SPACE-TIME ESTIMATES IN THE BESOV SPACES AND THE NAVIER-STOKES EQUATIONS*

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Abstract. In this paper, we establish the space-time estimates in the Besov spaces of the solution to the Navier-Stokes equations in $\mathbf{R}^n, n \geq 3$. As an application, we improve some known results about the regularity criterion of weak solutions and the blow-up criterion of smooth solutions. Our main tools are the frequency localization and the Littlewood-Paley decomposition.

Key words. Navier-Stokes equations, Besov space, Regularity criterion, Littlewood-Paley decomposition

AMS subject classifications. 35Q30, 35D10, 76D03, 76D05

1. Introduction. In this paper, we are concerned with the regularity of weak solutions and the blow-up phenomena of smooth solutions to the Navier-Stokes equations in $\mathbf{R}^n, n \geq 3$:

$$\text{(NS)} \quad \begin{cases} u_t - \Delta u + u \cdot \nabla u + \nabla \pi = 0, & \text{in} \quad \mathbf{R}^n \times (0, T), \\ \nabla \cdot u = 0, & \text{in} \quad \mathbf{R}^n \times (0, T), \\ u(0) = u_0(x), & \text{in} \quad \mathbf{R}^n, \end{cases} \quad (1.1)$$

where $u = (u^1(t,x), \cdots, u^n(t,x))$ and $\pi = \pi(t,x)$ denote the unknown velocity vector and the unknown scalar pressure of the fluid at the point $(t,x) \in (0,T) \times \mathbf{R}^n$, respectively, while $u_0 = (u_0^1(x), \cdots, u_0^n(x))$ is a given initial velocity vector satisfying $\nabla \cdot u_0 = 0$. Here we use the notation:

$$u \cdot \nabla v = \sum_{i=1}^{n} u^i \frac{\partial v}{\partial x_i}, \quad \nabla \cdot u = \sum_{i=1}^{n} \frac{\partial u^i}{\partial x_i},$$

for vector functions u, v.

For given $u_0 \in L^2(\mathbf{R}^n)$ with $\nabla \cdot u_0 = 0$, J. Leray and E. Hopf [17, 20](see also [21, 30]) constructed a global weak solution $u(t,x) \in L^\infty(0, \infty; L^2(\mathbf{R}^n)) \cap L^2(0, \infty; H^1(\mathbf{R}^n))$. It is well known that the weak solution is unique and regular in two spatial dimensions[11, 30]. In dimensions $n \geq 3$, however, the question of regularity of weak solutions has remained open in spite of much effort of mathematicians. There are two ways to develop the regularity theory for the Navier-Stokes equations. One is to give a regularity criterion on weak solutions and the other is to study a better partial regularity of weak solutions which are known to exist. For the latter, L. Caffarelli, R. Kohn, L. Nirenberg[4] proved that the one dimensional Hausdorff measure of the singular set is zero, which is the best result at present. J. Serrin[24, 25] is the pioneer in the former research, later on, Fabes, Jones and Riviere[13], Sohr[26], Giga[16], Struwe[27] and Takahashi[28] extended and improved Serrin's regularity criterion. In order to state their result, we first recall the definition of weak solutions to the Navier-Stokes equations.

DEFINITION 1.1. *Let $u_0 \in L^2(\mathbf{R}^n)$ with $\nabla \cdot u_0 = 0$. The function $u(t,x)$ will be called a weak solution of (NS) on $(0,T)$ if u satisfies the following properties:*

(1) $u \in L^\infty(0, T; L^2(\mathbf{R}^n)) \cap L^2(0, T; H^1(\mathbf{R}^n))$;
(2) $u_t - \Delta u + u \cdot \nabla u + \nabla \pi = 0 \quad in \quad \mathcal{D}'(\mathbf{R}^n \times (0, T))$;
(3) $\nabla \cdot u = 0 \quad in \quad \mathcal{D}'(\mathbf{R}^n \times (0, T))$;
(4) (*The Energy Inequality*)

$$\|u(t)\|_2^2 + 2 \int_0^t \|\nabla u(\tau)\|_2^2 d\tau \leq \|u_0\|_2^2, \qquad 0 < t < T.$$

Their result now reads:

PROPOSITION 1.1. *Suppose that $u(t, x)$ is a weak solution of (NS) on $(0, T)$, and satisfies*

$$\int_0^T \|u(t)\|_r^q dt < \infty \qquad for \quad \frac{2}{q} + \frac{n}{r} = 1 \quad n < r \leq \infty. \qquad (1.2)$$

Then u is regular on $(0, T]$, that is $u \in C^\infty((0, T] \times \mathbf{R}^n)$.

Instead of the solution itself, H. Beirão da Veiga[1] proved the following regularity criterion in terms of the gradient of the solution.

PROPOSITION 1.2. *Suppose that $u(t, x)$ is a weak solution of (NS) on $(0, T)$, and satisfies*

$$\int_0^T \|\nabla u(t)\|_r^q dt < \infty \qquad for \quad \frac{2}{q} + \frac{n}{r} = 2 \quad \frac{n}{2} < r \leq \infty. \qquad (1.3)$$

Then the weak solution u is regular in $(0, T]$.

REMARK 1.1.   *Since the gradient of velocity can be described by the vorticity through the singular integral operator, for example, for $n = 3$, $\nabla u = (-\triangle)^{-1} \nabla \, rot \, rot \, u$, Proposition 1.2 also states a condition for the vorticity $rot \, u$ if $r < \infty$.*

Recently H. Kozono, T. Ogawa, and Y. Taniuchi [19] refined the condition (1.3) to

$$\int_0^T \|rot \, u(t)\|_{\dot{B}_r^{0,\infty}}^q dt < \infty \qquad for \quad \frac{2}{q} + \frac{n}{r} = 2 \quad n \leq r \leq \infty. \qquad (1.4)$$

Here and thereafter, $\dot{B}_p^{s,q}$ stands for the homogenous Besov space, see Definition 1.2. Their proof is based on the logarithmic Sobolev inequality in the Besov spaces, and the authors established a priori estimate with a double exponential growth, that is

$$\|u(t)\|_{H^s} \leq \|u_0\|_{H^s} \exp\left\{ \exp\left(C \int_0^t (1 + \|rot u(\tau)\|_{\dot{B}_r^{0,\infty}})^q d\tau \right) \right\}, \qquad (1.5)$$

where $s > \frac{n}{2} + 1, \frac{2}{q} + \frac{n}{r} = 2, n \leq r \leq \infty$.

