of the equation

$$\Delta(u_0 + v + w_\eta) = \lambda_0 |u_0 + v + w_\eta|^{-(\nu+1)}(u_0 + v + w_\eta) \quad \text{in } B\backslash\{0\}. \tag{4.2}$$

To this end, we define, for all $(v, \eta) \in C^{2,\alpha}_{\mu,N} \times C^{2,\alpha}(S^{N-1})$

$$\mathcal{N}(v, \eta) \equiv \Delta(u_0 + v + w_\eta) - \lambda_0 |u_0 + v + w_\eta|^{-(\nu+1)}(u_0 + v + w_\eta).$$

It is easy to see that $\mathcal{N}$ is well defined from $C^{2,\alpha}_{\mu,N} \times C^{2,\alpha}(S^{N-1})$ into $C^{0,\alpha}_{\mu-2}$. In addition, $\mathcal{N}(0,0) = 0$ and $D\mathcal{N}|_{(0,0)}(v, 0) = \mathcal{L}v$. It follows easily from the implicit function theorem and Proposition 3.1 that all solutions of the equation (4.2) near $(0,0)$ are of the form (v_η, η) where $\eta \in C^{2,\alpha}(S^{N-1}) \to v_\eta \in C^{2,\alpha}_{\mu,N}$ is a regular mapping.

Thus, we can choose $\epsilon > 0$ sufficiently small, which satisfies that for any η satisfying $\|\eta\|_{C^{2,\alpha}(S^{N-1})} < \epsilon$, there is $v_\eta \in C^{2,\alpha}_{\mu,N}$ satisfying $\|v_\eta\|_{C^{2,\alpha}_\mu} \leq \Lambda\epsilon < 1/4$, where $\Lambda > 0$ is independent of ϵ, such that $u_\eta := u_0 + v_\eta + w_\eta$ is a solution of (4.2). It is easy to see that $v_\eta(0) + u_0(0) + w_\eta(0) = 0$. Since $u_0(x) = |x|^{2/(\nu+1)}$ and $v_\eta(x) \leq |x|^\mu/4$ with $2/(\nu+1) < \mu$, we know that $u_0(x) + v_\eta(x) + w_\eta(x) > 0$ for $x \in B_\delta \backslash \{0\}$, where $\delta > 0$ is a sufficiently small number. Note that $u_0(x) \geq \delta^{2/(\nu+1)}$ for $x \in B \backslash B_\delta$. By choosing $\epsilon > 0$ small enough, we obtain that

$$u_0 + v_\eta + w_\eta > 0 \text{ in } B\backslash\{0\}. \tag{4.3}$$

This implies that $u_\eta = u_0 + v_\eta + w_\eta$ is a nonnegative solution of the equation in (4.1) with $u_\eta(0) = 0$. Moreover,

$$u_\eta(\theta) = 1 + v_\eta(\theta) + \eta(\theta) \quad \text{for } \theta \in S^{N-1}. \tag{4.4}$$

Defining $\zeta(\theta) = v_\eta(\theta)$, we easily see that ζ is the required function. This completes the proof of Theorem 4.1. □

From Theorem 4.1 and Corollary 3.3, we easily obtain the following corollary.

Corollary 4.2. *Given $N = 2$ and $\nu > 3$, there exists $\epsilon > 0$ sufficiently small, such that, for any $\eta \in C^{2,\alpha}(S^1)$, if $\|\eta\|_{C^{2,\alpha}(S^1)} < \epsilon$, there exist a constant c_η satisfying $|c_\eta| \leq (\Lambda + 1)\epsilon < 1/2$ ($\Lambda > 0$ independent of ϵ) and a nonnegative solution u_η of the problem*

$$\Delta u = \lambda_0 u^{-\nu} \text{ in } B\backslash\{0\}, \quad u = 1 + c_\eta + \eta \text{ on } S^1 \tag{4.5}$$

with a nonremovable zero at 0.