On the other hand, for $u_0 \in H^s(\mathbf{R}^n), s \geq \frac{n}{2} - 1$, H. Fujita and T. Kato[14] proved that there exists $T = T(\|u_0\|_{H^s})$ and a unique strong solution $u(t)$ of (NS) on $[0, T)$ in the class

$$(CL)_s \quad u \in C([0, T); H^s) \cap C^1((0, T); H^s) \cap C((0, T); H^{s+2}).$$

It is a natural question whether $u(t)$ loses its regularity at $t = T$. Y. Giga[16] showed that if the above strong solution $u(t)$ satisfies the condition

$$\int_0^T \|u(t)\|_r^q dt < \infty \qquad for \quad \frac{2}{q} + \frac{n}{r} = 1, \quad n < r \le \infty,$$

then u can be continued to the solution in the class $(CL)_s$ beyond $t = T$. Recently, in the endpoint case(i.e. $r = \infty$), based on the bilinear estimates in the BMO space for the nonlinear term $u \cdot \nabla u$, H. Kozono and Y. Tanuchi[18] showed if

$$\int_0^T \|u(t)\|_{BMO}^2 dt < \infty,$$

then the strong solution $u(t)$ in the class $(CL)_s$ on $[0, T)$ can be continued to the one with the same regularity on for some $T' > T$. Moreover, the authors obtained the single exponential estimate of the solution. However, using the logarithmic Sobolev inequality, the L^∞-theory can only give a priori estimate with the double exponential growth.

The main purpose of this paper is to improve some known results about the regularity criterion of weak solutions and the blow-up criterion of smooth solutions. Instead of the logarithmic Sobolev inequality, our main tools are the frequency localization and the Littlewood-Paley trichotomy decomposition, which is a basic way to analysis bilinear expressions. These techniques have been widely used in the study of the fluid dynamics equations, see [6, 7, 12, 15, 22, 32] and reference therein.

Before stating main results, let's introduce some notations and definitions. Let $\mathcal{S}(\mathbf{R}^n)$ be the Schwartz class of rapidly decreasing functions. Given $f \in \mathcal{S}(\mathbf{R}^n)$, its Fourier transform $\hat{f}$ is defined by

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} e^{-ix \cdot \xi} f(x) dx.$$

Let us choose a nonnegative radial function $\phi \in \mathcal{S}(\mathbf{R}^n)$ such that

$$0 \le \hat{\phi}(\xi) \le 1, \quad \hat{\phi}(\xi) = \begin{cases} 1, & if \quad |\xi| \le 1, \\ 0, & if \quad |\xi| \ge 2. \end{cases}$$

We also introduce another function given by

$$\psi(x) = \phi(x) - 2^{-n}\phi(x/2).$$

Set

$$\phi_j(x) = 2^{jn}\phi(2^j x), \quad \psi_j(x) = 2^{jn}\psi(2^j x), \quad j \in \mathbb{Z}.$$

Then the Littlewood-Paley projection operators $P_{\le j}$ and P_j are respectively defined by

$$P_{\le j}f = \phi_j * f, \quad P_j f = \psi_j * f, \quad j \in \mathbb{Z}.$$

Informally, P_j is a frequency projection to the annulus $\{|\xi| \sim 2^j\}$, while $P_{\le j}$ is a frequency projection to the ball $\{|\xi| \lesssim 2^j\}$. We shall sometimes write $P_{<j}$ for $P_{\le j-1}$. We also define more general projections by

$$P_{k_1 \le \cdot \le k_2}f = \sum_{j=k_1}^{k_2} P_j f.$$

DEFINITION 1.2. *Let $s \in \mathbf{R}, p, q \in [1, \infty]$, the homogeneous Besov space $\dot{B}_p^{s,q}(\mathbf{R}^n)$ is defined by*

$$\dot{B}_p^{s,q}(\mathbf{R}^n) := \{f \in \mathcal{Z}'(\mathbf{R}^n); \|f\|_{\dot{B}_p^{s,q}} < +\infty\}.$$

Here

$$\|f\|_{\dot{B}_p^{s,q}} = \begin{cases} (\sum_{j=-\infty}^{+\infty} 2^{sjq}\|P_j f\|_p^q)^{\frac{1}{q}}, & if\ q < +\infty, \\ \sup_{j \in \mathbb{Z}} 2^{sj}\|P_j f\|_p, & if\ q = +\infty, \end{cases}$$

and $\mathcal{Z}'(\mathbf{R}^n)$ denotes the dual space of $\mathcal{Z}(\mathbf{R}^n) = \{f \in \mathcal{S}(\mathbf{R}^n); D^\alpha \hat{f}(0) = 0; \forall \alpha \in \mathbf{N}^n$ multi-index$\}$ and can be identified by the quotient space of $\mathcal{S}'/\mathcal{P}$ with the polynomials space $\mathcal{P}$. We refer to [2, 31] for more details.

We will also need a mixed space-time space $\widetilde{L}^r(a, b; \dot{B}_p^{s,q})$, whose norm is defined by

$$\|u\|_{\widetilde{L}^r(a,b;\dot{B}_p^{s,q})} := \begin{cases} (\sum_{j=-\infty}^{+\infty} 2^{sjq}\|P_j f\|_{L^r(a,b;L_x^p)}^q)^{\frac{1}{q}}, & if\ q < +\infty, \\ \sup_{j \in \mathbb{Z}} 2^{sj}\|P_j f\|_{L^r(a,b;L_x^p)}, & if\ q = +\infty. \end{cases}$$

For the convenience, we sometimes use $\widetilde{L}_T^r \dot{B}_p^{s,q}$ to denote $\widetilde{L}^r(0, T; \dot{B}_p^{s,q})$. For the introduction of this type of space, we refer to [8].

Now we state the space-time estimates in the Besov spaces of the solution to the Navier-Stokes equations, which might be of independent interest.

THEOREM 1.1. *Let $p, q, r, \sigma, \gamma, s$ satisfy*

$$p \in [2, \infty), q \in [1, \infty), (r, \sigma) \in [1, \infty], \gamma \in (-1, 1];$$
$$\frac{2}{q} + \frac{n}{r} = 1 + \gamma, \frac{n}{r} - \gamma + 1 + 2s > 0,\ and\ (r, \gamma) \neq (\infty, 1).$$

Assume that u is a smooth solution to the Navier-Stokes equations on $[0, T)$ with the initial data $u_0 \in \dot{B}_p^{s,\sigma}$. Then there holds

$$\|u\|_{\widetilde{L}_T^\infty \dot{B}_p^{s,\sigma}}^2 + \|u\|_{\widetilde{L}_T^2 \dot{B}_p^{s+1,\sigma}}^2 \leq C\|u_0\|_{\dot{B}_p^{s,\sigma}}^2 + C\|u\|_{\widetilde{L}_T^q \dot{B}_r^{\gamma,\infty}}^q \|u\|_{\widetilde{L}_T^\infty \dot{B}_p^{s,\sigma}}^2.$$