Theorem 4.3. *Given $N \geq 3$ and $\nu > 0$, or $N = 2$ and $0 < \nu \leq 3$, there exists $\epsilon > 0$ sufficiently small such that, for any $y \in B_\epsilon \subset B$, there exist $\zeta_y \in C^{2,\alpha}(S^{N-1})$ satisfying $\|\zeta_y\|_{C^{2,\alpha}(S^{N-1})} \leq (\Lambda+1)\epsilon < 1/2$ ($\Lambda > 0$ independent of ϵ) and a nonnegative solution u_y of the problem*

$$\Delta u = \lambda_0 u^{-\nu} \text{ in } B\backslash\{y\}, \quad u = 1 + \zeta_y \text{ on } S^{N-1}. \tag{4.6}$$

with a nonremovable zero at y.

Proof. Let $T:\ B \times B_{1/4} \to B$ be a $C^{2,\alpha}$ map which satisfies that, for all $y \in B_{1/4}$, $T(\cdot, y)$ is a $C^{2,\alpha}$ diffeomorphism from the unit ball into itself. Moreover, T satisfies that

$$T(x,y) = \begin{cases} x - y & \text{for all } x, y \in B_{1/4}, \\ x & \text{for all } x \in B\backslash B_{3/4} \text{ and all } y \in B_{1/4} \end{cases}$$

and

$$T(x,0) = x.$$

For $0 < 2/(\nu+1) < \mu < \gamma^+_{N+1}$, we define the nonlinear mapping

$$\mathcal{N}(v,y) = \Delta((u_0 + v) \circ T(\cdot, y)) \circ T^{-1}(\cdot, y) - \lambda_0 |u_0 + v|^{-(\nu+1)}(u_0 + v).$$

It is easy to see that $\mathcal{N}$ is well defined from $C^{2,\alpha}_{\mu,N} \times B_{1/4}$ into $C^{0,\alpha}_{\mu-2}$. In addition, $\mathcal{N}(0,0) = 0$ and

$$D\mathcal{N}|_{(0,0)}(v,0) = \mathcal{L}v.$$

It follows easily from the implicit function theorem and Proposition 3.1 that all solutions of the equation $\mathcal{N}(v,y)=0$ near $(0,0)$ are of the form (v_y,y) where

$$y\in B_{1/4}\to v_y\in C^{2,\alpha}_{\mu,N}$$

is some regular mapping. That is, we can choose $\epsilon>0$ which satisfies that, for any $y\in B_\epsilon$ there is $v_y\in C^{2,\alpha}_{\mu,N}$ satisfying $\|v_y\|_{C^{2,\alpha}_\mu}\le\Lambda\epsilon<1/4$, where $\Lambda>0$ is independent of ϵ, such that $u_y:=u_0+v_y$ is a solution of the equation

$$\Delta(u\circ T(\cdot,y))\circ T^{-1}(\cdot,y)=\lambda_0|u|^{-(\nu+1)}u\ \text{ in } B\backslash\{0\}. \tag{4.7}$$

Arguments similar to those in the proof of Theorem 4.1 imply that $u_y=u_0+v_y$ satisfies $u_y(y)=0$ and $u_y>0$ in $B\backslash\{y\}$. Define $\zeta_y(\theta)=v_y(\theta)$ for $\theta\in S^{N-1}$. Then $\|\zeta_y\|_{C^{2,\alpha}(S^{N-1})}\le(\Lambda+1)\epsilon$. This completes the proof. □

From Theorem 4.3 and Corollary 3.3, we easily obtain the following corollary.

Corollary 4.4. *Given $N=2$ and $\nu>3$, there exists $\epsilon>0$ sufficiently small such that, for any $y\in B_\epsilon\subset B$, there exists a nonnegative solution (λ,u_y) of the problem*

$$\Delta u=\lambda u^{-\nu}\ \text{ in } B\backslash\{y\},\quad u=1\ \text{ on } S^1. \tag{4.8}$$

with a nonremovable zero at y.