REMARK 1.2. *If the solution $u(t)$ satisfies*

$$A \equiv \int_0^T \|u(t)\|_{\dot{B}_r^{\gamma,\infty}}^q dt < +\infty,$$

i.e., $u(t) \in L^q(0, T; \dot{B}_r^{\gamma,\infty})$, then we can deduce the following single exponential estimate

$$\|u\|_{\widetilde{L}_T^\infty \dot{B}_p^{s,\sigma}}^2 + \|u\|_{\widetilde{L}_T^2 \dot{B}_p^{s+1,\sigma}}^2 \leq C\|u_0\|_{\dot{B}_p^{s,\sigma}}^2 \exp(C'A). \qquad (1.6)$$

In fact, take $\varepsilon_0 > 0$ such that $C\varepsilon_0 \leq \frac{1}{2}$, where C is the constant appearing in Theorem 1.4. If $A \leq \varepsilon_0$, then from Theorem 1.4 it follows that

$$\|u\|_{\widetilde{L}_T^\infty \dot{B}_p^{s,\sigma}}^2 + \|u\|_{\widetilde{L}_T^2 \dot{B}_p^{s+1,\sigma}}^2 \leq 2C\|u_0\|_{\dot{B}_p^{s,\sigma}}^2, \qquad (1.7)$$

since $\|u\|_{\widetilde{L}^q(0,T;B_r^{\gamma,\infty})} \leq \|u\|_{L^q(0,T;B_r^{\gamma,\infty})}$. *Otherwise, we choose* $0 = T_0 < T_1 < \cdots < T_N = T_{N+1} = T$ *so that*

$$\int_{T_i}^{T_{i+1}} \|u(t)\|_{\dot{B}_r^{\gamma},\infty}^q dt = \varepsilon_0, \quad i = 0, \cdots N-1, \quad \int_{T_N}^{T_{N+1}} \|u(t)\|_{\dot{B}_r^{\gamma},\infty}^q dt \leq \varepsilon_0.$$

Then

$$\int_0^T \|u(t)\|_{\dot{B}_r^{\gamma},\infty}^q dt = \sum_{i=0}^{N-1} \int_{T_i}^{T_{i+1}} \|u(t)\|_{\dot{B}_r^{\gamma},\infty}^q dt \geq N\varepsilon_0. \tag{1.8}$$

Repeating the argument that leads to (1.7) N times yields

$$\|u\|_{\widetilde{L}_T^\infty \dot{B}_p^{s,\sigma}}^2 + \|u\|_{\widetilde{L}_T^2 \dot{B}_p^{s+1,\sigma}}^2 \leq 2^N C \|u_0\|_{\dot{B}_p^{s,\sigma}}^2, \quad 0 \leq T \leq T_N. \tag{1.9}$$

Once we use (1.8) we get (1.6) with $C' = \varepsilon_0^{-1} \log 2$.
    If we take $p = \sigma = 2$ *in (1.6), we get*

$$\|u\|_{L_T^\infty H^s}^2 + \|u\|_{L_T^2 H^{s+1}}^2 \leq C\|u_0\|_{H^s}^2 \exp\left(C'A\right),$$

which is sharper than the estimate (1.5).

REMARK 1.3. *By using the analytic semigroup approach, Chen and Xin[10] derived similar estimates in the critical homogenous spaces and established the global existence of small regular solutions.*

REMARK 1.4. *Based on the characterization of Besov spaces in terms of the heat semigroup and Kato's method, Cannone [5] proved the global well-posedness of the Navier-Stokes equations with small data in the critical Besov spaces. If we apply Theorem 1.1 with* $p = r, q = 2, s = \frac{n}{r} - 1$, *we can provide a different proof of his result.*

As an application of Theorem 1.1, we obtain the blow-up criterion of smooth solutions to the Navier-Stokes equations.

THEOREM 1.2. *Let* $u_0 \in H^s(\mathbf{R}^n), s > \frac{n}{2} - 1$. *Assume that* u *is a strong solution of (NS) in the class* $(CL)_s(0,T)$. *If there exists a small* $\varepsilon_0 > 0$ *such that*

$$\limsup_{\varepsilon \to 0} 2^{j\gamma q} \int_{T-\varepsilon}^T \|P_j u(t)\|_r^q dt \leq \varepsilon_0,$$

where q, r, γ *satisfy* $\frac{2}{q} + \frac{n}{r} = 1 + \gamma, \frac{n}{1+\gamma} < r \leq \infty, -1 < \gamma \leq 1,$ *and* $(r,\gamma) \neq (\infty,1)$. *Then* u *can be continued to the strong solution in the class* $(CL)_s(0,T')$ *for some* $T' > T$.

REMARK 1.5. *Notice that if* $u(t) \in L^q(0,T;L^r), \frac{2}{q} + \frac{n}{r} = 1, n < r \leq \infty,$ *then*

$$\limsup_{\varepsilon \to 0} \int_{T-\varepsilon}^T \|P_j u(t)\|_r^q dt \leq \lim_{\varepsilon \to 0} \int_{T-\varepsilon}^T \|u(t)\|_r^q dt = 0,$$

therefore our result improves Giga's result[16].

REMARK 1.6. *For the incompressible Euler equation, Planchon [23] proved a blow-up criterion of smooth solutions in the same kind of mixed Besov space. Therefore our result can be regarded as an extension of his result.*

Next we apply Theorem 1.1 to the regularity criterion of weak solutions to the Navier-Stokes equations.

THEOREM 1.3. *Let q, r, γ be as in Theorem 1.2. Assume that $u(t)$ is a weak solution to the Navier-Stokes equations in $(0, T)$ with data $u_0 \in H^s, s > \frac{n-2}{2}$. If there exists a small ε_0 such that for any $t \in (0, T), u$ satisfies*

$$\lim_{\varepsilon \to 0} \sup_j 2^{j\gamma q} \int_{t-\varepsilon}^t \|P_j u(\tau)\|_r^q d\tau \leq \varepsilon_0, \tag{1.10}$$

then u is regular in $(0, T]$.

REMARK 1.7. *If $u(t) \in L^q(0, T; \dot{B}_r^{1,\infty}), q < +\infty$, then for $t \in (0, T)$*

$$\lim_{\varepsilon \to 0} \sup_j 2^{jq} \int_{t-\varepsilon}^t \|P_j u(t)\|_r^q dt \leq \lim_{\varepsilon \to 0} \int_{t-\varepsilon}^t \|u(t)\|_{\dot{B}_r^{1,\infty}}^q dt = 0,$$

thus, for $\frac{2}{q} + \frac{n}{r} = 2$ with $\frac{n}{2} < r < \infty$, our result is an improvement of the result given by H. Kozono, T. Ogawa, and Y. Taniuchi [19].