Proof. Using the same idea as in the proof of Theorem 4.3 and Corollary 3.3, we see that there exists $\epsilon>0$ such that, for any $y\in B_\epsilon$, there exist a constant c_y satisfying $|c_y|\le 1/2$ and a nonnegative solution $\tilde{u}_y$ of the problem

$$\Delta u=\lambda_0 u^{-\nu}\ \text{ in } B\backslash\{y\},\quad u=1+c_y\ \text{ on } S^1.$$

with a nonremovable zero at y. Setting $u_y=\tilde{u}_y/(1+c_y)$, we easily see that u_y satisfies (4.8) with $\lambda=\lambda_0(1+c_y)^{-(\nu+1)}$. Moreover, y is a nonremovable zero point of u_y. □

Theorem 4.5. *Given $N\ge 3$ and $\nu>0$, or $N=2$ and $0<\nu\le 3$, there exists $\epsilon>0$ sufficiently small such that, for any $\eta\in C^{2,\alpha}(S^{N-1})$, if $\|\eta\|_{C^{2,\alpha}(S^{N-1})}<\epsilon$, there exist $x_\eta\in B$; a constant c_η satisfying $|c_\eta|\le(\Lambda+1)\epsilon<1/2$ ($\Lambda>0$ independent of ϵ) and u_η a nonnegative solution of the problem*

$$\Delta u=\lambda_0 u^{-\mu}\ \text{ in } B\backslash\{x_\eta\},\ \ u=1+c_\eta+\eta\ \text{ on } S^{N-1} \tag{4.9}$$

with a nonremovable zero at x_η.

Proof. It is known from Proposition 2.1 that for $N\ge 4$ and $\nu>0$,

$$\gamma_1^+=\gamma_2^+=\ldots=\gamma_N^+=\frac{(1-\nu)}{(1+\nu)};$$

for $N=2$ and $0<\nu\le 1$,

$$\gamma_1^+=\gamma_2^+=\frac{(1-\nu)}{(1+\nu)};$$

for $N=2$ and $\nu>1$,

$$\gamma_1^-=\gamma_2^-=\frac{(1-\nu)}{(1+\nu)};$$

for $N = 3$ and $0 < \nu \leq 3$,

$$\gamma_1^+ = \gamma_2^+ = \gamma_3^+ = \frac{(1-\nu)}{(1+\nu)};$$

for $N = 3$ and $\nu > 3$,

$$\gamma_1^- = \gamma_2^- = \gamma_3^- = \frac{(1-\nu)}{(1+\nu)}.$$

We choose μ such that $0 < 2/(\nu+1) < \mu < \gamma_{N+1}^+$ and define the space $\mathbb{M}$ as follows:

$$\mathbb{M} = \text{span}\{\varphi_1(\theta), \varphi_2(\theta), \ldots, \varphi_N(\theta)\}.$$

Thanks to Corollary 3.2, for all $g \in C^{0,\alpha}_{\mu-2}$, the problem

$$\mathcal{L}w = g \text{ in } B\backslash\{0\}$$

has a solution in the space $(C^{2,\alpha}_\mu \oplus r^{(1-\nu)/(1+\nu)}\mathbb{M})_0$. Note that for $N \geq 4$ and $\nu > 0$; $N = 2$ and $0 < \nu \leq 1$; $N = 3$ and $0 < \nu \leq 3$, we use γ_1^+ in Corollary 3.2. For $N = 2$ and $1 < \nu \leq 3$; $N = 3$ and $\nu > 3$, we use γ_1^- in Corollary 3.2. It is clear that

$$\nabla|x|^{2/(\nu+1)} = \frac{2}{\nu+1}|x|^{(1-\nu)/(1+\nu)}\nabla|x|.$$

Given a function $\eta \in C^{2,\alpha}(S^{N-1})$ we have to find a solution $(v, y) \in C^{2,\alpha}_{\mu,0} \times \mathbb{R}^N$ of the equation

$$\Delta((u_0 + w_\eta + v) \circ T(\cdot, y)) \circ T^{-1}(\cdot, y) - \lambda_0 f(u_0 + w_\eta + v) = 0 \text{ in } B\backslash\{0\}$$

where $f(s) = |s|^{-(\nu+1)}s$. We define the nonlinear mapping

$$\mathcal{N}(v, y, \eta) = [\Delta((u_0 + w_\eta + v) \circ T(\cdot, y)) - \lambda_0 f((u_0 + w_\eta + v) \circ T(\cdot, y))] \circ T^{-1}(\cdot, y).$$