REMARK 1.8. *The known results of regularity criterion need to restrict γ to be non-negative, while our result allows γ to be negative. It seems to be impossible to obtain this result by using the logarithmic Sobolev inequality as in [19].*

2. Littlewood-Paley decomposition. Let's firstly introduce the Littlewood-Paly decomposition. Notice that that $P_j = P_{\leq j} - P_{<j}$. Also, if f is an L^2 function then $P_{\leq j} f \to 0$ in L^2 as $j \to -\infty$ and $P_{\leq j} f \to f$ in L^2 as $j \to +\infty$(this is an easy consequence of Parseval's theorem). By telescoping the series, we thus have the homogeneous Littlewood-Paley decompositin

$$f = \sum_{j=-\infty}^{+\infty} P_j f, \tag{2.1}$$

for all $f \in L^2$, where the summation is in the L^2 sense.

In order to analyze bilinear expressions such as the pointwise product $(f, g) \mapsto fg$, we introduce the Littlewood-Paley trichotomy decomposition[29], which is a modified version of Bony's paraproduct[3]. More precisely, one is interested in understanding the quantity $P_k(fg)$-the kth frequency component. One can split f, g into Littlewood-Paley pieces too, and write

$$P_k(fg) = \sum_{k', k'' \in \mathbb{Z}} P_k(P_{k'} f P_{k''} g).$$

Recall that $P_{k'} f$ has Fourier support in $2^{k'-1} \leq |\xi| \leq 2^{k'+1}$ and $P_{k''}$ has Fourier support in $2^{k''-1} \leq |\xi| \leq 2^{k''+1}$, so that $(P_{k'} f)(P_{k''} g)$ has Fourier support in the sum of these two annuli. This sum needs to intersect the annulus $2^{k-1} \leq |\xi| \leq 2^{k+1}$ in order for the summand to be non-zero. This has the following consequences. If

$k' < k - 5$, then the summand vanishes unless $k - 3 \leq k'' \leq k + 3$. Similarly, if $k'' < k - 5$, then the summand vanishes unless $k - 3 \leq k' \leq k + 3$. If $k' > k + 5$, then the summand vanishes unless $k' - 3 \leq k'' \leq k' + 3$. Putting this all together, we get

$$P_k(fg) = P_k \left\{ P_{<k-5} f P_{k-3 \leq \cdot \leq k+3} g + P_{k-3 \leq \cdot \leq k+3} f P_{<k-5} g \right.$$

$$\left. + P_{k-5 \leq \cdot \leq k+5} f P_{k-5 \leq \cdot \leq k+5} g + \sum_{k', k'' > k+5; |k' - k''| \leq 3} P_{k'} f P_{k''} g \right\}. \tag{2.2}$$

In other words, there are four distinct interactions of f and g which give a non-zero contribution to $P_k(fg)$:
- Low-high interactions: f has frequency $\ll 2^k$, g has frequency $\sim 2^k$.
- High-low interactions: f has frequency $\sim 2^k$, g has frequency $\ll 2^k$.
- Low-low interactions: f and g have frequency $\sim 2^k$.
- High-high interactions: f and g have frequency $\sim 2^{k'}$ for some $2^{k'} \gg 2^k$.

The equality (2.2) is called the Littlewood-Paley trichotomy decomposition, which plays an important role in the proof of Theorem 1.1.

Next we prove some lemmas that will be used in the proof of Theorem 1.1.

LEMMA 2.1. *Assume that* $f \in L^p(\mathbf{R}^n)$, *and* $\operatorname{supp} \hat{f} \subset \{\xi \in \mathbf{R}^n; |\xi| \sim 2^j\}, j \in \mathbb{Z}$. *Then there exists a constant C independent of f, j such that*

$$\|f\|_p \leq C 2^{-j|\alpha|} \sup_{|\beta| = |\alpha|} \|D^\beta f\|_p, \quad 1 \leq p \leq \infty. \tag{2.3}$$

$$\|D^\alpha f\|_q \leq C 2^{jn(\frac{1}{p} - \frac{1}{q}) + j|\alpha|} \|f\|_p, \quad 1 \leq p \leq q \leq \infty, \tag{2.4}$$

for any $\alpha \in \mathbf{N}_0^n$.

For a proof, we see [9].

LEMMA 2.2. *Let* $j \in \mathbb{Z}, 2 \leq q < \infty$, $f \in \mathcal{S}(\mathbf{R}^n)$ *and* $\operatorname{supp} \hat{f} \subset \{\xi \in \mathbf{R}^n; |\xi| \sim 2^j\}$. *Then we have the following inequality:*

$$\int_{\mathbf{R}^n} |f|^q dx \leq C 2^{-2j} \int_{\mathbf{R}^n} |\nabla f|^2 |f|^{q-2} dx. \tag{2.5}$$

Here C is a constant depending only on q, n.

Proof. Let $(\theta_i)_{1 \leq i \leq n}$ be a sequence of infinitely differential function on Σ^{n-1} satisfying the following properties:

$$\sum_{i=1}^n \theta_i(\xi) = 1, \quad \operatorname{supp} \theta_i \subset \left\{ \xi \in \Sigma^{n-1}; \frac{1}{2\sqrt{n}} \leq |\xi_i| \leq 1 \right\}.$$

We decompose

$$f(x) = \sum_{i=1}^n \left(\theta_i \left(\frac{\xi}{|\xi|} \right) \hat{f}(\xi) \right)^\vee (x) = \sum_{i=1}^n f_i(x).$$

Thus the L^q-norm of f can be written as

$$\int_{\mathbf{R}^n} |f|^q dx = \int_{\mathbf{R}^n} f f |f|^{q-2} dx = \sum_{i=1}^n \int_{\mathbf{R}^n} f_i f |f|^{q-2} dx.$$

For each $i \in \{1, \cdots, n\}$, we take a function F_i such that $\hat{F}_i = \frac{1}{-i\xi_i}\hat{f}_i(\xi)$, that is, $f_i(x) = -\partial_{x_i}F_i(x)$. Then we have

$$\int_{\mathbf{R}^n}|f|^q dx = \sum_{i=1}^n \int_{\mathbf{R}^n} -\partial_{x_i}F_i(x)f|f|^{q-2}dx = \sum_{i=1}^n \int_{\mathbf{R}^n} F_i\partial_{x_i}(f|f|^{q-2})dx,$$

where we have used the integration by parts in the last equality. It follows from the above equality and the Hölder inequality that

$$\int_{\mathbf{R}^n}|f|^q dx \le \sum_{i=1}^n \|F_i\|_q \left(\int_{\mathbf{R}^n}|\partial_{x_i}(f|f|^{q-2})|^{q'}dx\right)^{\frac{1}{q'}},$$

here q' is the conjugate exponent defined by $1/q + 1/q' = 1$. Notice that supp $\hat{f}_i \subset \{\xi \in \mathbf{R}^n; |\xi_i| \sim 2^j\}$, thus supp $\hat{F}_i \subset \{\xi \in \mathbf{R}^n; |\xi_i| \sim 2^j\}$, then by Lemma 2.1, we deduce that

$$\|F_i\|_q \le C2^{-j}\|\partial_{x_i}F_i\|_q = C2^{-j}\|f_i\|_q \le C2^{-j}\|f\|_q.$$