Obviously, $\mathcal{N}$ is well defined from $C^{2,\alpha}_{\mu,0} \times \mathbb{R}^N \times C^{2,\alpha}(S^{N-1})$ into the space $C^{0,\alpha}_{\mu-2}$. We notice that $\mathcal{N}(0,0,0) = 0$. Furthermore

$$D\mathcal{N}|_{(0,0,0)}(v, 0, 0) = \mathcal{L}v$$

and since $\Delta u_0 = \lambda_0 u_0^{-\nu}$ in B,

$$\begin{aligned} D\mathcal{N}|_{(0,0,0)}(0, z, 0) &= \Delta(Du_0|_x \circ D_yT|_{(x,0)}(z)) \\ &\quad + \lambda_0 \nu u_0^{-(\nu+1)}(Du_0|_x \circ D_yT|_{(x,0)}(z)) \\ &= \mathcal{L}(Du_0|_x \circ D_yT|_{(x,0)}(z)). \end{aligned}$$

Therefore,

$$D\mathcal{N}|_{(0,0,0)}(w, z, 0) = \mathcal{L}(w + Du_0|_x \circ D_yT|_{(x,0)}(z)).$$

Since $D_yT|_{(x,0)}(z) = 0$ if $x \in B\backslash B_{3/4}$ and since $D_yT|_{(x,0)}(z) = -z$ if $x \in B_{1/4}$ we see from [Re2] that $(C^{2,\alpha}_\mu \oplus r^{(1-\nu)/(1+\nu)}\mathbb{M})_0 = C^{2,\alpha}_{\mu,0} \oplus \text{span}\{Du_0|_x \circ D_yT|_{(x,0)}(z) : z \in \mathbb{R}^N\}$.

We can use the implicit function theorem to prove that all solutions $\mathcal{N}(v, y, \eta) = 0$ near $(0, 0, 0)$ are given by (v_η, y_η, η) where

$$\eta \in C^{2,\alpha} \to (v_\eta, y_\eta) \in C^{2,\alpha}_{\mu,0} \times \mathbb{R}^N$$

is a regular mapping. Therefore, arguments similar to those in the proof of Theorem 4.1 imply that $u_\eta := u_0 + w_\eta + v_\eta$ is a nonnegative solution of the equation in (4.9) which satisfies that $u_\eta = u_0 + w_\eta + v_\eta > 0$ in $B\backslash\{y_\eta\}$ and $u_\eta(y_\eta) = 0$. Moreover, $u_\eta(\theta) = 1 + c_\eta + \eta(\theta)$ for $\theta \in S^{N-1}$, where $c_\eta = v_\eta|_{S^{N-1}}$ is a constant. This completes the proof. □

The following corollary is an easy consequence of Theorem 4.5.

Corollary 4.6. *Given $N \geq 3$ and $\nu > 0$, or $N = 2$ and $0 < \nu \leq 3$, there exists $\epsilon > 0$ sufficiently small such that, for any constant ρ, if $|\rho| < \epsilon$, there exist $x_\rho \in B$ and (λ, u_ρ) a nonnegative solution of the problem*

$$\Delta u = \lambda u^{-\nu} \text{ in } B\backslash\{x_\rho\}, \quad u = 1 \text{ on } S^{N-1} \tag{4.10}$$

with a nonremovable zero at x_ρ.

Proof. It follows from Theorem 4.5 that for any constant ρ (since $\rho \in C^{2,\alpha}(S^{N-1})$) satisfying $|\rho| < \epsilon$, there exist $x_\rho \in B$; a constant c_ρ satisfying $|c_\rho| \leq (\Lambda + 1)\epsilon < 1/2$ and $\tilde{u}_\rho$ a nonnegative solution of the problem

$$\Delta u = \lambda_0 u^{-\nu} \text{ in } B\backslash\{x_\rho\}, \quad u = 1 + c_\rho + \rho \text{ on } S^{N-1} \tag{4.11}$$

with a nonremovable zero at x_ρ. Defining $u_\rho := \tilde{u}_\rho/(1 + c_\rho + \rho)$, we have that u_ρ satisfies the problem

$$\Delta u = \lambda u^{-\nu} \text{ in } B\backslash\{x_\rho\}, \quad u = 1 \text{ on } S^{N-1} \tag{4.12}$$

where $\lambda = \lambda_0(1 + c_\rho + \rho)^{-(\nu+1)}$. It is clear that x_ρ is a non removable zero of u_ρ. □

5. The case of $\psi(\theta) = C + \zeta(\theta)$. In this section we use the results obtained in Section 4 to consider the case that $\psi(\theta) = C + \zeta(\theta)$, where $C > 1$ or $0 < C < 1$, for $\theta \in S^{N-1}$ and $\zeta \in C^{2,\alpha}(S^{N-1})$ satisfying that $\|\zeta\|_{C^{2,\alpha}(S^{N-1})}$ is sufficiently small.