On the other hand, we have

$$\left|\partial_{x_i}\left(f|f|^{q-2}\right)\right| \le (q-1)|\nabla f||f|^{q-2}.$$

Putting together our estimates, we conclude that

$$\int_{\mathbf{R}^n}|f|^q dx \le C2^{-j}\|f\|_q \sum_{i=1}^n \left(\int_{\mathbf{R}^n}|\partial_{x_i}(f|f|^{q-2})|^{q'}dx\right)^{\frac{1}{q'}}$$

$$\le C2^{-j}\|f\|_q \left(\int_{\mathbf{R}^n}|\nabla f|^{q'}|f|^{(q-2)q'}dx\right)^{\frac{1}{q'}}$$

$$\le C2^{-j}\|f\|_q \left(\int_{\mathbf{R}^n}|\nabla f|^2|f|^{q-2}dx\right)^{\frac{1}{2}}\left(\int_{\mathbf{R}^n}|f|^q dx\right)^{\frac{1}{q'}-\frac{1}{2}},$$

which implies (2.5). $\square$

LEMMA 2.3 (Commutator estimate). *Let $k \in \mathbb{Z}, p, q, r$ satisfy $1 \le p, q, r \le \infty, \frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Then for any $f, g \in \mathcal{S}(\mathbf{R}^n)$, we have*

$$\|P_k(f\nabla g) - fP_k(\nabla g)\|_p \le C\|\nabla f\|_q\|g\|_r, \tag{2.6}$$

where C is a constant independent of f and g.

Proof. First we have

$$P_k(f\partial_{x_i}g) - fP_k(\partial_{x_i}g) = \int_{\mathbf{R}^n} 2^{kn}\psi(2^k y)(f(x-y) - f(x))\partial_{y_i}g(x-y)dy,$$

then by integration by parts, we get

$$P_k(f\partial_{x_i}g) - fP_k(\partial_{x_i}g)$$

$$= \int_{\mathbf{R}^n} 2^{(n+1)k}\partial_{y_i}\psi(2^k y)(f(x-y) - f(x))g(x-y)dy$$

$$- \int_{\mathbf{R}^n} 2^{kn}\psi(2^k y)\partial_{y_i}f(x-y)g(x-y)dy. \tag{2.7}$$

By the Minkowski inequality and the Hölder inequality, we can estimate the second term on the right hand side of (2.7) as follows:

$$\left\|\int_{\mathbf{R}^n} 2^{kn}\psi(2^k y)\partial_{y_i} f(x-y)g(x-y)dy\right\|_p \leq C\|\nabla f\|_q\|g\|_r. \qquad (2.8)$$

For the first term, by the fundamental theorem of calculus, we can rewrite it as

$$\int_{\mathbf{R}^n} 2^{(n+1)k}\partial_{y_i}\psi(2^k y)\int_0^1 y\cdot\nabla f(x-ty)\mathrm{d}t g(x-y)dy,$$

thus, in the same way as in (2.8), we have

$$\left\|\int_{\mathbf{R}^n} 2^{(n+1)k}\partial_{y_i}\psi(2^k y)(f(x-y)-f(x))g(x-y)dy\right\|_p$$
$$\leq \int_{\mathbf{R}^n} 2^{kn}|2^k y||\partial_{y_i}\psi(2^k y)|\int_0^1 \|\nabla f(x-ty)g(x-y)\|_p dt dy$$
$$\leq C\|\nabla f\|_q\|g\|_r. \qquad (2.9)$$

From (2.7)-(2.9), we get (2.6). □

3. Space-time estimates in the Besov spaces. In this section we prove Theorem 1.1.

Proof of Theorem 1.1. Taking the operation P_k on both sides of (1.1), we get

$$(P_k u^j)_t - \triangle P_k u^j + P_k(u^i\partial_{x_i} u^j) + \partial_{x_j} P_k\pi = 0, \quad (j=1,2,\cdots,n). \qquad (3.1)$$

Multiplying by $|P_k u^j|^{p-2}P_k u^j$ on both sides of (3.1), then integrating with respect to x over $\mathbf{R}^n$, we get the following equality

$$\int_{\mathbf{R}^n} (P_k u^j)_t|P_k u^j|^{p-2}P_k u^j dx$$
$$= \int_{\mathbf{R}^n} \triangle P_k u^j|P_k u^j|^{p-2}P_k u^j dx - \int_{\mathbf{R}^n} P_k(u^i\partial_{x_i} u^j)|P_k u^j|^{p-2}P_k u^j dx$$
$$- \int_{\mathbf{R}^n} \partial_{x_j} P_k\pi|P_k u^j|^{p-2}P_k u^j dx. \qquad (3.2)$$

In what follows, we are going to calculate the above four terms separately. Firstly, we have

$$\int_{\mathbf{R}^n} (P_k u^j)_t|P_k u^j|^{p-2}P_k u^j dx = \frac{1}{p}\frac{d}{dt}\int_{\mathbf{R}^n} |P_k u^j|^p dx. \qquad (3.3)$$

By integration by parts, we find

$$- \int_{\mathbf{R}^n} \triangle P_k u^j|P_k u^j|^{p-2}P_k u^j dx = (p-1)\int_{\mathbf{R}^n} |\nabla P_k u^j|^2|P_k u^j|^{p-2}dx,$$

then apply Lemma 2.2 to obtain

$$- \int_{\mathbf{R}^n} \triangle P_k u^j|P_k u^j|^{p-2}P_k u^j dx \geq C2^{2k}\int_{\mathbf{R}^n} |P_k u^j|^p dx. \qquad (3.4)$$

Using the Littlewood-Paley trichotomy decomposition with $f = u^i, g = \partial_{x_i} u^j$, we decompose the second term on the right hand of (3.2) as follows:

$$\int_{\mathbf{R}^n} P_k(u^i \partial_{x_i} u^j) |P_k u^j|^{p-2} P_k u^j \, dx$$

$$= \sum_{l < k-5} \int_{\mathbf{R}^n} P_k(P_{k-3 \le \cdot \le k+3} u^i P_l(\partial_{x_i} u^j)) |P_k u^j|^{p-2} P_k u^j \, dx$$

$$+ \sum_{l < k-5} \int_{\mathbf{R}^n} P_k(P_l u^i P_{k-3 \le \cdot \le k+3}(\partial_{x_i} u^j)) |P_k u^j|^{p-2} P_k u^j \, dx$$

$$+ \int_{\mathbf{R}^n} P_k(P_{k-5 \le \cdot \le k+5} u^i P_{k-5 \le \cdot \le k+5}(\partial_{x_i} u^j)) |P_k u^j|^{p-2} P_k u^j \, dx$$

$$+ \sum_{l,l' > k+5; |l-l'| \le 3} \int_{\mathbf{R}^n} P_k(P_l u^i P_{l'}(\partial_{x_i} u^j)) |P_k u^j|^{p-2} P_k u^j \, dx$$

$$\equiv I + II + III + IV. \tag{3.5}$$

Next we estimate each term of (3.5) separately below. By the Hölder inequality and Lemma 2.1, we obtain

$$|I| \le C \sum_{l < k-5} \|P_{k-3 \le \cdot \le k+3} u\|_p \|P_k u\|_p^{p-1} \|P_l(\partial_{x_i} u)\|_\infty$$

$$\le C \sum_{l < k-5} 2^{(\frac{n}{r}+1)l} \|P_l u\|_r \|P_{k-3 \le \cdot \le k+3} u\|_p \|P_k u\|_p^{p-1}. \tag{3.6}$$

For III, in the same way as in the proof of (3.6), we also have

$$|III| \le C \sum_{k-5 \le l \le k+5} 2^{(\frac{n}{r}+1)l} \|P_l u\|_r \|P_{k-5 \le \cdot \le k+5} u\|_p \|P_k u\|_p^{p-1}. \tag{3.7}$$

Since $\nabla \cdot u = 0$, we find

$$P_k(P_l u^i P_{l'}(\partial_{x_i} u^j)) = \partial_{x_i} P_k(P_l u^i P_{l'} u^j),$$

then by Lemma 2.1 and the Hölder inequality, we obtain

$$|IV| \le C 2^k \|P_k u\|_\infty \|P_k u\|_p^{p-2} \sum_{l,l' > k+5; |l-l'| \le 3} \|P_l u\|_p \|P_{l'} u\|_p$$

$$\le C 2^{(\frac{n}{r}+1)k} \|P_k u\|_r \|P_k u\|_p^{p-2} \sum_{l,l' > k+5; |l-l'| \le 3} \|P_l u\|_p \|P_{l'} u\|_p. \tag{3.8}$$

We next turn to the estimate of II. First, we observe

$$\int_{\mathbf{R}^n} P_l u^i P_k(P_{k-3 \le \cdot \le k+3} \partial_{x_i} u^j) |P_k u^j|^{p-2} P_k u^j \, dx$$

$$= \int_{\mathbf{R}^n} P_l u^i P_k(\partial_{x_i} u^j) |P_k u^j|^{p-2} P_k u^j \, dx$$

$$= \int_{\mathbf{R}^n} \partial_{x_i}(P_l u^i P_k u^j) |P_k u^j|^{p-2} P_k u^j \, dx$$

$$= -(p-1) \int_{\mathbf{R}^n} P_l u^i P_k(\partial_{x_i} u^j) |P_k u^j|^{p-2} P_k u^j \, dx = 0,$$

thus we have

$$II = \sum_{l<k-5} \int_{\mathbf{R}^n} \{ P_k(P_l u^i P_{k-3\le\cdot\le k+3}\partial_{x_i} u^j) - $$
$$- P_l u^i P_k(P_{k-3\le\cdot\le k+3}\partial_{x_i} u^j) \} |P_k u^j|^{p-2} P_k u^j dx.$$

On the other hand, by Lemma 2.3 and Lemma 2.1, we get

$$\left\| P_k(P_l u^i P_{k-3\le\cdot\le k+3}\partial_{x_i} u^j) - P_l u^i P_k(P_{k-3\le\cdot\le k+3}\partial_{x_i} u^j)\right\|_p$$
$$\le C\|P_l \nabla u\|_\infty \|P_{k-3\le\cdot\le k+3} u\|_p$$
$$\le C2^{(\frac{n}{r}+1)l}\|P_l u\|_r \|P_{k-3\le\cdot\le k+3} u\|_p,$$

from which we get

$$|II| \le C \sum_{l<k-5} 2^{(\frac{n}{r}+1)l}\|P_l u\|_r \|P_{k-3\le\cdot\le k+3}u\|_p \|P_k u\|_p^{p-1}. \tag{3.9}$$

Then by summing up (3.6)-(3.9), we have

$$\int_{\mathbf{R}^n} P_k(u^i\partial_{x_i} u^j)|P_k u^j|^{p-2} P_k u^j dx$$
$$\le C \sum_{l<k+5} 2^{(\frac{n}{r}+1)l}\|P_l u\|_r \|P_k u\|_p^{p-2}$$
$$\times (\sum_{m=k-5}^{k+5} \|P_m u\|_p^2 + \sum_{l,l'>k+5;|l-l'|\le 3} \|P_l u\|_p\|P_{l'} u\|_p). \tag{3.10}$$

Finally, we estimate the pressure term in (3.2). We recall the well-known representation of the pressure

$$\pi = (-\triangle)^{-1}\partial_{x_i}\partial_{x_m}(u^i u^m),$$

which gives

$$\partial_{x_j}\pi = (-\triangle)^{-1}\partial_{x_j}\partial_{x_i}\partial_{x_m}(u^i u^m) = (-\triangle)^{-1}\partial_{x_j}\partial_{x_m}(u^i\partial_{x_i} u^m).$$

Then exactly as in the derivation of (3.10), we have

$$\int_{\mathbf{R}^n} \partial_{x_j} P_k\pi|P_k u^j|^{p-2} P_k u^j dx$$
$$\le C \sum_{l<k+5} 2^{(\frac{n}{r}+1)l}\|P_l u\|_r \|P_k u\|_p^{p-2}$$
$$\times (\sum_{m=k-5}^{k+5} \|P_m u\|_p^2 + \sum_{l,l'>k+5;|l-l'|\le 3} \|P_l u\|_p\|P_{l'} u\|_p). \tag{3.11}$$

Then by summing up (3.2)-(3.4), (3.10), and (3.11), and doing summation for j, we finally obtain

$$\frac{d}{dt}\|P_k u\|_p^p + 2^{2k}\|P_k u\|_p^p$$
$$\le C \sum_{l<k+5} 2^{(\frac{n}{r}+1)l}\|P_l u\|_r \|P_k u\|_p^{p-2}$$
$$\times (\sum_{m=k-5}^{k+5} \|P_m u\|_p^2 + \sum_{l,l'>k+5;|l-l'|\le 3} \|P_l u\|_p\|P_{l'} u\|_p). \tag{3.12}$$

Set

$$G_k(t) = \sum_{m=k-5}^{k+5} \|P_m u(t)\|_p^2 + \sum_{l,l'>k+5; |l-l'|\leq 3} \|P_l u(t)\|_p \|P_{l'} u(t)\|_p.$$