By simple calculations, we easily know that $u_C(x) = C|x|^{2/(\nu+1)}$ satisfies the problem

$$\Delta u = \frac{2C^{\nu+1}(N + (N-2)\nu)}{(\nu+1)^2} u^{-\nu} \text{ in } B\backslash\{0\}, \quad u = C \text{ on } \partial B. \tag{5.1}$$

Define $\lambda_C = \frac{2C^{\nu+1}(N+(N-2)\nu)}{(\nu+1)^2}$ and the linear operator

$$\mathcal{L} : w \longmapsto \Delta w + \lambda_C \nu u_C^{-(\nu+1)} w.$$

Clearly we have

$$\lambda_C \nu u_C^{-(\nu+1)} = \frac{c_0}{r^2}, \tag{5.2}$$

where c_0 is same as in (2.1). Thus, $\mathcal{L}$ is exactly same as that we defined in Section 2. Thus, the indical roots of $\mathcal{L}$ are defined in (2.3). By arguments similar to those in the proofs of Theorems 4.1, 4.3, 4.5, we easily obtain the following results.

Theorem 5.1. *Given $N \geq 3$ and $\nu > 0$, or $N = 2$ and $0 < \nu \leq 3$, there exists $\epsilon > 0$ such that, for any $\eta \in C^{2,\alpha}(S^{N-1})$, if $\|\eta\|_{C^{2,\alpha}(S^{N-1})} < \epsilon$, there exist $\zeta_\eta \in C^{2,\alpha}(S^{N-1})$ satisfying $\|\zeta_\eta\|_{C^{2,\alpha}(S^{N-1})} \leq (\Lambda+1)\epsilon < C/2$ ($\Lambda > 0$ independent of ϵ) and a nonnegative solution u_η of the problem*

$$\Delta u = \lambda_C u^{-\nu} \ \text{ in } B\backslash\{0\}, \quad u = C + \zeta_\eta \ \text{ on } S^{N-1} \tag{5.3}$$

with a nonremovable zero at 0.

Corollary 5.2. *Given $N = 2$ and $\nu > 3$, there exists $\epsilon > 0$ sufficiently small, such that, for any $\eta \in C^{2,\alpha}(S^1)$, if $\|\eta\|_{C^{2,\alpha}(S^1)} < \epsilon$, there exist a constant c_η satisfying $|c_\eta| \leq (\Lambda+1)\epsilon < C/2$ ($\Lambda > 0$ independent of ϵ) and a nonnegative solution u_η of the problem*

$$\Delta u = \lambda_C u^{-\nu} \ \text{ in } B\backslash\{0\}, \quad u = C + c_\eta + \eta \ \text{ on } S^1 \tag{5.4}$$

with a nonremovable zero at 0.

Theorem 5.3. *Given $N \geq 3$ and $\nu > 0$, or $N = 2$ and $0 < \nu \leq 3$, there exists $\epsilon > 0$ sufficiently small such that, for any $y \in B_\epsilon \subset B$, there exist $\zeta_y \in C^{2,\alpha}(S^{N-1})$ satisfying $\|\zeta_y\|_{C^{2,\alpha}(S^{N-1})} \leq (\Lambda+1)\epsilon < C/2$ ($\Lambda > 0$ independent of ϵ) and a nonnegative solution u_y of the problem*

$$\Delta u = \lambda_C u^{-\nu} \ \text{ in } B\backslash\{y\}, \quad u = C + \zeta_y \ \text{ on } S^{N-1}. \tag{5.5}$$

with a nonremovable zero at y.