Then the Gronwall inequality gives

$$\|P_k u(t)\|_p^2 \leq \|P_k u(0)\|_p^2 e^{-2^{2k}t} + C \sum_{l<k+5} 2^{(\frac{n}{r}+1)l} e^{-2^{2k}t} * (\|P_k u\|_r G_k),$$

where the sign $*$ denotes the convolution of functions defined on $\mathbf{R}^+$. We take $L^\infty(0,T)$ norm to obtain

$$\|P_k u\|_{L_t^\infty L_x^p}^2 \leq \|P_k u(0)\|_p^2 + C \sum_{l<k+5} 2^{(\frac{n}{r}+1)l} \|P_l u\|_{L_t^q L_x^r} \|G_k\|_{L_t^{q'}}$$

$$\leq \|P_k u(0)\|_p^2 + C 2^{(\frac{n}{r}+1-\gamma)k} \|u\|_{\widetilde{L_T^q} \dot{B}_r^{\gamma,\infty}} \|G_k\|_{L_t^{q'}}, \tag{3.13}$$

where we have used the Hölder inequality. On the other hand, we take $L^2(0,T)$ norm to obtain

$$\|P_k u\|_{L_t^2 L_x^p}^2 \leq \|P_k u(0)\|_p^2 \|e^{-2^{2k}t}\|_{L_t^1} +$$

$$+ C \sum_{l<k+5} 2^{(\frac{n}{r}+1)l} \|e^{-2^{2k}t}\|_{L_t^1} \|P_l u\|_{L_t^q L_x^r} \|G_k\|_{L_t^{q'}}$$

$$\leq C 2^{-2k} \|P_k u(0)\|_p^2 + C 2^{(\frac{n}{r}-\gamma-1)k} \|u\|_{\widetilde{L_T^q} \dot{B}_r^{\gamma,\infty}} \|G_k\|_{L_t^{q'}}, \tag{3.14}$$

where we have used the Young inequality and Hölder inequality. Multiplying by 2^{2sk} on both sides of (3.13), and for the case of $\sigma \geq 1$ taking $l^{\frac{\sigma}{2}}$ quasi-norm with respect to k, we obtain

$$\|u\|_{\widetilde{L_T}^\infty \dot{B}_p^{s,\sigma}}^2 \leq \|u(0)\|_{\dot{B}_p^{s,\sigma}}^2$$

$$+ C \|u\|_{\widetilde{L_T^q} \dot{B}_r^{\gamma,\infty}} \left(\sum_k 2^{(\frac{n}{r}-\gamma+1+2s)\frac{k\sigma}{2}} \|G_k\|_{L_t^{q'}}^{\frac{\sigma}{2}} \right)^{\frac{2}{\sigma}}. \tag{3.15}$$

Multiplying by $2^{2(s+1)k}$ on both sides of (3.14), and taking $l^{\frac{\sigma}{2}}$ norm, we obtain

$$\|u\|_{\widetilde{L_T}^2 \dot{B}_p^{s+1,\sigma}}^2 \leq C \|u(0)\|_{\dot{B}_p^{s,\sigma}}^2$$

$$+ C \|u\|_{\widetilde{L_T^q} \dot{B}_r^{\gamma,\infty}} \left(\sum_k 2^{(\frac{n}{r}-\gamma+1+2s)\frac{k\sigma}{2}} \|G_k\|_{L_t^{q'}}^{\frac{\sigma}{2}} \right)^{\frac{2}{\sigma}}. \tag{3.16}$$

Next we will prove the following inequality

$$\left(\sum_k 2^{(\frac{n}{r}-\gamma+1+2s)\frac{k\sigma}{2}} \|G_k\|_{L_t^{q'}}^{\frac{\sigma}{2}} \right)^{\frac{2}{\sigma}} \leq C \|u\|_{\widetilde{L_T}^\infty \dot{B}_p^{s,\sigma}}^{\frac{2}{q}} \|u\|_{\widetilde{L_T}^2 \dot{B}_p^{s+1,\sigma}}^{\frac{2}{q'}}. \tag{3.17}$$

Notice that

$$G_k(t) \leq C \sum_{l\geq k-5} \|P_l u(t)\|_p^2.$$

If we let $\theta = \frac{n}{r} - \gamma + 1 + 2s(\theta > 0$, by assumptions), then we have

$$\left(\sum_k 2^{\theta \frac{k\sigma}{2}} \|G_k\|_{L_t^{q'}}^{\frac{\sigma}{2}}\right)^{\frac{2}{\sigma}}$$

$$\leq C \left(\sum_k 2^{\theta \frac{k\sigma}{2}} \left(\sum_{l \geq k-5} \|P_l u\|_{L_t^\infty L_x^p}^{\frac{2}{q}} \|P_l u\|_{L_t^2 L_x^p}^{\frac{2}{q'}}\right)^{\frac{\sigma}{2}}\right)^{\frac{2}{\sigma}}$$

$$= C \left(\sum_k 2^{\theta \frac{k\sigma}{2}} \left(\sum_{l \geq -5} \|P_{l+k} u\|_{L_t^\infty L_x^p}^{\frac{2}{q}} \|P_{l+k} u\|_{L_t^2 L_x^p}^{\frac{2}{q'}}\right)^{\frac{\sigma}{2}}\right)^{\frac{2}{\sigma}}$$

$$= C \left(\sum_k \left(\sum_{l \geq -5} 2^{-\theta l}(2^{(k+l)s}\|P_{l+k} u\|_{L_t^\infty L_x^p})^{\frac{2}{q}}(2^{(k+l)(s+1)}\|P_{l+k} u\|_{L_t^2 L_x^p})^{\frac{2}{q'}}\right)^{\frac{\sigma}{2}}\right)^{\frac{2}{\sigma}},$$

where we have used the fact $(\theta - 2s)q' = 2$ in the last equality. If $\sigma \geq 2$, we use the Minkowski inequality to obtain

$$\left(\sum_k 2^{\theta \frac{k\sigma}{2}} \|G_k\|_{L_t^{q'}}^{\frac{\sigma}{2}}\right)^{\frac{2}{\sigma}}$$

$$\leq C \sum_{l \geq -5} 2^{-\theta l} \left(\sum_k (2^{ks\sigma}\|P_k u\|_{L_t^\infty L_x^p}^{\sigma})^{\frac{1}{q}}(2^{k(s+1)\sigma}\|P_k u\|_{L_t^2 L_x^p}^{\sigma})^{\frac{1}{q'}}\right)^{\frac{2}{\sigma}}$$

$$\leq C \|u\|_{\widetilde{L_T}^\infty \dot{B}_p^{s,\sigma}}^{\frac{2}{q}} \|u\|_{\widetilde{L_T}^2 \dot{B}_p^{s+1,\sigma}}^{\frac{2}{q'}}.$$