Corollary 5.4. *Given $N = 2$ and $\nu > 3$, there exists $\epsilon > 0$ sufficiently small such that, for any $y \in B_\epsilon \subset B$, there exists a nonnegative solution (λ, u_y) of the problem*

$$\Delta u = \lambda u^{-\nu} \ \text{ in } B\backslash\{y\}, \quad u = C \ \text{ on } S^1. \tag{5.6}$$

with a nonremovable zero at y.

Theorem 5.5. *Given $N \geq 3$ and $\nu > 0$, or $N = 2$ and $0 < \nu \leq 3$, there exists $\epsilon > 0$ sufficiently small such that, for any $\eta \in C^{2,\alpha}(S^{N-1})$, if $\|\eta\|_{C^{2,\alpha}(S^{N-1})} < \epsilon$, there exist $x_\eta \in B$; a constant c_η satisfying $|c_\eta| \leq (\Lambda+1)\epsilon < C/2$ ($\Lambda > 0$ independent of ϵ) and u_η a nonnegative solution of the problem*

$$\Delta u = \lambda_C u^{-\nu} \ \text{ in } B\backslash\{x_\eta\}, \ \ u = C + c_\eta + \eta \ \text{ on } S^{N-1} \tag{5.7}$$

with a nonremovable zero at x_η.

We can also obtain the existence for a class of Dirichlet problems with constant boundary values.

Corollary 5.6. *Given $N \geq 3$ and $\nu > 0$, or $N = 2$ and $0 < \nu \leq 3$, there exists $\epsilon > 0$ such that, for any constant ρ, if $|\rho| < \epsilon$, there exist $x_\rho \in B$ and (λ_ρ, u_ρ) a nonnegative solution of the problem*

$$\Delta u = \lambda u^{-\nu} \ \text{ in } B\backslash\{x_\rho\}, \quad u = C \quad \text{on } S^{N-1} \tag{5.8}$$

with a nonremovable zero at x_ρ*.*

REMARK. It is easily seen from Theorems 5.1, 5.3 and 5.5 that the parameter λ depends upon the boundary value C. We can also obtain the existence for any $\lambda > 0$, but the boundary value changes. Indeed, for any fixed $\lambda > 0$, we easily know that $(\lambda, u_\lambda(r))$ is a nonnegative solution, with one isolated zero at 0, of the problem

$$\Delta u = \lambda u^{-\nu} \ \text{ in } B\backslash\{0\} \tag{5.9}$$

where

$$u_\lambda(r) = \Big[\frac{\lambda(\nu+1)^2}{2((N-2)\nu+N))}\Big]^{1/(\nu+1)} r^{2/(\nu+1)}.$$

It is clear that the boundary value of u_λ is the constant in the expression of $u_\lambda(r)$. Now we define

$$\mathcal{L} : w \longmapsto \Delta w + \lambda\nu u_\lambda^{-(\nu+1)} w.$$

It is clear that $\lambda\nu u_\lambda^{(-\nu+1)} = c_0 r^{-2}$ and $\mathcal{L}$ is exactly same as that we defined in Section 2. Thus, we can derive results similar to Theorems 5.1, 5.3, 5.5, but with different boundary conditions.

REFERENCES

[BBD] J. P. Burelbach, S. G. Bankoff and S. H. Davis, *Nonlinear stability of evaporating/condensing liquid films*, J. Fluid Mech., 195 (1988), pp. 463–494.

[BP1] A. L. Bertozzi and M. C. Pugh, *Long-wave instabilities and saturation in thin film equations*, Comm. Pure Appl. Math., 51:6 (1998), pp. 625–661.

[BP2] A. L. Bertozzi and M. C. Pugh, *Finite-time blow-up of solutions of some long-wave unstable thin film equations*, Indiana Univ. Math. J., 49:4 (2000), pp. 1323–1366.

[CR] M. G. Crandall and P. H. Rabinowitz and L. Tartar, *On a Dirichlet problem with a singular nonlinearity*, Comm. Part. Diff. Eqns., 2 (1977), pp. 193–222.

[De] M. del Pino, *A global estimate for the gradient in a singular elliptic problem*, Proc. Roy. Soc. Edinburgh Sect. A, 122 (1992), pp. 341–352.