where we have used the Hölder inequality for series in the last line. On the other hand, for $\sigma \leq 2$, we use the embedding $l^{\frac{\sigma}{2}} \hookrightarrow l^1$ to obtain

$$\left(\sum_k 2^{\theta \frac{k\sigma}{2}} \|G_k\|_{L_t^{q'}}^{\frac{\sigma}{2}}\right)^{\frac{2}{\sigma}}$$

$$\leq \left(\sum_k 2^{\theta \frac{k\sigma}{2}} \sum_{l \geq k-5} \|P_l u\|_{L_t^\infty L_x^p}^{\frac{\sigma}{q}} \|P_l u\|_{L_t^2 L_x^p}^{\frac{\sigma}{q'}}\right)^{\frac{2}{\sigma}}$$

$$\leq C \left(\sum_{l \geq -5} 2^{-\theta \sigma l/2} \sum_k (2^{ks\sigma}\|P_k u\|_{L_t^\infty L_x^p}^\sigma)^{\frac{1}{q}}(2^{k(s+1)\sigma}\|P_k u\|_{L_t^2 L_x^p}^\sigma)^{\frac{1}{q'}}\right)^{\frac{2}{\sigma}}$$

$$\leq C \|u\|_{\widetilde{L_T}^\infty \dot{B}_p^{s,\sigma}}^{\frac{2}{q}} \|u\|_{\widetilde{L_T}^2 \dot{B}_p^{s+1,\sigma}}^{\frac{2}{q'}}.$$

Putting together the above estimates, we obtain the inequality (3.17). Finally by summing up (3.15), (3.16), and (3.17), we get

$$\|u\|_{\widetilde{L_T}^\infty \dot{B}_p^{s,\sigma}}^2 + \|u\|_{\widetilde{L_T}^2 \dot{B}_p^{s+1,\sigma}}^2$$

$$\leq C \|u(0)\|_{\dot{B}_p^{s,\sigma}}^2 + C \|u\|_{\widetilde{L_T}^q \dot{B}_r^{\gamma,\infty}} \|u\|_{\widetilde{L_T}^\infty \dot{B}_p^{s,\sigma}}^{\frac{2}{q}} \|u\|_{\widetilde{L_T}^2 \dot{B}_p^{s+1,\sigma}}^{\frac{2}{q'}}.$$

Then apply the Young inequality to obtain

$$\|u\|^2_{\widetilde{L}_T^\infty \dot{B}_p^{s,\sigma}} + \|u\|^2_{\widetilde{L}_T^2 \dot{B}_p^{s+1,\sigma}} \le C\|u(0)\|^2_{\dot{B}_p^{s,\sigma}} + C\|u\|^q_{\widetilde{L}_T^q \dot{B}_r^{\gamma,\infty}}\|u\|^2_{\widetilde{L}_T^\infty \dot{B}_p^{s,\sigma}}.$$

This completes the proof of Theorem 1.1. ∎

4. Proof of Theorem 1.2 and Theorem 1.3. Let us firstly prove Theorem 1.2.

Proof of Theorem 1.2. Take $\varepsilon_0 > 0$ such that $C\varepsilon_0 \le \frac{1}{4}$, where C is the constant in Theorem 1.1. Then we choose $\varepsilon > 0$ such that

$$\sup_j 2^{j\gamma q} \int_{T-\varepsilon}^T \|P_j u(t)\|_r^q dt \le 2\varepsilon_0.$$

Thus by Theorem 1.4 with $p = \sigma = 2$, we have

$$\|u\|_{L^\infty([T-\varepsilon,T);H^s)} \le 2C\|u(T-\varepsilon)\|_{H^s}.$$

Hence by a standard argument of continuation of local solutions, the solution $u(t)$ can be continued after $t = T$. ∎

To complete the proof of Theorem 1.3, we will need the following two propositions.

PROPOSITION 4.1. *[16] Let $p > n$. Suppose $u_0 \in L^p(\mathbf{R}^n)$ satisfying $\nabla \cdot u_0 = 0$. Then there is $T_0 > 0$ and a unique mild solution of (NS) on $[0, T_0)$ such that $u \in C([0, T_0); L^p(\mathbf{R}^n))$; Furthermore, let $[0, T_*)$ be the maximal interval such that u solves the Navier-Stokes equations in $C([0, T_*); L^p(\mathbf{R}^n))$. Then*

$$\|u(t)\|_p \ge \frac{C}{(T_* - t)^{\frac{p-n}{2p}}}, \quad \forall\, 0 \le t < T_\star, \tag{4.1}$$

with constant C independent of T_* and p.

PROPOSITION 4.2. *[16] Let u, v be two weak solutions of (NS) on $(0, T)$. Suppose that $v \in L^r(0, T; L^p(\mathbf{R}^n))$, with $\frac{2}{r} + \frac{n}{p} = 1, p > n$. Then u agrees with v on $(0, T) \times \mathbf{R}^n$.*

Proof of Theorem 1.3. Since $u_0 \in H^s, s > \frac{n-2}{2}$, by the Sobolev imbedding theorem, there exists $p > n$ such that $u_0 \in L^p(\mathbf{R}^n)$. Then by Proposition 4.1, there is $T_0 > 0$ and a unique mild solution of (NS) on $[0, T_0)$ such that $v \in C([0, T_0); L^p(\mathbf{R}^n))$. Let $[0, T_*)$ be the maximal existence interval of v. Obviously v are both a weak solution and a regular solution. Thus by Proposition 4.2, u agrees with v on $(0, T_*) \times \mathbf{R}^n$. Now, if $T_* = T$, the proof is complete; Otherwise, by Proposition 4.1 again, we have

$$\|u(t)\|_p \ge \frac{C}{(T_* - t)^{\frac{p-n}{2p}}}, \quad \forall\, t \in [0, T_*). \tag{4.2}$$

On the other hand, by our assumptions, for $T_* \in (0, T)$, we can choose $\varepsilon_0, \varepsilon$ such that

$$\sup_j 2^{j\gamma q} \int_{T_*-\varepsilon}^{T_*} \|P_j u(t)\|_r^q dt \le 2\varepsilon_0 \le \frac{1}{2C},$$

where C is the constant in Theorem 1.1. Thus by Theorem 1.1 and the Sobolev embedding theorem, we have

$$\sup_{T_*-\varepsilon \le t < T_*} \|u(t)\|_p \le C \sup_{T_*-\varepsilon \le t < T_*} \|u(t)\|_{H^s} \le C\|u(T_* - \varepsilon)\|_{H^s} \le C, \tag{4.3}$$

which contradicts with (4.2). This completes the proof of Theorem 1.3. $\square$

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