[DH] M. A. del Pino and G. E. Hernandez, *Solvability of the Neumann problem in a ball for* $-\Delta u + u^{-\nu} = h(|x|)$, $\nu > 1$, J. Diff. Eqns., 124 (1996), pp. 108–131.

[FMP] G. Floes, G. A. Mercado and J. A. Pelesko, *Dynamics and touchdown in electrostatic MEMS*, Proceedings of ICMENS 2003, pp. 182–187.

[GHL] J. A. Gatica, G. E. Hernandez and P. Waltman, *Radially symmetric solutions of a class of singular elliptic equations*, Proc. Edinburgh Math. Soc., 33 (1990), pp. 169–180.

[Go] S. M. Gomes, *On a singular nonlinear elliptic problem*, SIAM J. Math. Anal., 17 (1986), pp. 1359–1369.

[GL] C. Gui and F. Lin, *Regularity of an elliptic problem with a singular nonlinearity*, Proc. Roy. Soc. Edinburgh Sect. A, 123 (1993), pp. 1021–1029.

[GG1] N. Ghoussoub and Y. Guo, *On the partial differential equations of electrostatic MEMS devices I: stationary case*, SIAM J. Math. Anal., 38:5 (2006/2007), pp. 1423–1449.

[GG2] N. Ghoussoub and Y. Guo, *On the partial differential equations of electrostatic MEMS devices II: dynamic case*, NoDEA Nonlinear Differential Equations Appl., 15:1-2 (2008), pp. 115–145.

[GG3] Y. Guo, *Global solutions of singular equations arising from electrostatic MEMS*, J. Diff. Eqns, 245:3 (2008), pp. 809–844.

[GPW] Y. Guo, Z. Pan and M. J. Ward, *Touchdown and pull-in voltage behavior of a MEMS device with varying dielectric properties*, SIAM J. Appl. Math., 66:1 (2005), pp. 309–338.

[GW1] Z. Guo and J. Wei, *Hausdorff dimension of zero set of weak solutions of a semilinear elliptic equation with singular nonlinearity*, Manuscripta Math., 120:2 (2006), pp. 193–209.

[GW2] Z. Guo and J. Wei, *Asymptotic behavior of touch-twon solutions and global bifurcations for an elliptic equation with a singular nonlinearity*, Comm. Pure Appl. Anal., 7:4 (2008), pp. 765–786.

[MP] R. Mazzeo and F. Pacard, *A construction of singular solutions for semilinear elliptic equation using asymptotic analysis*, J. Diff. Geometry, 44 (1996), pp. 331–370.

[MPU] R. Mazzeo, D. Pollack and K. Uhlenbeck, *Moduli spaces of singular Yamabe metrics*, J. Amer. Math. Soc., 9:2 (1996), pp. 303–344.

[LP1] R. S. Laugesen and M. C. Pugh, *Properties of steady states for thin film equations*, Europ. J. Appl. Math., 11 (2000), pp. 293–351.

[LP2] R. S. Laugesen and M. C. Pugh, *Finite-time blow-up of solutions of some long-wave unstable thin film equations*, Indiana Univ. Math. J., 49:4 (2000), pp. 1323–1366.

[LP3] R. S. Laugesen and M. C. Pugh, *Linear stability of steady states for thin film and Cahn-Hilliard type equations*, Arch. Ration. Mech. Anal., 154:1 (2000), pp. 3–51.

[Re1] Y. Rébaï, *Weak solutions of nonlinear elliptic equations with prescribed singular set*, J. Diff. Eqns., 127 (1996), pp. 439–453.

[Re2] Y. Rébaï, *Solutions of semilinear elliptic equations with one isolated singularity*, Diff. Integral Eqns., 12:4 (1999), pp. 563–581.

[WB] T. P. Witelski and A. J. Bernoff, *Dynamics of three-dimensional thin film rupture*, Physica D, 147 (2000), pp. 155–176.

[YD] S. G. Yiantsios and R. H. Davis, *Close approach and deformation of two viscous drops due to gravity and van der Waals forces*, J. Colloid Interf. Int. Sci., 144 (1991), pp. 412–433.

[YH] S. G. Yiantsios and B. G. Higgins, *Rupture of thin films: nonlinear stability analysis*, J. Colloid Interf. Int. Sci., 147 (1991), pp. 341–350.